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HIERARCHICAL MODELING AND MULTIRESOLUTION SIMULATION

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ABSTRACT

As systems become more complex, the need to explicitly account for uncertainty during modeling and simulation grows. The interactions between assumptions made in modeling different subsystems may greatly affect system behavior. Unless these assumptions are quantified and included in the simulation, results can be misleading or even completely wrong.

Piecewise linear (PL) modeling is proposed as a method for quantifying the uncertainty in a model. With PL models, sets of models with varying amounts of uncertainty are easily developed. Robust simulation is then used to account for uncertainty during analysis. Also, robust simulation allows dynamic selection of models. Through the use of PL modeling and robust simulation, unexpected model interaction can be predicted.

These techniques are demonstrated on 3 simple illustrative examples. A model library is developed for a saturation. This saturation is then used in a feedback system, and the simulation results of various models are examined. A final example demonstrates the benefits of changing model accuracy during simulation.

1 INTRODUCTION

Two trends in control design are placing new demands on modeling and simulation. First, component libraries are being developed to allow rapid prototyping of new designs. Second, as systems become more complex, models from different areas of engineering are being interconnected. For example, an engineer may choose a wing from a previous design, attach it to a new airplane model, and then simulate its performance during flight maneuvers. While conceptually simple, this is computationally very difficult. Aerodynamic coefficients are measured under static conditions. Structural models of the wing are inherently dynamic. How does the atmosphere interact with the structural modes of wing? How do the aerodynamic properties change as the wing flexes? Are these interactions important? This paper presents a modeling and simulation technique that addresses these types of issues.

In short, large systems can be constructed as the interconnection of simpler subsystems, but large system models are not simply the interconnection of simpler subsystem models. The assumptions made in modeling one subsystem may critical to a second subsystem's model. The solution to this problem is to explicitly define all assumptions as uncertainties in the model and use this information during simulation (Khatri 1996).

The first part of the solution, defining model assumptions, leads directly to model hierarchies. Once the assumptions are quantified, it is natural to create a set of models with varying degrees of accuracy. The second part of the solution, accounting for assumptions during simulation, requires new simulation techniques. Simulation results from any valid model should yield compatible results. Two simulation results for a system are compatible when their differences fall within explicit error bounds.

One implementation of this solution is to model using sets of uncertain piecewise linear (PL) systems and simulate using robust simulation. Sets of uncertain PL models are easily created from general nonlinear systems and admit a simple partial ordering of their accuracy. Robust simulation explicitly accounts for model uncertainty and allows for switching between models during a single simulation. These techniques present a practical, computationally tractable solution to the problems encountered when simulating large systems.

2 HIERARCHICAL MODELING

When creating a mathematical model of a physical system, any degree of complexity is possible. A resistor model can simply be $e = ir$ or can include a variety of parasitic effects. When developing models for interconnection with other systems, the degree of accuracy needed is not known *a priori*. Effects critical to one system may be superfluous to another. Thus, a set of models with varying degrees of accuracy are needed. Finally, the set of models must be useful for computation.

Model hierarchies are an area of active research in the computer graphics community. Wavelets are used to create sets of models of 3 dimensional objects for on-screen rendering. As an object moves toward the foreground of an image, more complex models are used to display finer detail (Schröder 1996).

Modeling for control has its own set of requirements. Primarily, simulation and analysis computations must grow reasonably as state dimension increases. In computer graphics, all problems are limited to 2 or 3 dimensions. Additionally, some measure of model accuracy is needed. Is a nonlinear 2 state model more accurate than a linear 5 state model? PL systems satisfy both of these requirements.

A PL system, described in more detail in (Kantner 1996) and (Sontag 1982), is a nonlinear system of the form

$$\begin{aligned} x[k+1] &= A_i x[k] + \bar{A}_i + B_i\,[u[k]; n[k]] \\ y[k] &= C_i x[k] + \bar{C}_i + D_i\,[u[k]; n[k]] \end{aligned} \quad , \quad (x[k], u[k]) \in R_i,\ i \in 1 \ldots l \qquad (1)$$

where R_i is one of a finite number of regions in the state and input space. $u[k]$ is the control input and $n[k]$ is a norm-bounded uncertainty or noise. To facilitate analysis, each R_i is a closed polyhedra and $|n[k]|_\infty \leq 1$.

This type of system is a conceptually simple extension of a linear system; each region R_i behaves as an affine system. PL systems are also easily simulated and implemented on digital computers. Identifying the appropriate R_i for a given $[x[k], u[k]]$ only requires matrix multiplication. By adding additional noise inputs, a PL system can approximate a nonlinear system to any degree of accuracy.

A partial ordering of model accuracy is determined by the number of piecewise linear regions, l, and the amount of noise in the model. Larger l and less noise lead to more accurate models. There are several methods for measuring the amount of noise in the system. For systems with a single noise input, one method is

$$\max_{i \in 1 \ldots l} \left| \begin{bmatrix} B_i \\ D_i \end{bmatrix} [0; 1] \right|_\infty .$$

More complex measures can weight the B_i and D_i or use other norms.

3 MODELING EXAMPLE

To demonstrate the ideas from the previous section, a model library is created for a simple static nonlinearity. The saturation

$$y = \frac{2}{\pi} \arctan u, \qquad (2)$$

shown in figure 1, can be described by a variety of PL systems. The simplest model treats it as one region, $l = 1$, with noise,

$$y = 0u + n, |n| \leq 1. \qquad (3)$$

While (3) does not describe the behavior of the nonlinearity, it does describe its bounds. This model is useful when the output of the saturation has little effect on the overall system.

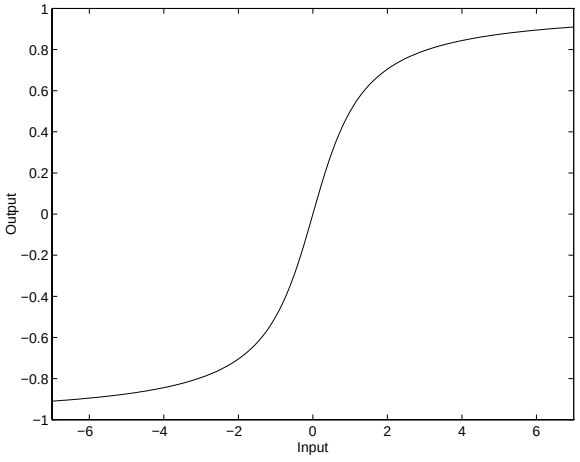

Figure 1: The Saturation Nonlinearity

A more common approximation for saturations is a 3-segment representation. When approximating (2) by a PL system with $l = 3$, some assumptions must be made. First, this approximation will hold for all values u. Second, the model will be symmetric about $u = 0$. Finally, the maximum size of the noise signal will be minimized. The model

$$y = \begin{cases} -0.87 + 0.13n & u < -2.32 \\ 0.38u + 0.13n & -2.32 \leq u \leq 2.32 \\ 0.87 + 0.13n & u > 2.32 \end{cases} , \qquad (4)$$

shown in figure 2, satisfies these requirements. Note that the saturated regions, $|u| > 2.32$ are not nominally equal to the saturated values. For this model, they are offset to reduce the amount of noise needed to cover the true saturation (2). As shown by the dotted lines, there is always a noise that makes the approximate system equal the true system.

A more accurate model can be created by using a 5-segment approximation. Using the same assumptions as before, the model, shown in figure 3, is

$$y = \begin{cases} -0.95 + 0.05n & u < -6.14 \\ 0.06u - 0.55 + 0.05n & -6.14 \leq u < -1.29 \\ 0.49u + 0.05n & -1.29 \leq u \leq 1.29 \\ 0.06u + 0.55 + 0.05n & 1.29 < u \leq 6.14 \\ 0.95 + 0.05n & u > 6.14 \end{cases} \qquad (5)$$

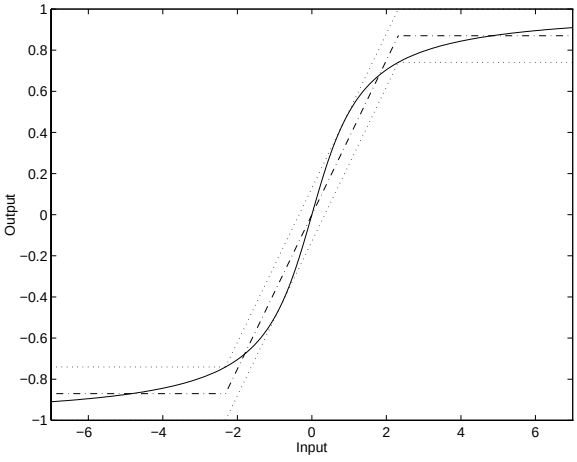

Figure 2: 3-Segment Saturation Approximation

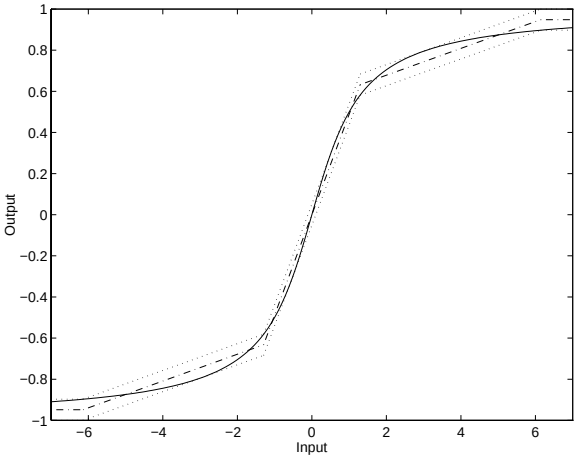

Figure 3: 5-Segment Saturation Approximation

All three of these models are valid representations of the original nonlinearity (2). The more complex models contain less uncertainty (noise), but may require more computation during simulation.

4 MULTIRESOLUTION SIMULATION

Traditional simulation techniques yield one result for an initial condition and noise signal. If several models exist for a system, and each yields a different simulation result, which one is correct? Also, the ability to change models during simulation is desired. One model may be accurate for one operating region and a second may describe a different region in detail.

When simulating uncertain systems, the result should be the set of all possible final conditions, not a single final condition. Since the system is uncertain, it is impossible to predict the final condition exactly. The simulation should also allow a set of initial conditions, so the results of a prior simulation can be used to initialize future runs. More accurate (less uncertain) models should yield smaller sets of final conditions than coarser models for the same set of initial conditions. Finally, all models should be compatible. For any set of initial conditions, the simulation results from all models should have some points in common. Robust simulation, described in (Kantner 1997) and (Kantner 1996), meets these requirements.

Robust simulation is the simulation of sets. For a given initial condition set and uncertainty description, the set of all possible final conditions is calculated. For model libraries that share the same state and input variables (or have mappings between different state and input descriptions), any model can be used at any time step without recalculating prior results. This allows the simulation technique to choose, at each time step, the most appropriate model for a given initial condition set, i.e. multiresolution simulation.

The ability to do dynamic model selection is a direct result of using robust simulation. The result at any time step can be used as the initial condition for simulating any model one time step forward. The technique does not provide guidelines for model selection; it only gives the ability to change models. As long as all models correctly describe the same system, the results are guaranteed, by construction of the algorithm, to be compatible.

This simulation technique exhibits polynomial computational growth in all variables. As problem size grows, computational cost grows reasonably. For each time step of simulation, roughly nl^2 linear programs must be solved. This technique is only applicable for models in the form of (1) and is conservative. The set of final conditions is guaranteed to contain all possible final conditions, but may also contain additional points that cannot be reached. If the results are too conservative, the simulation can be systematically refined until the exact solution is obtained.

5 SIMULATION EXAMPLE

Three 1-dimensional systems are used to demonstrate the multiresolution simulation ideas. While more complex examples can easily be solved, these simple problems illustrate the types of analyses possible. Robust simulation, the algorithm used during simulation, has already been successfully tested on higher order models (Kantner 1996).

5.1 Example 1

The first example demonstrates that in some circumstances, even the coarsest model may give acceptable results. Consider the stable system

$$x[k+1] = 0.4x[k] + 0.05\frac{2}{\pi}\arctan u[k] \qquad (6)$$

with the feedback law $u[k] = -x[k]$. This system is open loop stable, and the input has very little control authority. All simulations are for 5 time steps starting from $x[0] = 100$.

The first simulation of (6), whose results are shown in table 1, uses (3) to approximate the saturation. For this system, even the simplest model gives reasonable results. The final result, $0.942 \leq x[5] \leq 1.107$, differs by only 5% from its center value and, as expected, contains the exact result of 0.956. Note that for the first few time steps, the lower bound is, after rounding, identical to the exact value. This is because the worst case uncertainty (noise), which achieves the lower bound, is also the value needed to make the approximation match the true model.

Table 1: Simulation Results for Simplest Model

Time Step	Lower Bound	Exact	Upper Bound
0	100	100	100
1	39.95	39.95	40.05
2	15.93	15.93	16.07
3	6.32	6.32	6.48
4	2.47	2.48	2.64
5	0.94	0.96	1.11

Running the same simulation using the approximation (4) yields a final result of $0.942 \leq x[5] \leq 0.963$. This result set is much smaller because the saturation only takes on values between 1 and 0.74 in the saturation region. Finally, using (5) to approximate the saturation gives $0.953 \leq x[5] \leq 0.961$. As expected, the model with the least uncertainty gives the tightest bounds.

5.2 Example 2

The second example demonstrates that radically different results can be obtained from different, seemingly accurate, models. The nominally unstable system

$$x[k+1] = 1.1x[k] + \frac{2}{\pi}\arctan u[k] \tag{7}$$

is stabilized about the origin by the feedback law $u[k] = -x[k]$. What values can $x[50]$ attain if $7.5 \leq x[0] \leq 8.5$?

Table 2 shows results from the robust simulation using each of the three saturation approximations. Since the the coarsest approximation, (3), does not account for the sign of the saturation, that model does not result in a stable system. As shown in table 2, the simulation values differ greatly from the exact values.

For the 3-segment approximation, (4), which appears to model the nonlinearity fairly well, some states approach the origin and other states diverge. Model (5) shows that all states in the range $7.5 \leq x[0] \leq 8.5$ converge to the origin. For this example, the 3-segment saturation approximation is not sufficient; the 5-segment model is needed.

Table 2: Simulation Results for the Unstable System

Saturation Model	$\min x[50]$	$\max x[50]$
Model (3)	-280	2200
Model (4)	-0.47	140
Model (5)	-0.13	0.13
Exact	6.0 e-11	6.5 e-8

The results for models (4) and (5) demonstrate a limitation of the modeling framework. Since the uncertainty can enter as a constant offset, the simulation never converges to an equilibrium point, but always to a ball around an equilibrium point. For model (4), this set is $-0.47 \leq x \leq 0.47$. Even if $x[0] = 0$, the simulation result would be $|x[50]| \leq 0.47$. The size of this set is a function of the dynamics around the fixed point and the size of the model uncertainty. Model (5), which has much smaller uncertainty, reaches the ball $|x[50]| \leq 0.13$.

5.3 Example 3

A third example demonstrates multiresolution simulation. Consider the nominally unstable system with a quantized input

$$\begin{aligned} x[k+1] &= 1.9x[k] + q(u[k]) \\ q(u) &= u - (u + 0.05) \bmod 0.1 + 0.05. \end{aligned} \tag{8}$$

While q can be exactly represented as a PL mapping, it has 10 PL regions for every unit of state space modeled. A PL mapping valid for $|x[k]| < 10\ \forall k$ requires 200 PL regions. A much simpler model for the quantizer is

$$q(x) = x + 0.05n \tag{9}$$

Closing the loop with unity feedback and using approximation (9) for the quantizer gives

$$x[k+1] = 0.9x[k] + 0.05n[k]$$

Simulating the initial condition set $|x[0]| \leq 100$ for 1000 time steps gives the result $|x[1000]| \leq 0.50$, the smallest range attainable for this amount of uncertainty.

A substantially better result can be obtained by changing the quantizer approximation midway through the simulation. Another piecewise linear model for the quantizer is

$$q(x) = \begin{cases} x + 0.05n, & x \geq .55 \\ 0.5, & 0.45x \leq x < 0.55 \\ 0.4, & 0.35x \leq x < 0.45 \\ 0.3, & 0.25x \leq x < 0.35 \\ 0.2, & 0.15x \leq x < 0.25 \\ 0.1, & 0.05x \leq x < 0.15 \\ 0, & -0.05x \leq x < .05 \\ 0.1, & -0.15x \leq x < -0.05 \\ 0.2, & -0.25x \leq x < -0.15 \\ 0.3, & -0.35x \leq x < -0.25 \\ 0.4, & -0.45x \leq x < -0.35 \\ 0.5, & -0.55x \leq x < -0.45 \\ x + 0.05n, & x < -0.55 \end{cases} \tag{10}$$

This model exactly represents the quantizer near the origin, and uses the same approximation as (9) for $|x| > 0.55$.

After running the first 100 steps of the simulation using approximation (9), the range $|x[100]| \leq 0.503$ is obtained, roughly the smallest region attainable when using approximation (9). Since the region $|x[100]| \leq 0.503$ is valid for any model of the system, it can be used to initialize the simulation for a different model. Switching to approximation (10) for remainder of the simulation gives the result $|x[1000]| \leq 0.095$. This range is less than one fifth the size of the result when only model (9) is used.

The multiresolution simulation not only gives a better result, but also greatly reduces computational cost. While the same result could be obtained by using model (10) for the entire simulation, the computational cost is much greater. By using (9) for the first 100 steps, about 25000 fewer linear programs are solved. Since the number of linear programs solved grows as $\mathcal{O}(knl^2)$, this savings is even larger for more complex systems.

6 CONCLUSIONS

Piecewise linear modeling and robust simulation provide a practical numerical framework for analyzing uncertain nonlinear systems. Libraries of models for subsystem components are easily generated, and uncertainty is explicitly accounted for during the simulation process.

Multiresolution simulation that explicitly includes uncertainty is demonstrated. As models are changed, potential errors are accounted for and propagated through the simulation. Even for the simple demonstration system, dramatic reductions in computation are achieved.

While no guidelines for model selection are given, the effects of poor decisions are shown. Results can vary greatly with only small changes in model accuracy. Methods for systematically choosing the appropriate model for simulation are also being developed.

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