*Proceedings of the 1999 Winter Simulation Conference*
*P. A. Farrington, H. B. Nembhard, D. T. Sturrock, and G. W. Evans, eds.*

# SIMULATING A NONSTATIONARY POISSON PROCESS USING BIVARIATE THINNING: THE CASE OF "TYPICAL WEEKDAY" ARRIVALS AT A CONSUMER ELECTRONICS STORE

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## ABSTRACT

We present a case study in which *thinning* is applied to simulate time-varying arrivals at a consumer electronics store. The underlying simulation was developed to support an analysis of new staffing schedules for retail sales associates, given proposed changes in store layout and operating procedures. A principal challenge was developing a modeling approach for customer arrivals, where it was understood that the arrival rate varied by time-of-day and by day-of-the-week, as well as seasonally. An analysis of arrival data supported a conjectured "typical weekday" as one basic arrival model. For this model, arrivals were assumed to be nonstationary Poisson, with a piecewise-linear arrival rate independently modulated by hour and by day. Arrival data were filtered and independent hourly and daily thinning factors computed. In the simulation, potential arrivals were generated with a mean equal to the minimum average interarrival rate, determined from the average arrival count for the hour/day time block with unit thinning factors. Candidate arrivals were then thinned using a bivariate acceptance probability equal to the product of the corresponding hourly and daily thinning factors.

## 1 BACKGROUND

We are perhaps all too familiar with the regular ebb and flow of demand in service systems. Examples abound--the seasonal crush of holiday shopping, the daily grind of morning and evening rush hours, savings on evening and weekend telephone calls, the wisdom of dinner-hour reservations at your favorite restaurant, the annoyance of rolling electricity brownouts, and the economy of airfares for Saturday-night layovers. Indeed, WSC is scheduled during the early weeks of December at least in part because of the slack demand (and favorable scheduling and pricing) of hotel facilities at this time of year.

Given the ubiquity and impact of such systems, it is not surprising that there is a significant literature on simulating input processes that vary in intensity over time (Çinlar, 1975; Johnson, *et al*., 1994; Kaminsky and Rumpf, 1977; Kelton, *et al*., 1998; Klein and Roberts, 1984; Law and Kelton, 1991; Leemis, 1991; Lewis and Shedler, 1979). The purpose of this paper is to add a case study to this literature, based on recent experience. The study also provides a unique, bivariate extension of the standard *thinning* technique developed by Lewis and Shedler (1979) for generating nonstationary Poisson arrivals.

The client for this case was a major chain of consumer electronics outlets. In response to slipping market share, the corporate office proposed a change in retail store layout and operating procedures intended to streamline customer service. The study objective was to determine the potential impact of the new operating procedures on the workload of sales associates, as well as to design new staffing schedules and compensation strategies to account for these impacts.

A key challenge was making sense of the arrival process of retail customers at a so-called megastore. Arrivals and departures were registered using a beam counter at the store entrance and total counts, aggregated in one-hour time buckets, were recorded and stored. While these data had some known deficiencies (e.g., arrivals and departures where indistinguishable in the data set and customers shopping as units were observed largely as component individuals), adjustments for these defects could be made based on data from secondary recording devices, direct observation, expert knowledge, and common sense. The principal concern was developing a modeling approach that addressed the time-varying nature of the data. It was understood that the rate of arrivals varied (at least) by time of day and day of the week, as well as seasonally.

In Section 2 we describe the modeling approach adopted for this study, including the conjecture that a basic model could be developed which adequately captures arrivals during a "typical weekday". The data treatment developed to test this conjecture is outlined in Section 3. Implementation of the result using a bivariate thinning scheme in the simulation is presented in Section 4. Results and conclusions are drawn in the final section.

## 2 MODELING APPROACH

Available data supported the intuition that arrival patterns (as well as store hours) differed between weekdays, on the one hand, and Saturdays and Sundays, on the other. Data also supported the intuition that arrival rates varied seasonally, with acute increases in arrivals during weeks associated with holiday shopping. Given these obvious differences, it was apparent that different staffing schedules would be required for each of these different periods. Indeed, current staffing practices reflected these differences.

On this basis, we decided to focus the initial input analysis on determining whether or not the data supported the idea of a "typical weekday" as one basic arrival model. This model could then be used to study staff scheduling during the majority of the year. Alternate arrival models would be developed and applied to study other prototypical periods of operation.

## 3 FILTERING THE DATA

Figure 1 shows the seventy-one continuous weeks of raw arrival data where made available for this study. The data included arrival counts for each hour of store operation, for each day in the week (Sunday through Saturday), beginning in late winter 1997 and ending in mid-summer 1998. These data were subjected to the following (admittedly *ad hoc*) reduction.

*Step 1.* The raw data set was pruned to weekdays by eliminating all weekends from the data set.

*Step 2.* The resulting weekday data set again was pruned to "typical weeks" by eliminating all weeks in which arrivals for any day were greater than two standard deviations above the yearly mean. The data set retained comprised 55 weeks of arrivals, for each of the five weekdays Monday through Friday, for each hour of the 11 hours of store operation 10 a.m. through 9 p.m.

*Step 3.* A subjective check was performed on the 16 weeks eliminated (and 55 weeks retained) to determine plausible explanations for significant differences in shopping patterns. As expected, weeks surrounding the holidays in November and December, and extending into January, were pruned. Also pruned were several weeks in late spring, corresponding to school graduations. Unexpectedly (to the analysts, but confirmed by domain experts), the weeks surrounding Valentines Day were revealed as particularly good times for consumer electronics sales and likewise pruned.

*Step 4.* The typical-weekday data set was sorted lexicographically by (1) day of the week, then by (2) hour of day, and then by (3) date. A 55-week moving average of these data was plotted, revealing clear daily trends across the week, as well as clear hourly tends within each day. As shown in Figure 3, arrivals on any day typically rose during the morning hours, leveled in the afternoon, dipped modestly during the dinner hours, and then rose again in the evening. Mean daily arrivals fell from Monday through mid-week, then rose to weekly highs on Fridays.

## 4 ESTIMATING THE RATE FUNCTION

Based on expert judgement that the vast majority of arrivals were essentially independent, we assumed that a nonstationary Poisson process adequately modeled typical weekday arrivals. Based on the trends observed in the pruned data set, we concluded that the "typical weekday" arrival conjecture, with piecewise-constant rates modulated the by hour of the day and scaled by the day of the week, was in fact adequate to our study objectives.

The data were blocked accordingly and the mean $\bar{x}_{ij}$ of each hour ($i$=10,…,20)/day ($j$=2,…,6) block was calculated from the reduced data set. The average arrival count then was calculated for each hour of the day and, again, for each day of the week:

$$\bar{x}_i = \frac{1}{5}\sum_{j=2}^{6} \bar{x}_{ij}\ ,\ \forall i = 10,\cdots,20$$

$$\bar{x}_j = \frac{1}{11}\sum_{i=10}^{20} \bar{x}_{ij}\ ,\ \forall j = 2,\cdots,6$$

Independent hourly and daily thinning factors were computed:

$$\eta_i = \bar{x}_i \Big/ \max_i \left(\bar{x}_i\right), \forall i = 10,\cdots,20$$

$$\delta_j = \bar{x}_j \Big/ \max_j \left(\bar{x}_j\right), \forall j = 2,\cdots,6$$

and the maximum arrival rate for all hour/day blocks was estimated:

$$\lambda_{max} = \bar{x}_{ij}\ \ for\,(i,j)\,such\ \ that\ \ \eta_i\delta_j = 1$$

In the simulation, customer arrivals where generated using an exponential interarrival rate with mean $1/\lambda_{max}$ and then thinned with the bivariate acceptance probability

$$p_{ij} = \lambda_{ij} / \lambda_{max} = \eta_i \delta_j$$

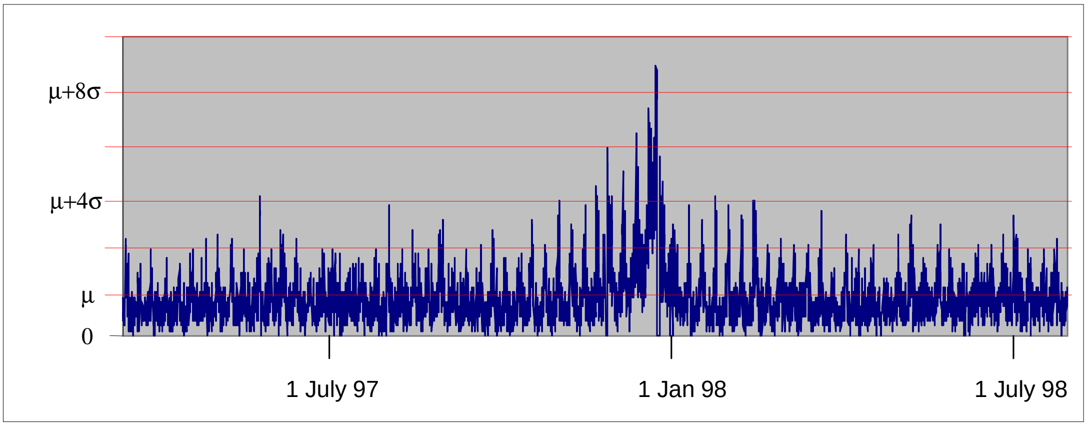


Figure 1: Raw Data (Arrival Counts by Hour for Seventy-One Consecutive Weeks)

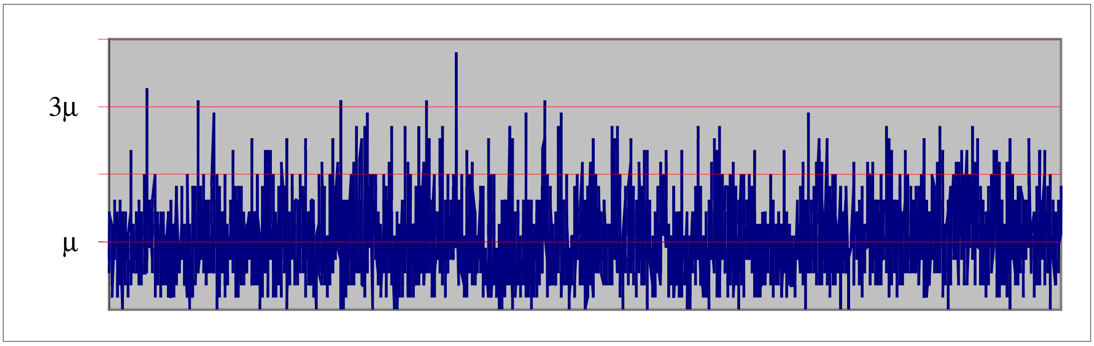


Figure 2: Raw Data Pruned to Typical Weekdays

Values for the day/hour block indices where determined from the simulation clock $t$ (in minutes):

$$k = 1 + a\,int(t/3300)$$
$$j = 2 + a\,int(t - 3300(k-1))/\,660$$
$$i = 10 + a\,int(t - 3300(k-1) - 660(j-2))/\,60$$

where $k$ indexes the week and the function $aint(z)$ truncates the real number $z$ to an integer.

## 5 RESULTS AND CONCLUSIONS

The adequacy of the arrival model developed can be assessed qualitatively by comparing the graphs presented in Figures 3 and 4. Figure 3 shows the hourly means for

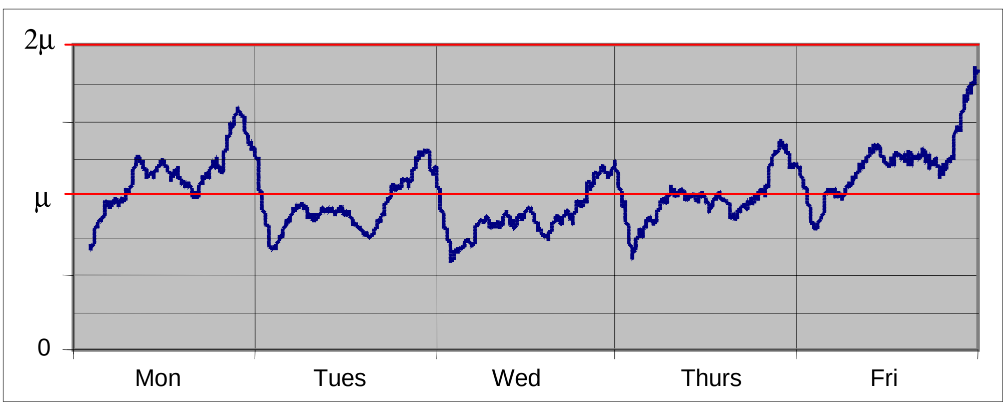


Figure 3: Typical Weekday Data Set, Reordered Lexicographically by Day of the Week, Hour of the Day, and Date and Then Smoothed Using a 55-week Moving Average

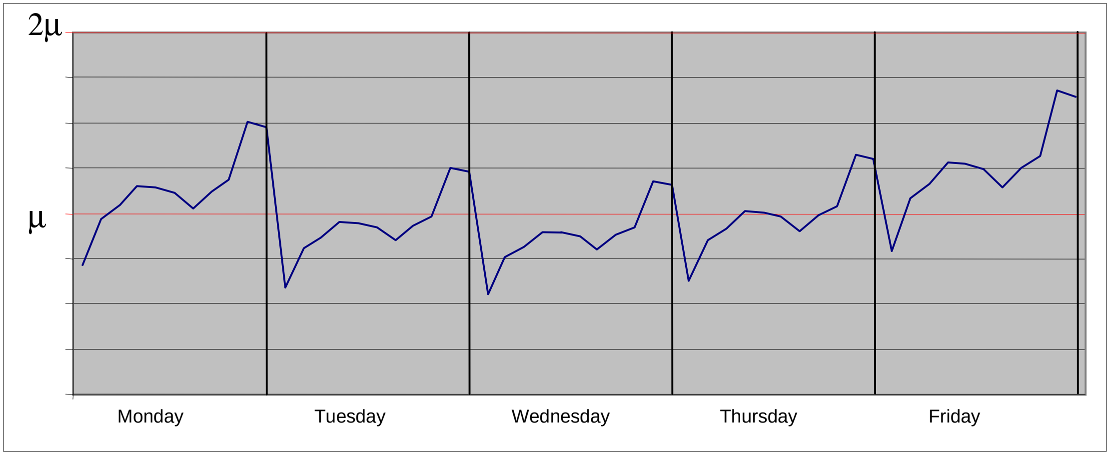


Figure 4: Estimated Rate Function Plotted with Linear Interpolation between Piecewise Constant Values

the typical-weekday data set (1) lexicographically ordered by day of the week, hour of the day, and date and then (2) smoothed using a 55-week moving average. Every 55th observation (beginning with the initial point plotted) therefore is the hour/day block mean $\bar{x}_{ij}$ for the corresponding block. For visual comparison with the moving average plotted in Figure 3, Figure 4 shows the estimated rate function plotted with linear interpolation between piecewise constant values.

Quantitative comparisons of the data and fit were also applied. Given the known defects in the original data set and the *ad hoc* data treatment, the model appears to be more than adequate to support its application to analysis of the underlying staffing problem.

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