Proceedings of the 1995 Winter Simulation Conference
ed. C. Alexopoulos, K. Kang, W. R. Lilegdon, and D. Goldsman

SEQUENTIAL EXPERIMENTATION FOR ESTIMATING THE OPTIMAL DELAY OF ACTIVITIES IN PERT NETWORKS

Arnold H. Buss

Operations Research Department
Naval Postgraduate School
Monterey, CA 93943-5000, U.S.A.

ABSTRACT

We apply Response Surface Methodology (RSM) utilizing simple fractional designs to optimize the Expected Present Value (EPV) of a stochastic project network with respect to the delay of activities. Such delay could increase EPV by postponing negative cash flows, possible at the expense of also delaying the final payment for the project. The problem was challenging due to a combination of high variability with relatively flat objective function near the optimum. Comparisons with the true optimal solutions, where available, indicated the robustness of the approach.

1 INTRODUCTION

Analyzing project networks with stochastic activity durations is a quite difficult task. That projects do indeed have random durations was recognized early on with the development of the Project Evaluation and Review Technique (PERT) (Malcolm, *et al* 1959) in which activity durations were assumed to have beta distributions. The PERT approach to the project duration (using only activities on the deterministic critical path with a normal approximation) has been shown to be extremely unconservative, giving wildly optimistic approximations to the true project's duration. To date, the Markov PERT Network (MPN) model of Kulkarni and Adlakha (1986) is the only one to give exact solutions for projects with arbitrary precedence relations. Independent exponential activity durations must be assumed, however, in order to apply the MPN.

The difficulty is compounded when costs and revenues are introduced and a present value criterion is utilized. While there have been a number of papers on the deterministic problem, to date the only analytic results for the stochastic EPV problem are in Buss and Rosenblatt (1995), who extend Kulkarni and Adlakha's MPN model.

The assumption of exponential activity durations may be restrictive in certain situations. The memoryless property implies that no work actually gets done on an activity until it is actually completed. Consequently, there is concern over utilizing solutions obtained by Markov-based analysis in situations where the exponential assumption is not justified. A principal motivation for this work is to study the robustness of the MPN solutions when activity durations are not necessarily exponential.

We utilize RSM to estimate the optimal delay of activities for projects with stochastic activity durations. Due to a combination of high variability and flatness of the objective near the optimum, only the gradient search step proved to be useful in improving the solution for the networks studied.

In one set of experiments we use exponential activities to compare with analytic solutions, thus giving some validation to the RSM procedure. In a second set of experiments we use more realistic distributions to examine the robustness of the exponential solutions. Since projects typically have many activities, use of fractional designs becomes critical. Since the decision variables are constrained to be non-negative, the main design point of interest is not the center point, but rather a corner point. Finally, the optimal solutions typically have most variables at zero. Thus, most constraints will be binding at or near the optimum.

The focus of this paper is on how the optimal delays and estimated EPVs are affected by changes in the distributions of the activity durations. Consequently, our sequential procedure is not necessarily the best possible, nor do we fully utilize variance reduction techniques. In future work we will incorporate more refined designs and examine the impact of various variance reduction strategies, particularly blocking along the lines of Schruben and Margolin (1978).

2 PROBLEM STATEMENT

A project consists of a number of indivisible activities that are subject to precedence constraints. Each activity has costs associated with performing that activity. A cash payment is received upon completion of the project, which occurs when all activities have been finished. We assume that the cash outflow associated with each activity occurs at the activity's completion. This is without loss of generality, since more complex patterns of cash flows distributed throughout the activity may be converted to a single cash flow at the activity's completion having the same EPV as the original. We could also consider intermediate cash inflows called milestone payments, which are sometimes made upon the completion of a predetermined subset of activities. The objective is to maximize expected present value (EPV) of the project.

If activity durations are deterministic, then (in the absence of intermediate milestone payments) then it is clear that every activity should begin at its respective latest start time. This is because delaying a negative cash flow increases present value as long as another positive cash flow is not also delayed. In the deterministic case, starting activities as late as possible postpones the negative cash flows of the activities without postponing the positive cash flow upon completion of the project. Only activities on the critical path will be not be delayed from their early start times in the optimal schedule.

For stochastic activity durations, however, the situation is not so straightforward. Delaying *any* activity beyond its early start time results in delaying the expected completion to the project, and thereby decrease the contribution to EPV made by the revenue for the project. An activity should be delayed only if the increase to EPV due to postponing that activity's negative cash flow more than counteracts the loss due to postponing the positive cash flow.

2.1 Markov Project Network Analysis

The first step in a Markov Project Network (MPN) is to generate the state space from the precedence relations of the network. Each activity is: (i) *active*, currently being processed; (ii) *dormant*, completed processing, but at least one immediate successor has at least one immediate predecessor still uncompleted; (iii) *idle*, neither active nor dormant. The state of the MPN consists of a vector of the status of each activity. See Kulkarni and Adlakha (1986) for more details. The resulting infinitesimal generator Q can be made upper triangular. The expected present value of the project without delay is

$$\Pi = (rI - Q)^{-1} f \tag{1}$$

where r is the continuous discount rate and f is a vector on the MPN state space of weighted cash flows. For state i, f_i is the sum of the cash flows associated with each active activity in i, weighted by the inverse of the mean duration of that activity. See Buss and Rosenblatt (1995) for further details. From Equation (1) the EPV for the project when certain activities are delayed may be derived. The resulting expression may then be optimized with respect to activity delays. Furthermore, the derivative with respect to the delay of activity j, $\partial\Pi/\partial d_j$, may also be derived for each activity. These derivatives provide useful information as to which activities are desirable for delay. No such expressions are available for arbitrary networks with non-exponential activity durations, however, and we must resort to simulation.

3 PROCEDURE

To obtain simulation estimates of the optimal delays for activities having non-exponential distributions we utilized a variant of Response Surface Methodology (see, for example, Myers 1971).

1. Estimate the gradient.
2. Check for curvature using replications of center points.
3. If curvature is evident, estimate second order model.
4. Else, project gradient onto non-negativity constraints.
5. Perform experiments in the direction of gradient until no improvement is evident.
6. Fit quadratic to the previous step's experiments.
7. Update new center to optimum of fitted quadratic and repeat Steps 1–7 until curvature or maximum number of iterations is reached.

Instead of performing experiments in the gradient direction "until there is no improvement," we fit a quadratic to the points and optimize to get the new center point. This was necessitated by the extremely high levels of variance exhibited by our models and appeared to produce better results. Furthermore, we found that the high variability and flatness of the objective function near the optimum led to no further improvement in the solution in Steps 6–7. Therefore, we confined our analysis to the gradient estimation and search steps.

The non-negativity of activity delays made the problem a constrained one, necessitating the gradient projection step after the initial estimation. Furthermore, at or near the optimal solution there are typically very few activities that are delayed by a positive amount. That is, the non-negativity constraints tended to be binding for most variables.

4 EXPERIMENTAL DESIGNS

There is a potential delay associated with each activity, so the dimension of the search space becomes quite large even for moderate size problems. To minimize the number of runs we employed Resolution III designs together with replications of center points to check for curvature. For example, Table 2 shows the design we utilized for the 5 activity network, a 2^{5-2} design with four center points. Designs for even moderate size projects are outside the range of most tables. For example, the 21 activity network studied below required a 2^{21-16} design to provide estimates for all main effects.

The most natural starting point was the origin (0 delay for all activities). Since this point (as with most subsequent points) was on the boundary of the feasible region, we had to think of the "center point" as the one with all factors at their lowest level. This is in contrast to the typical experimental situation, in which the "center point" is in fact the center point of the design. Thus, for our initial step the center of the design had all activities delayed by the respective halfwidths.

Table 1: Parameters for the Five Activity Network

Activity	Mean Duration	Cost ($\times 1000$)	Immediate Predecessors
1	3	20	
2	8	10	
3	2	50	1
4	9	5	1
5	3	5	2 3

$R = 105,000$
$r = 0.01$

Table 2: Design for Five Activity Network

Run	d_1	d_2	d_3	d_4	d_5	EPV
1	−	−	−	+	+	3281
2	+	−	−	−	+	3962
3	−	+	−	−	−	4110
4	+	+	−	+	−	3650
5	−	−	+	+	−	4356
6	+	−	+	−	−	4918
7	−	+	+	−	+	3928
8	+	+	+	+	+	3337
9	0	0	0	0	0	4021
10	0	0	0	0	0	4379
11	0	0	0	0	0	4026
12	0	0	0	0	0	3828
Grad:	24	−186	192	−287	−316	
Projected:	24	0	192	0	0	

5 RESULTS

5.1 Five Activity Network

We first estimated the optimal delays in a five activity network with the parameters shown in Table 1 and exponential activity durations. This network is analyzed in Buss and Rosenblatt (1995), so we could compare the RSM results with the true optimal value. Each design point was replicated 500 times, and the half width of the design was 0.5.

We first performed the procedure on the five activity network with exponential activity durations. In this case we have the true optimal delays available, so we can evaluate the efficacy of the procedure. The initial EPV for zero delays was estimated to be \$3843, compared with a theoretical value or \$4181. The gradient search portion terminated with $d_1 = 1.226, d_3 = 8.144$, at which time curvature was significant. The estimated EPV for these delays was \$5863, obtained by averaging the response at the center points. The estimated optimal delays were $d_1 = 1.729, d_3 = 5.911$ with an EPV of \$5253.

The design and results for the first step for the exponential five activity project are shown in Table 2. The gradient is first estimated, then projected onto the non-negativity constraints. For this first step, this amounts to simply replacing the negative derivatives with zeroes. In this case the estimated gradient corresponded reasonably well with the true gradient.

Although we only utilized the gradient search portion of RSM, there was considerable improvement in EPV due to delay. Figure 1 plots the best EPV and the zero delay EPV against activities' CV for log normally distributed activities. The improvement in EPV was about \$1300 for those CV's greater than 0. As mentioned previously, the second order model led to no further improvement. It is possible that the introduction of more sophisticated variance reduction strategies would lead to some improvement in the fi-

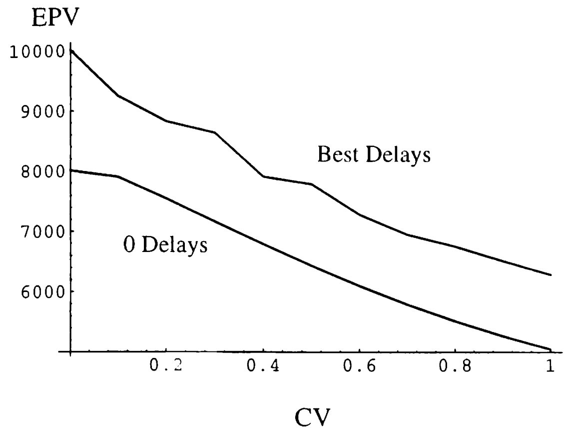

Figure 1: EPV vs CV for Log Normal Activities for the Five Activity Network

nal stage.

5.2 Twenty-one Activity Network

The second network we considered was a 21 activity network with parameters shown in Figure 3. Activity durations were taken to be gamma with cv of 0.577.

In Table 4 we show the delays obtained by RSM together with the derivatives for the MPN model for those activities which had positive delay at termination of the simulation. All activities not listed had zero delay from RSM and negative derivatives from the MPN model. Note that RSM had activities 1 and 4 delayed, whereas they had negative derivatives in the MPN. The magnitude of the derivatives for these two activities were small, however, and the estimated delays also small compared with the others.

6 DISCUSSION

The preliminary set of experiments presented in the previous section indicate the efficacy of RSM in producing substantial improvements in EPV for projects by delaying activities. The methodology is of greatest utility for non-Markovian project networks for which the exponential activity assumption is not deemed acceptable. Furthermore, these experiments indicate a robustness to the MPN solution with respect to departures from the exponential assumption. Although EPV increased when the activity CVs decreased, the delays produced by RSM remained quite similar to the MPN optimal delays.

Table 3: Parameters for the Twenty-one Activity Network

Activity	Mean Duration	Cost ($\times$1000)	Immediate Predecessors
1	1.0	18	
2	0.5	14	1
3	1.0	18	1
4	0.5	4	1
5	3.0	13	3 4
6	3.5	10	5
7	0.5	20	4
8	0.5	3	7
9	3.0	19	8
10	4.0	5	6 9
11	1.0	1	6 9
12	2.0	14	2 10 11
13	2.0	1	12
14	2.0	15	12
15	2.0	13	4
16	0.1	16	2 13 14 15
17	0.5	19	16
18	1.0	14	17
19	1.0	1	17
20	2.0	17	19
21	2.0	13	18 20

$R = 290,000$
$r = 0.01$

Table 4: Delays and Derivatives for the Twenty-one Activity Network

Activity	(MPN) $\partial\Pi/\partial d_j$	(RSM) Delay
1	−40.51	0.240
2	137.85	2.461
4	−17.13	0.065
7	48.90	1.239
15	119.73	1.160

7 CONCLUSIONS

We have applied a variation on standard RSM to the problem of delaying activities in stochastic project networks with the objective of maximizing EPV. Simulation was necessary due to lack of analytic results for general activity distributions. For exponential activity durations RSM produced solutions very close to the true optima. For non-exponential distributions, RSM produced solutions with substantial improvement and provided evidence of the robustness of the MPN solutions.

ACKNOWLEDGEMENTS

Support for this work from the Naval Postgraduate School is gratefully acknowledged.

REFERENCES

Buss, A. H. and M. J. Rosenblatt. 1995. Activity Delay in Stochastic Project Networks, *Operations Research*, forthcoming.

Kulkarni, V. G. and V. G. Adlakha. 1986. Markov and Markov-Regenerative PERT Networks, *Operations Research*, **34**, 769–781.

Law, A. and D. Kelton. 1991. *Simulation Modeling and Analysis, Second Edition*, McGraw-Hill, NY.

Malcolm,D. G., J. H. Roseboom, C. E. Clark, and W. Fazar. 1959. Applications of a Technique for Research and Development Program Evaluation, *Operations Research*, **7**, 646–669.

Schruben, L. W. and B. H. Margolin. 1978. Pseudorandom Number Assignment in Statistically Designed Simulation and Distribution Sampling Experiments, *Journal of the American Statistical Association*, **73**, 504–520.

AUTHOR BIOGRAPHY

ARNOLD BUSS is currently a Visiting Assistant Professor in the Operations Research Department at the Naval Postgraduate School. He received his PhD in Operations Research from Cornell University. He is not nearly as funny as he thinks he is.