Proceedings of the 1994 Winter Simulation Conference
ed. J. D. Tew, S. Manivannan, D. A. Sadowski, and A. F. Seila

CONSISTENCY OF OVERLAPPING BATCH VARIANCES

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ABSTRACT

When the simulation point estimator is the sample variance, its standard error can be estimated using batch variances, which are analogous to batch means for the standard error of the sample mean. We suggest a modification to the definition of OBV for analytical tractability and to improve its statistical properties. We discuss conjectures about when overlapping batch variances (OBV) is consistent. In particular, we argue that OBV seems likely to be consistent (almost) whenever overlapping batch means (OBM) is consistent. Both the definition modification and the consistency conjecture seem relevant to all overlapping batch statistics (OBS) estimators where the point estimator can be interpreted as the mean of a related stochastic process.

1 INTRODUCTION

Steady-state simulation experiments produce output data $\{X_1, X_2, \ldots, X_n\}$, from which a point estimator $\hat{\theta}$ is calculated. Typical point estimators are the sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

for the process mean μ and the sample variance

$$S^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

for the process variance σ^2. We consider the simulation output analysis problem of determining the standard error of the point estimator S^2.

Schmeiser, Avramidis, and Hashem (1990) discuss overlapping batch statistics (OBS) to estimate the standard error of general point estimators $\hat{\theta}$ from stationary output processes. The OBS estimator of $\mathrm{var}(\hat{\theta})$ is, for a given batch size m,

$$\hat{V}_S(m) = \frac{m}{(n-m)(n-m+1)} \sum_{j=1}^{n-m+1} (\hat{\theta}_j - \hat{\theta})^2 ,$$

where the j^{th} batch statistic $\hat{\theta}_j$ is the point estimator calculated from the j^{th} batch of observations: $\{X_j, X_{j+1}, \ldots, X_{j+m-1}\}$. The standard error of $\hat{\theta}$ is then estimated by $\hat{V}_S(m)^{1/2}$.

OBM and OBV are special cases of OBS. The OBM estimator of $\mathrm{var}(\bar{X})$ is

$$\hat{V}_M(m) = \frac{m}{(n-m)(n-m+1)} \sum_{j=1}^{n-m+1} (\bar{X}_j - \bar{X})^2 ,$$

where $\bar{X}_j = m^{-1} \sum_{i=j}^{j+m-1} X_i$. The OBV estimator of $\mathrm{var}(S^2)$ is

$$\hat{V}_V(m) = \frac{m}{(n-m)(n-m+1)} \sum_{j=1}^{n-m+1} (S_j^2 - S^2)^2 ,$$

where $S_j^2 = (m-1)^{-1} \sum_{i=j}^{j+m-1} (X_i - \bar{X}_j)^2$.

For OBM and OBV, as for all OBS estimators, the choice of batch size m is central to the statistical performance. For any particular estimator, the appropriate batch size is a function of simulation run length n and unknown properties of the output process $\{X_i\}$. Although quite a lot is known for OBM, little is known about OBV, including even the fundamental issue of whenever OBV provides a consistent estimator of $\lim_{n\to\infty} n\,\mathrm{var}(S^2)$. (In interesting cases $\lim_{n\to\infty} \mathrm{var}(\hat{\theta}) = 0$, in which case $\hat{V}_S(m)$ converges trivially to $\lim_{n\to\infty} \mathrm{var}(\hat{\theta})$. Therefore, we are interested in estimating $\lim_{n\to\infty} n\,\mathrm{var}(\hat{\theta})$ consistently.)

In Section 2 we review some results for OBM, ending with a conjecture that summarizes various conditions for consistency. In Section 3 we argue for modifying the definition of OBV, both to simplify analysis and to improve statistical performance. In Section 4 we discuss a sequence of conjectures that argue that OBV is consistent for $\lim_{n\to\infty} n\,\mathrm{var}(S^2)$ when OBM is consistent for $\lim_{n\to\infty} n\,\mathrm{var}(\bar{X})$.

2 CONSISTENCY OF OBM

Under mild conditions, $n\hat{V}_M(m)$ is an mse-consistent estimator of $\lim_{n\to\infty} n\,\mathrm{var}(\bar{X})$; that is, the bias and variance go to zero as $n \to \infty$. Roughly, if the batch-size rule satisfies $m \to \infty$ and $n/m \to \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty} m\,\mathrm{bias}[n\hat{V}_M(m)] = -\gamma_1\sigma^2$$

and

$$\lim_{n\to\infty} \frac{n}{m}\mathrm{var}[n\hat{V}_M(m)] = \frac{4}{3}(\gamma_0\sigma^2)^2 ,$$

where the sum of autocorrelations is

$$\gamma_0 = \sum_{h=-\infty}^{\infty} \rho_h = 1 + 2\sum_{h=1}^{\infty} \rho_h$$

and the weighted sum of autocorrelations is

$$\gamma_1 = \sum_{h=-\infty}^{\infty} |h|\rho_h = 2\sum_{h=1}^{\infty} h\rho_h$$

and $\rho_h = \mathrm{corr}(X_i, X_{i+h})$.

Damerdji (1991), Pedrosa (1994), and Song and Schmeiser (1994) provide various conditions under which these two limits hold. Without specifying precise conditions, we combine these results as

Conjecture 1. For covariance-stationary data $\{X_i\}$, if $0 < \sigma^2 < \infty$, $\gamma_1 < \infty$, and m and n/m go to infinity as $n \to \infty$, then $n\hat{V}_M(m)$ is an mse-consistent estimator of $\lim_{n\to\infty} n\,\mathrm{var}(\bar{X})$.

3 REDEFINING OBV

The original definition of OBV has two disadvantages. First, the use of $m-1$, rather than m, in the definition of the batch variances S_j^2 and of $n-1$, rather than n, in the definition of the grand variance S^2 complicates analysis with essentially no statistical benefit, because simulation batch sizes and run lengths are typically quite big. The usual purpose — obtaining an unbiased estimator — is not compelling, because the nonzero autocorrelations cause some bias and because the smaller mean squared error (mse) obtained by using the sample size is appealing (Ceylan and Schmeiser 1993).

Second, statistical performance and analysis tractability are improved by using the grand sample mean $\bar{X}$ rather than the batch sample means $\bar{X}_j$ in the definition of S_j^2; that is, now

$$S_j^2 = \frac{1}{m} \sum_{i=j}^{j+m-1} (X_i - \bar{X})^2$$

and

$$S^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 .$$

The grand sample mean is available and is a better estimator of μ than the batch mean. Computation is still possible in $O(n)$ time.

4 CONSISTENCY OF OBV

Using the new definition of OBV, we now propose a conjecture that says that OBV works under conditions similar to those needed for OBM. Let α_4 denote $\mathrm{E}[((X_i - \mu)/\sigma)^4]$, the fourth standard moment of X_i.

Conjecture 2. For covariance-stationary data $\{X_i\}$, if $0 < \sigma^2 < \infty$, $1 < \alpha_4 < \infty$, $\gamma_1 < \infty$, and m and n/m go to infinity as $n \to \infty$, then $n\hat{V}_V(m)$ is an mse-consistent estimator of $\lim_{n\to\infty} n\,\mathrm{var}(S^2)$.

Conjecture 2 adds only the requirement that the fourth moment be finite and greater than one. For $\alpha_4 = 1$, the data are binary, which causes $\mathrm{var}(S^2) = O(n^{-2})$ rather than $O(n^{-1})$ (Ceylan and Schmeiser 1993).

The argument for Conjecture 2 rests on viewing OBV as an application of OBM to the data process of squared deviations from the sample mean.

Result 3. OBV applied to the data process $\{X_i\}$ is algebraically equivalent to OBM applied to the data process $\{(X_i-\bar{X})^2\}$.

The proof is trivial because under the new definition the sample batch variances S_j^2 and the sample variance S^2 are sample averages of the squared deviations from the sample mean.

Because of Result 3, OBV is consistent for $\{X_i\}$ if and only if OBM applied to the squared-deviations process is consistent. Let tildes on process properties denote the squared-deviations process; for example, $\tilde{\sigma}^2$ is the variance of $(X_i - \bar{X})^2$.

Conjecture 4. For covariance-stationary data $\{(X_i - \bar{X})^2\}$, if $0 < \tilde{\sigma}^2 < \infty$, $\tilde{\gamma}_1 < \infty$, and m and n/m go to infinity as $n \to \infty$, then $n\hat{V}_V(m)$ is an mse-consistent estimator of $\lim_{n\to\infty} n\,\mathrm{var}(S^2)$.

Result 5 and Conjecture 6 provide the two conditions needed for Conjecture 4 to be applicable: that the squared-deviations process is covariance stationary and that the weighted sum of autocorrelations $\tilde{\gamma}_1$ is finite. Let $\bar{\alpha}_4$ be the fourth standard moment of $\bar{X}$, which asymptotically has the value three if $\bar{X}$ is asymptotically normal.

Result 5. Assume that γ_1, α_4, and $\bar{\alpha}_4$ are finite. The squared-deviations process $\{(X_i - \bar{X})^2\}$ is asymptotically covariance stationary if the original data process $\{X_i\}$ is covariance stationary. That is,

$$5(a)\ \lim_{n\to\infty} \mathrm{E}[(X_i - \bar{X})^2] = \mathrm{E}[(X_i - \mu)^2] = \sigma^2 \ ,$$

$$5(b)\ \lim_{n\to\infty} \mathrm{var}[(X_i - \bar{X})^2] = \mathrm{var}[(X_i - \mu)^2] = (\alpha_4 - 1)\,\sigma^4 \ ,$$

and

$$5(c)\ \lim_{n\to\infty} \mathrm{cov}[(X_i - \bar{X})^2, (X_{i+h} - \bar{X})^2] = \mathrm{cov}[(X_i - \mu)^2, (X_{i+h} - \mu)^2]\ ,\ h = 1, 2, \ldots,$$

independent of i.

The proof of Result 5, which is not given here, consists of expanding terms and taking expected values. Conjecture 6 says that the squared-deviations process has sufficient mixing if the original data process mixes sufficiently.

Conjecture 6. If $\gamma_1 < \infty$, then $\tilde{\gamma}_1 < \infty$.

The argument for Conjecture 6 consists of examples where the result holds, and the lack of a counterexample. For processes with normal marginal distributions, $\tilde{\rho}_h = \rho_h^2$ and therefore $0 < |\tilde{\rho}_h| < |\rho_h|$ for $h = 1, 2, \ldots$ (Patel and Read 1982). In the appendix we show that the conjecture holds for first-order moving-average processes with arbitrary marginal distributions on the error terms. Monte Carlo simulation results are consistent with the conjecture for steady-state $M/M/1$ queue waiting time processes. Numerical integration results indicate that $|\tilde{\rho}| < |\rho|$ for bivariate lognormal data. We have no counterexample for $|\tilde{\rho}_h| < |\rho_h|$.

5 SUMMARY

Via a sequence of six results and conjectures, we have argued that using OBV to estimate the standard error of the sample variance is reasonable whenever using OBM to estimate the standard error of the sample mean is reasonable. The argument is quite heuristic, based on viewing the sample variance as a sample mean. (See also Priestley 1992, p. 321.) It nevertheless adds confidence to our empirical experience. The only counterexample, symmetric binary marginal distributions, is pathological and easy to identify in practice.

The points made here about OBV estimators seem likely to be relevant to the more general OBS estimators. Grand point estimates should be used where possible. For OBS estimators where the point estimator can be viewed as a sample mean of a related process, application seems likely to work well. In addition, the ideas of this paper also apply to other batching estimators (Song and Schmeiser 1993).

APPENDIX

We show here that Conjecture 6, that γ_1 finite implies $\tilde{\gamma}_1$ finite, holds for the steady-state MA(1) process

$$X_i = \varepsilon_i + a\,\varepsilon_{i-1} \ ,$$

where the error terms $\{\varepsilon_i\}$ are independent with zero mean, variance σ_ε^2, and fourth standardized moment α_4^ε.

Because $\rho_h = \tilde{\rho}_h = 0$ for $h = 2, 3, \ldots$, the argument reduces to showing that $|\tilde{\rho}_1| \le |\rho_1|$.

$$
\begin{aligned}
|\tilde{\rho}_1| &= |\operatorname{corr}(X_1^2, X_2^2)| \\
&= \left| \frac{E(X_1^2 X_2^2) - E(X_1^1)E(X_2^2)}{\operatorname{var}(X_1^2)} \right| \\
&= \left| \frac{a^2(\alpha_4^\varepsilon - 1)}{(1+a^2)^2(\alpha_4 - 1)} \right| \\
&\le |2 \operatorname{corr}^2(X_1, X_2)| \\
&\le |\operatorname{corr}(X_1, X_2)| \\
&= |\rho_1| \;.
\end{aligned}
$$

The first two and last equalities are by definitions. The third equality follows from $\sigma^2 = \sigma_\varepsilon^2(1+a^2)$ and $E(X_1^2 X_2^2) = (a^2\alpha_4^\varepsilon + a^4 + a^2 + 1)\sigma_\varepsilon^4$ (Song 1988). The first inequality follows from

$$\frac{\alpha_4^\varepsilon - 1}{\alpha_4 - 1} \le 2 \;,$$

which follows from (Song 1988)

$$\alpha_4^\varepsilon = \alpha_4 + \frac{2a^2}{1+a^4}(\alpha_4 - 3) \;.$$

The last inequality is true because $\rho_1 \le 1/2$.

ACKNOWLEDGMENTS

This work is partially supported by grant DDM-93-00058 from the National Science Foundation to Purdue University. We acknowledge helpful discussions with David Goldsman.

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