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HIGHER-ORDER CUMULANT SPECTRAL-BASED STATISTICAL TESTS OF PSEUDO-RANDOM VARIATE GENERATORS

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ABSTRACT

Higher-order cumulant spectral based tests of Gaussianity, independence, linearity, and stationarity are used to analyze the higher-order statistical properties of pseudo-random number generators. The tests are applied to three uniform and four Gaussian pseudo-random number generators.

1 INTRODUCTION

In the simulation of systems with stochastic characteristics, one needs to be able to generate sequences of random variables from prescribed probability distributions. This need is fundamental to all simulation models that require some form of stochastic innovation, input, or shock. Algorithms are required to generate random variates from probability distributions with statistical characteristics that facilitate the needs of the simulation model under analysis. The typical procedure for generating a stream of random variables consists of first, generating a sequence of uniform(0,1) variates, and then obtaining a sequence of variates with the desired statistical characteristics by an appropriate transformation.

The decision not to reject the appropriateness of the statistical properties of a generated sequence of pseudo random variates is driven by the level of reliability required to validate the accuracy of the generator. Today, many practitioners assume this issue has been addressed and answered adequately, i.e. there are "good" sources of pseudo random variate generators capable of generating sequences of random variates with the required statistical properties. This acceptance may have been driven by the frequent use of the Gaussian distribution as a variate generating source. Also, the fact that most statistical tests of random variate generators use mean- or covariance-based measures of statistical reliability has helped increase the acceptance. These tests may not be sufficient even with Gaussian variates where the acceptance of the null of Gaussianity is based on the ability to show that the cumulants of orders greater than two are statistically equal to zero. This is especially true in the generation of sequences of non-Gaussian/nonlinear variates that require statistically reliable estimates of higher-order cumulants of orders three and greater.

The higher-order statistical properties of available pseudo random variate generators may not be reliable enough to ensure that their inherent shortcomings are not the causes of deviations in the simulation model or statistical test from the expected behavior. Also, existing tests of pseudo random variate generators that consider moments of orders greater than two are either theoretical or difficult to implement. With the extension of power spectral techniques to higher-orders, there are tests that can address the issue of the appropriateness of random number generators and their higher-order statistical properties.

Statistical tests analyzing the higher-order statistical characteristics of pseudo random variate generators are developed and implemented using higher-order cumulant spectral functions. The specific statistical issues to be addressed concerning pseudo random variate generators include: 1) the independence of the generated variates , 2) the stationarity of the variates, and 3) the inherent randomness of the variates. What is meant by the last issue is that the Fourier transform of the particular cumulant of a distribution function of interest is equal to zero not in a statisical sense, but in a determinisitc sense with probability one. The statistical properties of generated Gaussian variates resulting from transforming the pseudo uniform variates will also be studied.

2 CUMULANTS, MOMENTS, AND CUMULANT SPECTRAL FUNCTIONS

Let $\{X(t): t \in T\}$ denote a univariate random process such that all moments are assumed to exist and to be bounded, where T is an index set on the non-negative integers. The n-th order moment (cumulant) function is defined as the coefficient of the n-th order term in the Taylor series expansion of the joint characteristic function (of the logarithm of the m-th order joint characteristic function). From their "parallel" definitions, the n-th order joint cumulant function can be expressed in terms of joint moment functions of order n or less. Relationships between moments and cumulants of any order can be derived [David and Barton (1962)]. Let $C_n(X(t_1),...,X(t_n)) = C_n(t_1,...t_n) = C_n(t^n)$ denote the n-th order cumulant function, where $t^m = \{t_1 \in T,....,t_m \in T\}$ is a vector of equally spaced time points and $n \le m$. The n-th order moment function can be expressed as

$$M_n(X(t_1),...,X(t_n)) = M_n(t^n)=E\left(X(t_1)X(t_2)\cdots X(t_n)\right).$$

The Fourier transform of the n-th order cumulant is the n-th order cumulant spectrum. The latter can be expressed as the the n-th order moment spectrum and terms which are products of moment spectra of order less than n. Assuming the integrability of the n-th order cumulant as a function of the evolution times, the n-th order cumulant spectrum is defined as:

$$S_{C_n}(f^n) = \int_{R^n} C_n\left(t^n\right)\exp(-i2\pi(f_1t_1+f_2t_2+...+f_nt_n))\prod_{i=1}^{n} dt_i. \quad (1)$$

where $f^n = \{f_1, f_2,...., f_n\}$ is the n-th order frequency set, R^n is the n-dimensional real frequency space, i.e. $(-\infty,\infty)^n$, and the subscript C on the left hand side denotes a cumulant spectral function. The strict stationarity assumption and the requirement of a cumulant mixing condition (Brillinger [2]) form the basis for the use of the asymptotic properties of the n-th order cumulant spectral estimates under the various null hypotheses. From this, the n-th order cumulant function can be written as a function of the n-1 time differences $u_{j-1} = (t_j - t_1)$, $j = 2,..., n$ as $C_n(0,....,t_n - t_1) = C_n(u_1,...., u_{n-1}) = C_n(u^{n-1})$. The (n-1)-th order stationary spectrum is the Fourier transform of the n-th order cumulant function where the support is constrained to the stationary manifold, $f_1+f_2+\cdots+f_n=0$:

$$S_{s_{n-1}}(f^{n-1})=\int_{R^{n-1}} C_n(u^{n-1})\exp(-i2\pi(f_1u_1+f_2u_2+...+f_{n-1}u_{n-1}))\prod_{i=1}^{n-1} du_i. \quad (2)$$

where the subscript S on the left hand side of equation (2) denotes a stationary spectral function. The support set is the principal domain of an n-th order cumulant spectral function. The principal domain is the minimal region in R^n necessary for a complete representation of the specific n-th order spectral function of interest (Dalle Molle and Hinich [1991]). Calculation of any n-th order spectral estimate defined not in the n-th order principal domain, but still within R^n is not required. The subset of the principal domain that is the intersection of the support sets of the n-th order cumulant spectrum and the n-th order stationarity constraint is called the stationary set. The set theoretic difference between the support sets of the principal domain of the n-th order cumulant spectrum and the corresponding (n-1)-th order stationary set is called the n-th order transient set. The transient sets of orders greater than two that are used in the stationarity tests correspond to the transient sets of the Hinich-Wolinsky [1988], which are subsets of the corresponding transient sets of the higher-order cumulant spectra.

3 N-TH ORDER CUMULANT SPECTRAL-BASED TESTS

A framework is outlined for an n-th order joint cumulant spectral based set of tests for Gaussianity, independence, linearity, and stationarity. The statistical framework for detecting positive support of an n-th order cumulant spectral estimate is an extension of the Hinich bispectral-based Gaussianity test (Hinich [1982]). The version of the linearity test used in this paper is from Dalle Molle and Hinich [1989]. The different tests result from how the support of the n-th order cumulant spectral function is decomposed (Dalle Molle and Hinich [1991]). The conceptual foundations for the stationarity test are a generalization of the Hinich-Wolinsky [1988] test for transients and are extended to the n-th order in Dalle Molle and Hinich [1991].

The n-th order cumulant spectral-based test statistic can be defined as $\hat{\xi}_n(f^n)=2|\hat{\upsilon}_n(f^n)|^2$ where $\hat{\upsilon}_n(f^n)=\hat{S}_{c_n}(f^n)/\hat{\sigma}_{n_{ls}}$, $\hat{S}_{c_n}(f^n)$ is calculated using an arithmetic frame average, and $\hat{s}^2_{n_{ls}}$ is an estimate of the large sample variance which is derived in Rosenblatt [1985] and is defined by:

$$\sigma^2_{n_{ls}} = \left[\frac{L_F^{n-2}}{N_F}\right] \beta_n(f^n) \prod_{i=1}^{n} S_{s_1}(f_i)\left(1+O\left(\frac{1}{L_F}\right)\right) \qquad (3),$$

where N_F is the number of partitions of a frame size L_F that a sample of size N can be partitioned into.

$\beta_n(f^n)$ is a scale factor which is a function of when any of the possible subsets of the set of n frequency indices in equation (3) are either the same or complex conjugates of one another. An estimate of the large sample variance is determined by substituting estimators for the 1st-order stationary spectrum (i. e. the power spectrum) at each of the n frequencies.

A global test statistic for the nulls of Gaussianity or stationarity can be thought of as a test for positive support and defined as a collection of realizations of the test statistics $\hat{\xi}_n(f^n)$ belonging to the n-th order stationary or transient sets, depending on the test. The application of the asymptotic properties of the n-th order cumulant spectral estimates to the large sample properties of the statistic $\hat{\xi}_n(f^n)$ is a straightforward extension of the heuristic derivation in Hinich [1982]. The asymptotic properties of the estimate $\hat{S}_{c_n}(f^n)$ are examined in Rosenblatt [1985]. If the random process $\{X(t): t \in Z\}$ is strictly stationary, then each n-th order cumulant spectral estimate $\hat{S}_{c_n}(f^n)$ in the n-th order transient set should not be significantly different from zero. Under the null hypotheses of Gaussianity or stationary, the global test statistic is:

$$\xi_n^g = \sum \sum \cdots \sum \xi_n(f^n) \qquad (4)$$

where the realizations take values over all frequency sets in the appropriate n-th order support set. The estimate of equation (4) is asymptotically distributed as a central chi-squared variate with 2r degrees of freedom where r is the number of frequency sets in the n-th order support set.

For large sample sizes, it is convenient to use a normal approximation for the sum of the chi-squares in equation (4). The chi-squared variate defined in equation (4) can be transformed to a standardized Gaussian random variate using the Fisher transformation ;

$$Z = \sqrt{2\xi_n^g} - \sqrt{4r-1}. \qquad (5)$$

The rejection of the null hypothesis is a function of the significance level required.

Under the null hypothesis of Gaussianity, estimates of the n-th order cumulant spectrum in the n-th order stationary set for $n \geq 3$ should have an expected value of zero. An n-th order Gaussianity test is developed from a global statistic derived from the properties of the collection of n-th order cumulant spectral estimates in the n-th order stationary set.

Under the null hypothesis, if a random process is stationary, then any n-th order cumulant spectral estimate defined in the appropriate n-th order transient set has an expected value of zero. An n-th order cumulant spectral-based stationarity test is a global statistic derived from properties of the collection of n-th order cumulant spectral estimates in the n-th order transient set.

The null hypothesis that a random process is linear, although not necessarily Gaussian, implies that the non-centrality parameter $2|\upsilon_{f_1 \ldots f_n}|^2$ of any order cumulant spectra is a constant, say, λ_0 which is the mean of the r estimators of the form $2|f_1 \ldots f_n|^2$. Under the null hypothesis of linearity, each observation of the statistic $2|\upsilon_{f_1 \ldots f_n}|^2$ is an independent sample from a non-central chi-squared distribution $\chi^2(2, 2|\upsilon_{f_1 \ldots f_n}|^2)$, where $2|\upsilon_{f_1 \ldots f_n}|^2$ is the non-centrality parameter. A sample of size ρ of these statistics should have the sample dispersion as a $\chi^2(2, \lambda_0)$ distribution. Each observed $2|\upsilon_{f_1 \ldots f_n}|^2$ is an independent sample from a non-central chi-squared distribution, each with a the non-centrality parameter $2|\upsilon_{f_1 \ldots f_n}|^2$. To reject linearity, the sample dispersion should be greater in value than the dispersion required under the null.

The test statistic for linearity is based on a comparison of the magnitude of the dispersion in the sample fractiles from the empirical distribution of the individual Gaussianity test statistics relative to the actual fractiles that are required under the null hypothesis, i.e. $(\xi_s-\xi_a)/\sigma_q$ where ξ_a are the actual fractiles and ξ_s the sample fractiles of the CDF of $\chi^2(2, \lambda_0)$, r the sample size, and σ_q the standard deviation of the uniform order statistic and is defined as

$\sigma_q^2 = (\xi_a(1-\xi_a))/\rho$. The test statistic is a standardized normal variate (i.e. N(0,1)) within a 100(1-α) percent symmetric confidence interval about the mean. The null hypothesis is rejected if the values of the test statistic are outliers of the standardized normal distribution. The fractile used is the 80-th fractile and was chosen based on previous simulations.

The test for independence is a joint test. If the variates of a random process are generated independently, then for any n-th order cumulant spectral estimate defined in an n-th order stationary set, the real part is constant and the imaginary part is equal to zero. This is shown in Brillinger and Rosenblatt [1967]. Thus, first we test for a constant real part using the linearity test statistic, and then, if linearity is not rejected, we test if imaginary part of the Gaussianity statistic is zero valued in the n-th order stationary set.

4 TESTING PSEUDO RANDOM NUMBER GENERATORS

As test cases, algorithms for three uniform and four Gaussian variate generators are examined. Two different implementations of the linear congruential generator (LCG) and a Tausworthe generator are the uniform generators considered. The algorithms for the Gaussian variate generation are Box-Muller, inversion, acceptance-rejection, and the sum of twelve uniform(0,1) random variates.

The first of the LCG's considered can be found in Bratley, Fox, and Schrage [1987]. The recursion is $Z_i = (16807*Z_{i-1})\mathrm{mod}(2^{31}-1)$ where $Z_i = 1, 2,\ldots,$ is a sequence of positive integers with $0 \leq Z_i \leq m-1$ for all i. The motivation for choosing this generator is its widespread use and prior extensive testing of its statistical properties. This uniform(0,1) generator is also used as the feed for the Gaussian variate generation. The second LCG is the infamous RANDU version, $Z_i = (65539*Z_{i-1})\mathrm{mod}(2^{31})$, which produces an observably nonrandom sequence of variates. The algorithm for Tausworthe generator is the version found in Bratley, Fox, and Schrage [1987]. Although Tausworthe generators are not as widely employed as the LCG's, they are commonly used because of their superior performance with respect to their potential to generate cycles of variates with a large period, independent of the word-size of the computer.

For the Gaussian variate generation, three of the more commonly used generation schemes are used: 1) Box-Muller, 2) an inversion, and 3) an acceptance-rejection. The Box-Muller procedure is an exact transformation which utilizes all uniform(0,1) variates generated. The usefulness of the inversion procedure is its potential for provable variance reduction. The acceptance-rejection scheme can generate variates in a rapid fashion. Additionally, the approach of summing twelve uniform(0,1) random variates is included even though it is typically not recommended for use. The implementations for all four Gaussian variate generators are found in Bratley, Fox, and Schrage [1987]. The LCG generator found in Bratley, Fox, and Schrage [1987] and used in the tests of the uniform generators is used as the feed for the Gaussian variate generation.

5 ANALYSIS OF PSEUDO RANDOM NUMBER GENERATORS

For the 2nd-order cumulant spectrum estimation procedures see Dalle Molle and Hinich (1989). The simulation study consists of running 250 realizations of each of the seven generators. The sample sizes studied are 1000 and 10000. The results are presented in the form of z-statistics derived from the fractiles of the empirical distribution of the realizations of the individual tests statistics. The z-statistic is a standardized normal variate (i.e. N(0,1)) within a 100(1-α) percent symmetric confidence interval about the mean. The z-statistics derived from the Gaussianity and stationarity global test statistics and the decomposition of the test statistics into its imaginary part are one-sided. This is because the Gaussianity and stationarity global test statistics (under the null hypotheses) are transformed chi-squared which are non-negative. The individual linearity test statistics are two-sided. The significance of the last two statements comes from the interpretation of the z-statistics from the empirical distribution of the 250 realizations. The interpretation for large negative values of the Gaussianity and stationarity z-statistics is that the generator of interest may not have higher-order random characteristics, i.e. the process is deterministic with respect to that order cumulant spectral function in the sense that the level of random noise is almost non-existent. For the linearity test statistic, the interpretation of the large negative values is analogous to that of the large positive values, i.e. one rejects the null hypothesis. The two-sided z-statistics that will be referred to in this section are ±1.96 (5 % level), and ±2.325 (1 % level). In the tables presented, four fractiles, i.e. 0.60, 0.69, 0.77, and 0.89, are reported, and were chosen since they are fairly robust to small samples.

The results for the 2nd-order cumulant spectral test for stationarity are in Table 1. For the sample size of 1000, all the uniform generators do not reject stationarity, but all Gaussian generators do. For the sample sizes of 10000, all generators reject stationarity. Table 2 contains the results for the stationarity test using the Hinich-Wolinsky 3rd-order transient set, a subset of the complete 3rd-order transient set. For the sample size of 1000, all uniform generators reject stationarity. The stationarity test

Table 1: Second Order Cumulant Spectral Stationary Test Statistic

Frac	Z-value													
	1000 Obs							10000 Obs						
	Taus	LCG	RAN	Sum	B-M	A/R	Inv	Taus	LCG	RAN	Sum	B-M	A/R	Inv
0.60	1.38	1.69	-0.08	6.82	6.01	6.69	7.30	3.61	3.96	3.57	8.64	7.58	7.65	7.64
0.69	0.91	1.52	0.00	5.98	4.33	5.52	5.85	3.04	2.91	3.67	7.65	6.41	6.80	7.23
0.77	0.87	1.01	1.49	5.27	2.93	5.26	5.16	1.81	2.23	1.64	6.49	5.53	6.09	6.33
0.89	-1.17	0.39	-0.30	4.30	3.23	3.53	3.81	1.36	1.19	0.56	4.24	3.57	4.11	4.72

Table 2: Third Order Cumulant Spectral Stationary Test Statistic

Frac	Z-value													
	1000 Obs							10000 Obs						
	Taus	LCG	RAN	Sum	B-M	A/R	Inv	Taus	LCG	RAN	Sum	B-M	A/R	Inv
0.60	-6.72	-7.93	-7.13	-2.21	-2.67	-2.48	-2.84	-6.05	-7.28	-6.82	-0.11	0.55	-0.36	-0.28
0.69	-6.20	-8.35	-7.44	-2.78	-2.24	-2.16	-2.66	-6.10	-6.67	-7.12	0.24	0.43	-1.01	-0.45
0.77	-6.54	-9.21	-8.36	-2.00	-2.30	-2.72	-2.85	-5.41	-6.62	-7.25	0.51	0.79	0.49	0.28
0.89	-6.36	-10.1	-9.32	-3.93	-3.42	-4.10	-3.21	-4.36	-5.52	-7.00	-0.33	-1.17	-1.12	-2.50

Table 3: Fourth Order Cumulant Spectral Stationary Test Statistic

Frac	Z-value													
	1000 Obs							10000 Obs						
	Taus	LCG	RAN	Sum	B-M	A/R	Inv	Taus	LCG	RAN	Sum	B-M	A/R	Inv
0.60	11.8	11.4	11.4	0.04	3.34	2.31	0.91	-19.3	-19.3	-19.3	-2.51	-1.47	-0.59	0.03
0.69	9.90	9.57	9.49	0.22	3.70	2.46	0.88	-23.7	-23.7	-23.7	-2.90	-1.12	-0.56	0.60
0.77	8.14	7.96	8.00	-0.33	2.26	2.36	0.93	-29.2	-29.2	-29.2	-2.84	-1.85	-1.10	0.62
0.89	5.31	5.22	5.29	-0.73	2.19	2.30	0.22	-46.0	5.43	5.43	-3.28	-2.83	-1.45	-1.70

Table 4: Fourth Order Cumulant Spectral Stationary Test Statistic (Imaginary Part)

Frac	Z-value													
	1000 Obs							10000 Obs						
	Taus	LCG	RAN	Sum	B-M	A/R	Inv	Taus	LCG	RAN	Sum	B-M	A/R	Inv
0.60	-12.6	-12.5	-12.5	0.42	2.53	2.45	-0.19	-16.1	-18.0	-17.8	-3.35	-0.73	0.61	-0.17
0.69	-13.4	-14.5	-13.6	-0.20	2.69	3.33	0.54	-18.6	-21.4	-21.1	-2.68	-0.41	1.15	-0.36
0.77	-15.0	-16.7	-16.2	0.49	1.63	2.19	0.74	-22.1	-25.6	-25.0	-3.30	-0.56	0.45	-1.15
0.89	-18.8	-19.7	-19.5	-1.28	2.41	1.85	-1.25	-30.3	-36.7	-35.9	-4.23	-1.87	-1.09	-2.75

Table 5: Third Order Cumulant Spectral Gaussianity Test Statistic

Frac	Z-value													
	1000 Obs							10000 Obs						
	Taus	LCG	RAN	Sum	B-M	A/R	Inv	Taus	LCG	RAN	Sum	B-M	A/R	Inv
0.60	-11.9	-12.4	-11.5	-3.48	-3.36	0.00	-3.59	-10.4	-9.21	-11.3	0.99	-1.31	0.68	0.35
0.69	-13.2	-13.4	-12.9	-2.82	-1.75	-0.33	-2.71	-10.0	-10.9	-11.2	1.20	-1.08	0.96	1.51
0.77	-14.7	-14.7	-12.7	-2.43	-1.87	-0.41	-2.62	-11.2	-11.3	-11.3	0.55	-0.09	0.49	1.34
0.89	-16.0	-13.2	-12.5	-3.45	-2.86	-0.80	-3.10	-11.1	-11.8	-15.3	-1.33	0.07	-0.65	-1.14

Table 6: Fourth Order Cumulant Spectral Gaussianity Test Statistic

Frac	Z-value													
	1000 Obs							10000 Obs						
	Taus	LCG	RAN	Sum	B-M	A/R	Inv	Taus	LCG	RAN	Sum	B-M	A/R	Inv
0.60	12.9	12.9	12.9	-13.3	-13.5	-13.3	-13.8	-19.3	-19.3	-19.3	-10.5	-7.22	-9.72	-8.51
0.69	10.5	10.5	10.5	-15.3	-14.2	-14.9	-15.5	-23.7	-23.7	-23.7	-11.2	-7.22	-9.40	-8.22
0.77	8.55	8.54	8.55	-15.8	-15.0	-15.2	-16.0	-29.2	-29.2	-29.2	-12.2	-6.50	-8.19	-6.29
0.89	5.43	5.43	5.43	-16.9	-12.7	-11.5	-12.4	-46.0	-46.0	-46.0	-8.02	-3.87	-4.50	-5.23

Table 7: Third Order Cumulant Spectral Independence Test Statistic

Frac	Z-value (Gaussian Imag Part/Linearity (Frac: 0.8)													
	1000 Obs							10000 Obs						
	Taus	LCG	RAN	Sum	B-M	A/R	Inv	Taus	LCG	RAN	Sum	B-M	A/R	Inv
0.60	-9.03	-9.77	-7.40	-2.90	-2.39	-1.67	-2.04	-9.38	-8.27	-6.98	0.28	-2.23	-0.89	-0.59
	-12.2	-13.6	-11.6	-7.14	-8.54	-8.17	-9.26	-9.47	-7.53	-8.20	-4.99	-5.12	-3.36	-3.62
0.69	-9.32	-10.5	-8.43	-2.83	-2.18	-1.98	-1.34	-8.85	-7.53	-8.38	-0.60	-1.12	-0.42	-0.42
	-13.9	-16.1	-12.4	-8.30	-8.98	-9.11	-9.63	-9.56	-6.98	-8.96	-5.98	-3.69	-4.03	-3.63
0.77	-10.1	-10.5	-9.55	-2.33	-1.17	-1.21	-0.63	-7.29	-7.13	-8.78	-0.22	0.61	0.23	-0.50
	-15.1	-16.2	-13.9	-8.57	-10.5	-9.18	-10.4	-10.3	-6.84	-9.80	-6.73	-4.48	-5.33	-4.18
0.89	-11.0	-10.1	-11.3	-3.16	-1.42	-2.49	-2.95	-9.31	-5.64	-8.08	-2.77	0.48	0.49	-1.19
	-20.9	-21.9	-17.8	-11.5	-11.3	-11.3	-13.4	-11.0	-5.96	-12.1	-6.41	-6.23	-7.26	-5.19

Table 8: Fourth Order Cumulant Spectral Independence Test Statistic

Frac	Z-value (Gaussian Imag Part/Linearity (Frac: 0.8)													
	1000 Obs							10000 Obs						
	Taus	LCG	RAN	Sum	B-M	A/R	Inv	Taus	LCG	RAN	Sum	B-M	A/R	Inv
0.60	-19.1	-19.2	-19.1	-18.6	-18.5	-18.4	-18.7	-19.3	-19.3	-19.3	-19.3	-19.2	-19.3	-19.3
	-0.21	-1.60	-1.96	-10.5	-8.37	-9.42	-11.3	-2.06	-4.32	-3.51	-8.77	-5.12	-4.06	-5.19
0.69	-23.3	-23.5	-23.4	-22.4	-22.0	-22.0	-22.7	-23.7	-23.7	-23.7	-23.6	-23.6	-23.6	-23.6
	-1.94	-2.95	-3.44	-10.8	-8.89	-10.2	-12.6	-1.89	-4.80	-4.17	-8.39	-5.45	-4.56	-4.67
0.77	-28.6	-28.9	-28.8	-26.6	-26.1	-26.2	-27.3	-29.2	-29.2	-29.2	-29.1	-29.0	-29.1	-28.9
	-3.70	-4.51	-4.83	-11.5	-9.99	-11.3	-12.4	-2.25	-5.78	-4.16	-7.96	-5.92	-4.93	-4.41
0.89	-44.3	-44.8	-44.8	-36.8	-36.5	-37.1	-37.4	-46.0	-46.0	-46.0	-45.3	-44.8	-45.1	-44.6
	-5.58	-8.18	-8.04	-13.4	-11.4	-13.1	-12.5	-3.87	-6.15	-6.72	-5.98	-6.08	-5.82	-4.64

using the Gaussian generators is generally rejected at the 1% level, but always at the 5 % level. For sample size of 10000, all the uniform generators do not reject stationarity; the negative magnitude of the z-statistics indicates the absence of random characteristics. However, all the Gaussian generators do not reject the 3rd-order stationarity at either significance level.

The results for the 4th-order cumulant spectral test of stationarity using the Hinich-Wolinsky 4th-order transient set are presented in Table 3. For a sample size of 1000, all generators reject stationarity. For a sample size of 10000, all the uniform generators do not reject stationarity, but the negative magnitude of z-statistics suggests a lack of randomness. All the Gaussian generators, except the sum of twelve uniform(0,1) scheme, do not reject at either significance level. Table 4 reports results of the imaginary part of the 4th-order cumulant spectral test of stationarity. It is important to note that for both sample sizes, the negative magnitude of the test statistics derived from the uniform generators imply the lack of random characteristics. The implication of this observation is that as the variates are generated, the random structure, initially only in the real part of the 4th-order cumulant spectral estimate, vanishes completely.

Table 5 presents results for the 3rd-order Gaussianity test statistic. Note that since the 3rd-order Gaussianity test statistics is a decomposition of the skewness function, it can be interpreted as a test to detect a symmetry in a distribution function. The Gaussian and uniform distributions are symmetric. For both sample sizes, all the uniform generators should not and do not reject Gaussianity, i.e. the symmetric property of the uniform generators is not rejected. The negative magnitude of the z-statistics suggests the absence of randomness. For a sample size of 1000, all the Gaussian generators do not reject Gaussianity. Only the magnitude of the z-statistics of the acceptance-rejection scheme has values that are not on the borderline of being considered absent of random characteristics. For a sample size of 10000, all Gaussian generators do not reject Gaussianity at either significance level. The results from the imaginary part of the 3rd-order Gaussianity test statistic are presented in Table 7. Note that Gaussianity is not rejected, but there is an indication that the generators lack random characteristics because of the negative magnitude of the z-statistics.

In Table 6 the results from the 4th-order Gaussianity test statistic are reported. For the sample size of 1000, all uniform generators reject Gaussianity, as expected. For the sample size of 10000, Gaussianity is not rejected, but the negative magnitude of the z-statistics imply the absence of random structure in the generated variates. The Gaussian generators using both sample sizes do not reject Gaussianity, but the negative magnitude of the z-statistics is in accord with the notion that the generators may not produce a random process. Table 8 presents the results from the imaginary part of the 4th-order Gaussianity test statistic. The negative magnitudes of the z-statistics indicates the lack of randomness

Tables 7 and 8 present the results for the 3rd- and 4th-order tests of independence, i.e. first if the linearity test statistic is not rejected, then examine the imaginary part of the Gaussianity test statistic. If the imaginary part is statistically not different from zero, then the independence cannot be rejected. The analysis of the results in these two tables is straightforward: for both the 3rd- and 4th-order tests and for both sample sizes, independence is rejected for all generators.

6 CONCLUSION

Using higher-order cumulant spectral functions, a study of the higher-order statistical characteristics of pseudo random variate generators was undertaken. Statistical properties of uniform and Gaussian generators were examined. Results of these analyses are summarized as follows: 1) The commonly assumed property of independence between the generated variates is rejected for all generators; 2) The negative magnitude of the z-statistics indicates that these generators generally may not have the statistical properties required by the specific distribution of concern, i.e. the generators lack random characteristics; and 3) stationarity, a crucial assumption with respect to the reliability of the statistics of the simulation output, is in general rejected. Finally, a question open for further investigation concerns what happens to the random characteristics of the generated sequence, as the number of variates generated is increased.

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REFERENCES

Bratley, P., B. L. Fox, and L. E. Schrage. 1987. *A Guide to Simulation*, New York, New York: Springer-Verlag.

Brillinger, D. R. 1975. *Time Series, Data Analysis and Theory*, New York, New York: Holt, Rinehart, and Winston.

Brillinger, D. R. and M. Rosenblatt. 1967. "Asymptotic Theory of K-th Order Spectra," in *Spectral Analysis of Time Series*, edited by B. Harris, New York, New York: John Wiley and Sons Inc., 153-232.

Dalle Molle, J. W. and M.J. Hinich. 1989. "Trispectral Analysis", Technical Report No.

Dalle Molle, J. W. and M.J. Hinich. 1991. "Cumulant Spectra-Based Tests for the Detection of a Coherent Signal in Noise", *Proceedings of the International Signal Processing Workshop on Higher-Order Statistics*, Chamrousse, France.

David, F. N. and D. E. Barton. 1962. *Combinatorial Chance*, London, England: C. Griffin Limited.

Hinich, M.J. 1982. "Testing for Gaussianity and Linearity of a Stationary Time Series", *Journal of Time Series Analysis*, Vol. 3, No. 3, 169-176.

Hinich, M.J. and M. A. Wolinsky. 1988. "A Test for Aliasing Using Bispectral Analysis", *Journal of the American Statistical Association*, 83(402), 499-502.

Rosenblatt, M. 1985. *Stationary Sequences and Random Fields*, Boston, Mass: Birkhauser Inc..

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