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DISCRETE OPTIMIZATION IN SIMULATION: A METHOD AND APPLICATIONS

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ABSTRACT

Consider the problem of using simulation to optimize the performance of a stochastic system with respect to a number of decision variables. In the past, a considerable effort has been spent on the development of methods for solving such problems in the case when all the decision variables are continuous. However, the case when the decision variables are *discrete* has received very little attention to date. In this paper, we discuss a method for discrete simulation optimization and show how this method can be applied to optimize a special class of objective functions.

1 INTRODUCTION

In recent years, the problem of applying simulation to optimize the performance of a stochastic system with respect to continuous decision parameters has received considerable attention. One approach to solving this problem is to use a stochastic approximation algorithm (see for instance Robbins and Monro (1951), Venter (1967), Ruppert (1985) and Andradóttir (1991a, 1992a)) to locate a root of the gradient of the performance measure with respect to the decision parameters. For a recent survey of gradient estimation techniques, the reader is referred to L'Ecuyer (1991), and for results that guarantee the convergence of this simulation optimization procedure, the reader is referred to Glynn (1986) and Andradóttir (1991b).

On the other hand, the problem of applying simulation to optimize the performance of a stochastic system with respect to *discrete* decision parameters has received very little attention to date. Yet, many important stochastic optimization problems, such as the problem of determining the optimal buffer allocation in a manufacturing network, involve discrete decision parameters. When the number of alternatives is small, methods for selecting the best system (see Goldsman, Nelson and Schmeiser (1991)) can be used to solve such discrete optimization problems, and recently, Yan and Mukai (1992) have proposed a method for discrete stochastic optimization that is related to simulated annealing. This paper is concerned with a new method for discrete stochastic optimization. The organization of the paper is as follows: in Section 2, we present this new method, and in Section 3, we show how it can be applied to optimize a special class of objective functions. Finally, Section 4 contains some concluding remarks.

2 A METHOD FOR DISCRETE OPTIMIZATION IN SIMULATION

Consider the following optimization problem:

$$\min_{n \in \mathcal{N}} f(n) = E\{X_n\}, \tag{1}$$

where $\mathcal{N} = \{1, 2, \ldots\}$ and $\{X_n\}$ is a sequence of random variables. Let $\mathcal{S} \subset \mathcal{N}$ denote the set of local minimizers of the function f. When the expected values $E\{X_n\}$, $n = 1, 2, \ldots$, can be evaluated analytically, standard integer programming techniques can be used to solve the optimization problem (1). Unfortunately, these techniques cannot be applied when simulation is used to estimate the values of the objective function f, since they are not designed to handle noisy function evaluations. We present a new method that is designed to locate an element of the set $\mathcal{S}$ in the situation where the expected values $E\{X_n\}$, $n = 1, 2, \ldots$, cannot be evaluated analytically. We will need the following assumptions:

Assumption 1: For each $n \in \mathcal{N}$, there exist random variables Y_n, Y_n^+ such that $P\{Y_n^+ > Y_n\} > 0$, $P\{Y_n^+ < Y_n\} > 0$ and

$$\begin{aligned} f(n+1) > f(n) &\Rightarrow P\{Y_n^+ > Y_n\} > P\{Y_n^+ < Y_n\} \\ f(n+1) < f(n) &\Rightarrow P\{Y_n^+ > Y_n\} < P\{Y_n^+ < Y_n\} \\ f(n+1) = f(n) &\Rightarrow P\{Y_n^+ > Y_n\} = P\{Y_n^+ < Y_n\}. \end{aligned}$$

Assumption 2: There exists $N \in \mathcal{N}$ such that

$$\sup_{n \geq N} \frac{P\{Y_n^+ < Y_n\}}{P\{Y_n^+ > Y_n\}} < 1.$$

For all $n \in \mathcal{N}$, the random variables Y_n and Y_n^+ can be used to compare state n with state $n+1$: if $Y_n < Y_n^+$, then by Assumption 1, we have reason to believe that $f(n) < f(n+1)$, so since we are interested in minimizing the function f, we have reason to believe that state n is better than state $n+1$. Similarly, if $Y_n > Y_n^+$, then we have reason to believe that state $n+1$ is better than state n, and if $Y_n = Y_n^+$, then we have reason to believe that states n and $n+1$ are equally good. Observe that Assumptions 1 and 2 imply that $f(n+1) > f(n)$ for all $n \geq N$, so the set $\mathcal{S}$ is necessarily finite. The algorithm for solving the optimization problem (1) is given below:

Algorithm 1

Step 0: *Select a starting point $k_0 \in \mathcal{N}$. Let $N_{k_0} = 1$ and $N_n = 0$ for all $n \in \mathcal{N}, n \neq k_0$. Let $m = 0$ and $k_m^* = k_0$.*

Step 1: *Generate a Bernoulli random variable J_m such that $J_m = 1$ with probability $\frac{1}{2}$ and $J_m = 0$ with probability $\frac{1}{2}$. If $J_m = 1$, then go to step 2. If $J_m = 0$ and $k_m \neq 1$, then go to step 3. Finally, if $J_m = 0$ and $k_m = 1$, then let $k_{m+1} = k_m$ and go to step 4.*

Step 2: *Generate an observation Z_m of Y_{k_m} and an observation Z_m^+ of $Y_{k_m}^+$. If $Z_m^+ < Z_m$, then let $k_{m+1} = k_m + 1$. Otherwise, let $k_{m+1} = k_m$. Go to step 4.*

Step 3: *Generate an observation Z_m of Y_{k_m-1} and an observation Z_m^+ of $Y_{k_m-1}^+$. If $Z_m^+ > Z_m$, then let $k_{m+1} = k_m - 1$. Otherwise, let $k_{m+1} = k_m$. Go to step 4.*

Step 4: *Let $m = m+1$ and $N_{k_m} = N_{k_m} + 1$. If $N_{k_m} > N_{k_{m-1}^*}$, then let $k_m^* = k_m$. Otherwise, let $k_m^* = k_{m-1}^*$. Go to step 1.*

The essential idea behind Algorithm 1 is to define a random walk $\{k_m\}$ on the state space $\mathcal{N}$ having the property that if k_m^* is the state that the random walk has visited most often after m iterations of the algorithm, $m = 0, 1, \ldots$, then the sequence $\{k_m^*\}$ converges almost surely to an element of the set $\mathcal{S}$ as m goes to infinity. In each iteration m of Algorithm 1, we compare state k_m with either state $k_m + 1$ or with state $k_m - 1$. We then let k_{m+1} be equal to k_m if the two states appear to be equally good, otherwise we let k_{m+1} be the better one of the two states. It is possible to show that if Assumptions 1 and 2 are satisfied, then the sequence $\{k_m^*\}$ converges almost surely to an element of $\mathcal{S}$ (see Andradóttir (1992b)).

The major difficulty in applying Algorithm 1 is to find random variables Y_n, Y_n^+, for all $n \in \mathcal{N}$, that satisfy Assumptions 1 and 2. In some cases, it is possible to let $Y_n = X_n$ and $Y_n^+ = X_{n+1}$ for all n. In other cases, the choice of Y_n and Y_n^+ is less straightforward. In Section 3, we propose choices of Y_n and Y_n^+ that work on a particular class of problems.

3 APPLICATIONS

Consider the following optimization problem:

$$\min_{n \in \mathcal{N}} f(n) = an + bE\{I_n\}, \tag{2}$$

where $\{I_n\}$ is a sequence of indicator random variables and $a \geq 0$, $b > 0$. Optimization problems of this form arise frequently in situations where it is desired to determine the number of units that minimizes total cost, where costs arise from two sources: there is a cost a associated with each unit and a cost b that is incurred if a certain event takes place. Example 1 describes a particular instance of an optimization problem of the form (2).

Let I be a Bernoulli random variable with $p = P\{I = 1\} = a/(a+b)$, and consider

$$\begin{aligned} Y_n &= (1-I)I_n, \\ Y_n^+ &= I + (1-I)I_{n+1}. \end{aligned}$$

Observe that if $a = 0$ then $f(n) = E\{X_n\}$ with $X_n = bI_n$, and $Y_n = X_n/b$ and $Y_n^+ = X_{n+1}/b$ for all $n \in \mathcal{N}$.

Let $x_n = E\{I_n\}$, $p_n = P\{Y_n^+ < Y_n\}$ and $q_{n+1} = P\{Y_n^+ > Y_n\}$, for all n. Then, if I, I_n and I_{n+1} are independent, we have

$$\begin{aligned} p_n &= (1-p)x_n(1-x_{n+1}) \\ q_{n+1} &= p + (1-p)(1-x_n)x_{n+1} \end{aligned}$$

for all n, so that

$$q_{n+1} - p_n = p + (1-p)[x_{n+1} - x_n] \tag{3}$$

for all n. Assume that $0 < x_n < 1$ for all n. Since $b > 0$, it is clear that $p < 1$, so $p_n, q_{n+1} > 0$ for all n. In order to show that Assumption 1 is satisfied, it therefore suffices to show that

$$\text{sign}(q_{n+1} - p_n) = \text{sign}(f(n+1) - f(n)), \tag{4}$$

for all n, where $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = -1$ if $x < 0$, and $\text{sign}(0) = 0$. If $f(n+1) > f(n)$ for some

n, then $a + bx_{n+1} > bx_n$ or $x_{n+1} - x_n > -\frac{a}{b}$, so that

$$q_{n+1} - p_n > p - (1-p)\frac{a}{b} = 0$$

by equation (3). On the other hand, if $f(n+1) < f(n)$ for some n, then $a + bx_{n+1} < bx_n$ or $x_{n+1} - x_n < -\frac{a}{b}$, so that

$$q_{n+1} - p_n < p - (1-p)\frac{a}{b} = 0$$

by equation (3). Finally, if $f(n+1) = f(n)$ for some n, then $a + bx_{n+1} = bx_n$ or $x_{n+1} - x_n = -\frac{a}{b}$, so that

$$q_{n+1} - p_n = p - (1-p)\frac{a}{b} = 0$$

by equation (3). This shows that equation (4) holds for all n, so Assumption 1 is satisfied whenever $0 < x_n < 1$ for all n.

It remains to show that Assumption 2 is satisfied. We have

$$\frac{P\{Y_n^+ < Y_n\}}{P\{Y_n^+ > Y_n\}} = \frac{p_n}{q_{n+1}} = 1 - \frac{q_{n+1} - p_n}{q_{n+1}} \qquad (5)$$

for all n. Assume there exists $N \in \mathcal{N}$ such that

$$\inf_{n \geq N} (x_{n+1} - x_n) > -\frac{a}{b} \qquad (6)$$

Then, from equation (3)

$$\begin{aligned} \inf_{n \geq N} (q_{n+1} - p_n) &= p + (1-p) \inf_{n \geq N} (x_{n+1} - x_n) \\ &> p - (1-p)\frac{a}{b} = 0 \end{aligned}$$

since $p < 1$. Therefore, from equation (5)

$$\begin{aligned} \sup_{n \geq N} \frac{P\{Y_n^+ < Y_n\}}{P\{Y_n^+ > Y_n\}} &= 1 - \inf_{n \geq N} \frac{q_{n+1} - p_n}{q_{n+1}} \\ &\leq 1 - \inf_{n \geq N} (q_{n+1} - p_n) \\ &< 1. \end{aligned}$$

We have shown that Assumption 2 is satisfied whenever the inequality (6) holds.

Example 1 Consider the problem of determining the number of components of a parallel system that minimizes the expected total cost, when the cost of each component is a and a cost b is incurred if the system fails to function for T units of time. Let T_n denote the lifetime of a parallel system of n components. This problem is of the form (2), where $I_n = I_{\{T_n < T\}}$.

The remainder of this section will be devoted to the application of Algorithm 1 to the problem described above in the case when all components are independent with lifetime that is exponentially distributed with parameter λ. In this case, the objective function is easy to compute:

$$f(n) = an + b\left(1 - e^{-\lambda T}\right)^n$$

so the solution to the optimization problem (2) can be determined analytically. We will compare this analytical value with the results obtained from Algorithm 1. Observe that $x_n = (1 - e^{-\lambda T})^n$ for all n, so $0 < x_n < 1$ for all n, and $x_n \to 0$ as $n \to \infty$. Since

$$x_{n+1} - x_n = -e^{-\lambda T}\left(1 - e^{-\lambda T}\right)^n = -x_n e^{-\lambda T}$$

for all n, this shows that when $a, b > 0$, there exists $N \in \mathcal{N}$ such that the inequality (6) is satisfied, so both Assumptions 1 and 2 are satisfied on this problem.

Let $\lambda = 0.1$, $T = 10$, $a = 1$ and $b \in \{35, 350, 3500\}$. Then it is easy to show that when $b = 35$, the function f is minimized at $n = 6$, when $b = 350$, f is minimized at $n = 11$, and when $b = 3500$, f is minimized at $n = 16$. To test the performance of Algorithm 1 on these three problems, we run it 1000 times with the starting point $k_0 = 1$. The average performance of the algorithm is shown in Tables 1, 2 and 3. In all three cases, the sequence $\{k_m^*\}$ generated by Algorithm 1 moves towards the minimizer of the function f as m increases, but the empirical convergence rate depends on the value of the parameter b.

Table 1: Performance of Algorithm 1 when $b = 35$

Iteration	90% Confidence Interval
1,000	$6.040 \pm 8.663 \times 10^{-2}$
5,000	$6.079 \pm 5.544 \times 10^{-2}$
10,000	$6.079 \pm 4.419 \times 10^{-2}$
20,000	$6.097 \pm 3.466 \times 10^{-2}$
50,000	$6.058 \pm 2.626 \times 10^{-2}$
100,000	$6.057 \pm 1.740 \times 10^{-2}$
200,000	$6.021 \pm 9.098 \times 10^{-3}$
500,000	$6.002 \pm 2.325 \times 10^{-3}$

4 CONCLUSION

We have presented a new method for discrete stochastic optimization and shown how this method can be applied to solve a particular class of simulation optimization problems. Unlike methods for selecting the best system, this new method can handle a large number of alternatives, and it spends most of the computational effort close to the optimal solution of the problem it is being applied to (in fact, the estimate of the solution is always the parameter value

Table 2: Performance of Algorithm 1 when $b = 350$

Iteration	90% Confidence Interval
1,000	$8.271 \pm 1.086 \times 10^{-1}$
5,000	$10.427 \pm 1.018 \times 10^{-1}$
10,000	$10.827 \pm 9.064 \times 10^{-2}$
20,000	$11.002 \pm 7.999 \times 10^{-2}$
50,000	$11.066 \pm 5.832 \times 10^{-2}$
100,000	$11.071 \pm 4.515 \times 10^{-2}$
200,000	$11.058 \pm 3.719 \times 10^{-2}$
500,000	$11.069 \pm 2.735 \times 10^{-2}$

Table 3: Performance of Algorithm 1 when $b = 3500$

Iteration	90% Confidence Interval
1,000	$8.614 \pm 1.117 \times 10^{-1}$
5,000	$11.692 \pm 1.106 \times 10^{-1}$
10,000	$13.077 \pm 1.089 \times 10^{-1}$
20,000	$14.190 \pm 1.100 \times 10^{-1}$
50,000	$15.480 \pm 1.010 \times 10^{-1}$
100,000	$15.808 \pm 9.076 \times 10^{-2}$
200,000	$15.980 \pm 7.719 \times 10^{-2}$
500,000	$16.030 \pm 6.045 \times 10^{-2}$

that the algorithm has visited most often). However, more work is needed in order to determine the scope of applicability of this new method.

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