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SYSTEM IDENTIFICATION USING FREQUENCY DOMAIN METHODOLOGY

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ABSTRACT

We propose a method for using Frequency Domain Methodology to identify a linear meta-model. This involves multiple Frequency Domain runs to eliminate the effect of gain. The method uses the Fourier transform of the output process, rather than the periodogram or smoothed periodogram estimators. The identification focuses on estimating the coefficients of the model, and is therefore similar in spirit to Response Surface Methodology. The method is tested on two models, one with known structure and the other with unknown structure. The procedure does well at fitting the coefficients, including obtaining the correct sign. The procedure may be extended to identify the stochastic structure of the as well.

1. INTRODUCTION

Frequency Domain Methodology (FDM) has been proposed as a technique for factor screening [Schruben & Cogliano 1987] and for gradient estimation [Jacobson 1988]. A major advantage of FDM is its ability to give information in a *region* of the parameter space, rather than at a single factor combination as for conventional simulation experiments. Thus, FDM offers the potential to give the experimenter the same information as in a conventional factorial experiment with a fraction of the number of runs. In this paper we propose a method for identifying a linear meta-model from a relatively small number of frequency domain runs. As with previous researchers, we must contend with the effects of *gain*, the differential response of systems to various frequencies. Gain is the frequency domain analog of autocorrelation, which complicates simulation analysis in the time domain.

In this paper we restrict ourselves to models which are linear in the factors as well as the coefficients. Models which are polynomial in the factors may also be considered with essentially the same analysis. Other models, may be also considered [Schruben, Heath, and Buss 1988; Jacobson 1988; Sanchez and Buss 1987], again without essential change in the analysis.

The idea of FDM is to oscillate the design factors during the run, rather than holding them fixed as conventional simulation runs do. In this way, factors at different levels in all combinations occur during the run. Each factor is oscillated at a unique frequency, its *driving frequency*, and the resulting output has (potential) peaks at related frequencies, the *term indicator frequencies*. The analysis is in the frequency domain, which isolates the effects of each factor and combination of factors. This approach raises a number of issues. First, there is the validity of the meta-model selected. Given the validity of the meta-model, there are design issues for FDM. This involves choosing the appropriate region of the parameter space, the driving frequencies and amplitudes for the experimental factors, and the appropriate method for estimating the frequency content of the output.

Schruben and Cogliano [1987] and most subsequent researchers have used smoothed periodogram estimators. Others have more recently proposed using harmonic analysis [Jacobson 1990]. In this paper we will use the Fourier transform of the output, appropriately scaled, and use a least squares approach to fit the coefficients of the meta-model as well as the effects of gain at the term indicator frequencies. The reason for using the unsmoothed Fourier transform is that we are primarily interested in determining the coefficients of the meta-model, rather than the complete spectrum. To this effect, we are interested in estimating the frequency content of the output at a relatively small number of frequencies. The reasons for using smoothed spectral estimators have to do with statistical difficulties when trying to estimate the spectral density, which we are not attempting here. On the other hand, in the extensions we will indicate how a reasonable estimate of the spectrum may be obtained from a Frequency Domain Experiment.

2. THE MODEL

We begin with a collection of m experimental factors each of which takes values in a finite interval on the real line. We do not consider qualitative factors. We may thus assume without loss of generality that the factor space is $[-1,1]^m$. However, we note that for most experimental situations, the factors in fact represent deviations from certain non-zero nominal values. This is important in our approach to estimating the response coefficients.

Denoting the output series as $\{Y(t)\}$, we postulate a linear relationship between the steady state mean of $\{Y(t)$ and the experimental factors. That is, if factor j is at level x_j for $j = 1,\ldots,m$, then the mean response is given by

$$\mu_Y \equiv EY(t) = \sum_{j=1}^{m} b_j x_j \tag{2.1}$$

for some coefficients $b_1,\ldots,b_m$. We further assume that the response of the system is linear when the factors are oscillated and that there is a common impulse response function $g(k)$ to all factors. That is, if factor j is oscillated at frequency ω_j, then the steady state output is given by

$$Y(t) = \mu_Y + \sum_{j=1}^{m} \beta_j \sum_{k=1}^{\infty} g(k) \cos 2\pi\omega_j(t-k) + \epsilon(t), \tag{2.2}$$

where $\{\epsilon(t)\}$ is a stationary zero-mean process with absolutely continuous spectral density $f_\epsilon(\omega)$. We will also need $\{g(t)\}$ to be *strictly stable*; that is, $\sum_{k=1}^{\infty} k|g(k)| < \infty$ [Ljung 1987]. Letting

$$\hat{g}(\omega) = \sum_{k=1}^{\infty} g(k)e^{-2\pi i\omega k}, \tag{2.3}$$

we see that holding each factor constant at some level x_j results in the response in Equation (2.2):

$$Y(t) = \mu_Y + \hat{g}(0)\sum_{j=1}^{m} \beta_j x_j + \epsilon(t). \tag{2.4}$$

Observe that $b_j = \beta_j \hat{g}(0), j = 1, \ldots, m$. Our objective is to estimate the $\hat{g}(0)\beta_j$'s, since they will identify the deterministic portion of the system. We may also desire information about the stochastic portion of the system. In the current setting, this involves identifying the noise process $\{\epsilon(t)\}$, which may be done using standard methods after suitably adjusting the output.

There are several consequences of our assumption about the spectrum of $\{\epsilon(t)\}$. The primary implication for Frequency Domain Methodology is that there is no inherent periodic term in the disturbance. Thus, all peaks in the output spectrum correspond to cyclic components which have been induced by the oscillation of input factors. If the experimenter suspects the non-oscillated system to have periodic components, then this will affect both the frequency selection procedure, since these frequencies must be avoided, as well as the asymptotic properties of the spectral estimators. In our method, this means that the part of the Fourier transform due to the noise process has variance which decays as $1/N$ (see Equation (3.4) below).

An important feature of the output spectrum for this model is that there will be terms involving both $\hat{g}(\omega)$ and $f_\epsilon(\omega)$, the Fourier transforms of, respectively, the sequence $\{g(k)\}$ and the covariance of $\{\epsilon(t)\}$. If there is no oscillation, then $\hat{g}(\omega)$ is not present in the output spectrum. On the other hand, $f_\epsilon(\omega)$ will be in every estimate of the output spectrum. The importance of this is that the system will typically respond differently to different frequencies, a phenomenon called *gain*. This clearly will impact a Frequency Domain Experiment, since each factor will be oscillated at a different frequency. If gain is not adequately accounted for, then there is a confound between the effect of the factor and of the system at the corresponding frequency. That is, if we observe a peak at a term indicator frequency, we won't know if it is due to the factor or to system gain. However, in this context gain only refers to $f_\epsilon(\omega)$, not $\hat{g}(\omega)$. Even if the effects of $f_\epsilon(\omega)$ are removed (such as in the standard "Signal to Noise Ratio" estimate), $\hat{g}(\omega)$ is still in the spectrum. Our method, which we present in the following section, reduces the effect of $f_\epsilon(\omega)$ by using an estimator for which its effects diminish with larger samples. The effects of $\hat{g}(\omega)$, and the corresponding interaction with the β_j coefficients, is then handled using a least squares approach.

3. IDENTIFICATION

Assume that the m factors, m driving frequencies and m amplitudes have been selected. As suggested in Schruben and Cogliano [1987], we will *always* take driving frequencies to be Fourier frequencies; that is, frequencies of the form $2\pi k/N$, where N is the sample size. With the above model assumptions, we first compute the discrete Fourier transform of the output series, after first centering to zero mean:

$$\hat{Y}_N(\omega) \equiv \frac{1}{\sqrt{N}}\sum_{t=1}^{N} Y(t)e^{-2\pi i\omega t} \tag{3.1}$$

$$= \frac{\sqrt{N}}{2}\sum_{j=1}^{m} \beta_j \hat{g}(\omega_j)\delta(\omega - \omega_j) + \hat{\epsilon}_N(\omega) \tag{3.2}$$

where $\hat{\epsilon}_N(\omega) \equiv \sum_{t=1}^{N} \epsilon(t)e^{-2\pi\omega t}/\sqrt{N}$ is the Fourier transform of the error process, and

$$\delta(\omega) = \begin{cases} 1, & \text{if } \omega = 0 \\ 0, & \text{if } \omega \neq 0 \end{cases}$$

is the discrete Delta function. Thus, for the term indicator frequencies $\omega_1, \ldots, \omega_m$ we have

$$\frac{2}{\sqrt{N}}\hat{Y}_N(\omega_j) = \beta_j \hat{g}(\omega_j) + \frac{2}{\sqrt{N}}\hat{\epsilon}_N(\omega_j). \tag{3.3}$$

In the above we have assumed that $\mu_Y = 0$; in practice, we will have to estimate μ_Y by the sample mean $\bar{Y}$ and apply the analysis to $Y(t) - \bar{Y}$.

Observe that the (unknown) gain enters in each estimate of the output spectrum and that, with our model assumptions, $\hat{g}(\omega)$ is determined solely by the system response to oscillation. The response at the term indicator frequencies apparently has little to do with the frequency response of $\{\epsilon(t)\}$. However, we have [Ljung 1987]

$$\mathrm{Var}\left[2\hat{\epsilon}_N(\omega_j)/\sqrt{N}\right] = O(1/N) \tag{3.4}$$

so that $2\hat{Y}_N(\omega_j)/\sqrt{N}$ is consistent for $\beta_j\hat{g}(\omega_j)$. However, since $\hat{g}(\omega_j)$ is still in each expression at ω_j, this fact does not solve the problem associated with gain.

We will use multiple runs with different factor-frequency assignments in order to get around the effects of gain. Each run will be a signal run in which the factors are oscillated, but the driving frequencies will be assigned to different factors for each run. We propose the following approach. A set of driving frequencies is selected on the basis of the number of factors, the order of the model, and the criterion for spacing of term indicator frequencies. See Jacobson, Buss, and Schruben [1990] for some approaches to frequency selection. These are then adjusted to the closest Fourier frequency [Morrice 1990]. The factor-driving frequency assignments for the first run are selected randomly from the chosen set of frequencies. The remaining $m - 1$ runs are performed by cyclically permuting the driving frequency assignments in the first run. In this way we obtain estimates at each term indicator frequency-factor combination. Note that this multiple run approach is similar in spirit to ideas in Schruben and Cogliano [1987] and Jacobson [1988]. The difference here is in how the output is analyzed.

Let Y_{ij} denote the (complex) value of $2\hat{Y}_N(\omega_j)/\sqrt{N}$ for the run in which ω_j is the term indicator frequency for β_i. Denoting $\hat{g}(\omega_j)$ by γ_j, we will first estimate $\beta_1, \ldots, \beta_m$ and $\gamma_1, \ldots, \gamma_m$ by a least-squares solution; namely we seek to minimize

$$\sum_{ij} |\beta_i\gamma_j - Y_{ij}|^2. \tag{3.5}$$

Differentiating Expression (3.4), we obtain the following "normal" equations

$$\beta_i = \frac{\sum_j \Re(\bar{\gamma}_j Y_{ij})}{\sum_j |\gamma_j|^2} \tag{3.6}$$

$$\gamma_j = \frac{\sum_i \beta_i Y_{ij}}{\sum_i \beta_i^2} \tag{3.7}$$

where $\Re(\cdot)$ denotes the real part of a complex quantity. Although there does not appear to be a general closed-form solution to Equations (3.5) and (3.6), they are easily solved numerically.

Having obtained the solutions $\tilde{\beta}_1, \ldots, \tilde{\beta}_m$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ to Equations (3.5) and (3.6), we note that the $\tilde{\beta}$'s should be correct in the relative magnitudes, but not necessarily accurate estimates of the original β's. Recall that we are really interested in $\beta_j \hat{g}(0)$, for $j = 1, \ldots, m$. To obtain an estimate of $\hat{g}(0)$, we return to the original parameter region of the model. Let $c_1, \ldots, c_m$ be the actual center points in the region (in most practical applications, these are non-zero). Then since the steady-state mean of $\{Y(t)\}$ is

$$EY(t) = \sum_{j=1}^{m} \beta_j c_j \hat{g}(0), \tag{3.8}$$

we can estimate $\hat{g}(0)$ by

$$\frac{\bar{Y}}{\sum_{j=1}^{m} \tilde{\beta}_j c_j}. \tag{3.9}$$

Our estimated meta-model is therefore

$$E[Y] = \bar{Y} + \sum_{j=1}^{m} b_j x_j \tag{3.10}$$

where

$$b_j = \frac{\bar{Y}\tilde{\beta}_j}{\sum_{q=1}^{m} \tilde{\beta}_q c_q} \tag{3.11}$$

for $j = 1, \ldots, m$.

4. EXPERIMENTAL RESULTS

We will consider two examples to test the approach developed in the preceding section. The first example is a model which meets the assumptions exactly. The second is a model of a serial production line with stochastic processing times.

4.1 A Linear Model

We first examine a linear model for which the assumptions are met. We take $\{Y(t)\}$ to be of the form

$$Y(t) = \sum_{j=1}^{2} \beta_j \sum_{k=1}^{3} (c_j + a_j \cos 2\pi\omega_j(t-k))\, g(k) + \epsilon(t) \tag{4.1}$$

in which $g(1) = 1.$, $g(2) = 0.8$, $g(3) = 0.6$, $\beta_1 = 2.$, $\beta_2 = 1.$, $c_1 = c_2 = a_1 = a_2 = 1$, and $\{\epsilon(t)\}$ is a MA(1) process with parameter $\alpha = 0.8$. Observe that the corresponding Response Surface Model is therefore given by

$$E[Y] = 7.2 + 4.8x_1 + 2.4x_2. \tag{4.2}$$

The system was simulated for 4096 ($= 2^{12}$) observations, the resulting estimates being $\bar{Y} = 7.231$, $\tilde{\beta}_1 = 1.349$, $\tilde{\beta}_2 = 0.651$, $\tilde{g}(0) = 3.616$, $\tilde{\gamma}_1 = 0.593 - 1.188i$, $\tilde{\gamma}_2 = 0.512 - 0.123i$, so the fitted model is

$$E[Y] = 7.231 + 4.876x_1 + 2.355x_2. \tag{4.3}$$

We remind the reader that the estimates $\tilde{\beta}_j$'s and $\tilde{g}(0)$ are not necessarily accurate for the β_j's and $\hat{g}(0)$, but that $\tilde{\beta}_j\tilde{g}(0)$ does appear to accurately estimate $\beta_j \hat{g}(0)$.

One difficulty encountered with the use of smoothed periodogram estimators, which is shared by methods utilizing the periodogram, involves the sign of the coefficients, since the periodogram removes information about the sign. We therefore tested the approach in this paper with the above model, but with $\beta_1 = -1$. The true Response Surface is therefore

$$E[Y] = 2.4 + 4.8x_1 - 2.4x_2. \tag{4.4}$$

Again, 4096 observations were generated. The estimates were $\bar{Y} = 2.368$, $\tilde{\beta}_1 = 4.044$, $\tilde{\beta}_2 = -2.044$, $\tilde{g}(0) = 1.190$, $\tilde{\gamma}_1 = 0.332 - 0.663i$, $\tilde{\gamma}_2 = 0.277 - 0.598$, so the fitted model is

$$E[Y] = 2.368 + 4.808x_1 - 2.441x_2. \tag{4.5}$$

Thus, the method was able to detect the sign of the coefficient of x_2.

4.2 A Production Line

Next, we consider a production line consisting of machines in a series having stochastic processing times. Following standard assumptions, the first machine always has work and the second machine is never blocked. There is finite buffer capacity between the machines, however, which can lead to blocking and starving. Our experimental factors will be the mean processing times of each machine, and the output process will be the sequence of throughput times for each job.

We first consider a line with two machines and no buffer capacity between them. Thus, if a job finishes on the first machine while the second machine is busy, the first machine becomes blocked. We take the processing times are distributed as gamma random variables with mean 100. and coefficient of variation 0.3 for each machine. The amplitudes for the means will be 10. and the frequencies will be 585/4096 and 878/4096, with a run length of 4096. The fitted Response Surface Model is

$$E[Y] = 217.153 + 0.648x_1 + 1.523x_2 \tag{4.6}$$

in which x_i is the deviation from 100 for the mean of machine i. To test the fitted model, various combinations of means were separately estimated using conventional simulation runs. The results are summarized in Table 4.1. below. Observe that the fitted model is very close to the values obtained by the actual runs. The worst cases are near the boundaries of the parameter region.

Table 1. Predicted vs Actual Values

μ_1	μ_2	Estimated	Actual	% Error
90	110	225.900	228.514	1.1%
100	110	232.383	232.928	0.2%
105	105	228.010	227.516	0.2%
105	110	235.625	235.516	0.0%
110	90	208.407	208.581	0.1%
110	100	223.637	223.012	0.3%
110	105	231.252	230.572	0.3%
110	110	238.867	238.350	0.2%

Naturally, models such as this can be highly nonlinear. Therefore, it might be more accurate to employ higher order meta-models, particularly if the range of possible values is large. Nevertheless, the linear model presented here has done quite well. Experiments performed on other factors, such as the coefficient of variation, produced similar results.

5. SUMMARY AND RESEARCH DIRECTIONS

We have presented a method which identifies a linear model using Frequency Domain Methodology. Although more runs are required for this procedure than conventional FDM, the information obtained is more detailed. The method appears to work well in the examples presented. By focusing on the coefficient estimation alone, we were able to use the Fourier transform of the output series, rather than a periodogram estimator. The primary reason for this is that spectral estimators are oriented toward obtaining the entire spectrum, while for our purposes it suffices to obtain information at a small set of predetermined frequencies. Thus, for example, the asymptotic independence of an estimator at distinct frequencies is not the liability here that it is if we were attempting to estimate the spectral density. However, we can still estimate the disturbance spectrum using smoothed periodogram estimators as follows. First remove the Fourier transform points corresponding to the term indicator frequencies, then smooth the resulting periodogram. Observe that this estimates the spectrum of $\{\epsilon(t)\}$, but does not contain any information about $\hat{g}(\omega)$.

As discussed before, the method can easily be extended to higher order meta-models with relatively little change. This is being pursued in related work. Although we have used m runs, it is possible to determine the required estimates with as few as two runs. In this situation, the runs must be performed so that each factor has a unique *pair* of frequencies for the two runs. It is expected that this approach would result in considerably worse estimates than the m run method. Naturally, experiments with the number of runs between 2 and m could also be done.

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REFERENCES

Box, G. and N. Draper (1987), *Empirical Model Building and Response Surfaces*, John Wiley, New York, NY.

Brigham, E.O. (1988), *The Fast Fourier Transform and its Applications*, Prentice Hall, Englewood Cliffs, NJ.

Cogliano, V.J. (1982), "Sensitivity Analysis and Model Identification in Simulation Studies," PhD Thesis, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.

Jacobson, S. (1988), "Discrete-Event Computer Simulation Response Optimization in the Frequency Domain," PhD Thesis, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.

Jacobson, S. (1990), "Input/Output Model Extensions for Gradient Estimation in the Frequency Domain," presented at the 29*th* TIMS/ORSA Joint National Meeting, Las Vegas. NV.

Jacobson, S., A. Buss, and L. Schruben (1990), "Driving Frequency Selection for Frequency Domain Simulation Experiments," *Operations Research*, to appear.

Ljung, L. (1987), *System Identification: Theory for the User*, Prentice-Hall, Englewood Cliffs, NJ.

Morrice, D. (1990) *Personal Communication.*

Morrice, D. and L. Schruben (1988), "The False Negatives Problem," Technical Report #784, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.

Priestley, M. (1981), *The Spectral Analysis of Time Series*, Academic Press, New York, NY.

Schruben, L. and V. Cogliano (1987), "An Experimental Procedure for Simulation Response Surface Model Identification," *Communications of the Association for Computing Machinery*, 30 (8), 716–730.

Schruben, L., D. Heath, and A. Buss (1988), "A Dynamic Response Surface Model for Frequency Domain Simulation Experiments," Technical Report #808, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.