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MULTIVARIATE INFERENCES FOR REGENERATIVE SIMULATIONS TO A SPECIFIED PRECISION LEVEL

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ABSTRACT

When making an inference from a simulation output, the problem we usually encounter is the autocorrelation among observations. In this paper we proposed a technique using regenerative methods to overcome this problem and establish simultaneous confidence intervals for some correlated variables to a prespecified relative precision level. Five simulation models are used to test the performance of this technique and the coverage rate is the criterion. From the empirical results, the performance of this thchnique is quite satisfactory.

1. INTRODUCTION

Among those developed methods for establishing confidence intervals for simulation results,scholars seldom study the multivariate method : a method which considers several variables' joint effect while building up its joint confidence intervals. But we strongly believe that in this world,it is not one but many variables which are factors of decision-making applied in the simulation technology in order to solve practical problems. By considering the total effect from those variables can we make the best decision while dealing with practical problems.

When one wanted to estimate several population parameters during a simulation run, the old method to do this is to build up each variable's confidence interval individually. For example, if we want to observe five variables, we then build up each variable's 90 percent confidence range; but under this situation, the probability of covering the system's true value from each variable's confidence interval is only greater than or equal to 0.5. This can be proved by Bonferroni Inequality.

Suppose that I_l, $l = 1, 2, \ldots, K$ are the $1-\alpha_l$ confidence interval for the measure of performance μ_l. Then the probability that all K confidence intervals simultaneously contain their respective true measures satisfies the inequality :

$$P\{\mu_l \in I_l, \text{ for all } l = 1, 2, \ldots, K\} \geqq 1-\sum_{l=1}^{K}\alpha_l$$

whether or not the I's are independent.

This result is known as the Bonferroni inequality. Thus we know, when the number of variables we want to observe increases, the probability for all variables' confidence intervals to cover the system's true values decreases. This will influence the quality of decision-making seriously. So, using a multivariate method to build up a simultaneous confidence interval is indeed necessary.

In the field of multivariate simulation output analysis, Dr. Robert Chen in 1985 and 1986 had developed a technique to examine whether multivariate batch means vectors are independent from each other or not. This technique has made possible the use of multivariate batch means method to build up a joint confidence interval.

Consequently, S.J. Liue and L.Y. Wu, who under Dr. Chen's guidence,examined multivariate batch means method empirically by using a sequential method and a fixed-sample-size method. Their results were satisfying.

The original idea of regenerative method was brough up by Cox & Smith in 1961, but did not have any in-depth research until Crane & Iglehart (1974, 1975) and Fishman (1973, 1974) gave a series of studies in regenerative method and established a theoretical structure.

Seila (1982) had developed the theory of univariate regenerative method into a multivariate one, but did not do empirical test in his article. However, his artical is the base of this research. In this research, we not only improve the theorem of multivariate regenerative method, but also deveolp an algorithm for practical use. To those models which possess regenerative characteristics, we analyze their multivariate statistics from the simulation output, and build up its multivariate confidence interval. To our awareness, the concept of precision is very important in a confidence interval. If the interval is too wide which means the precision is too low, then the interval has little value to the decision makers. In our research, we use a sequential method to control the width of the confidence interval and set up a relative precision $\gamma = 0.1$. This research could help those simulation experimenters concentrate on their model-building. Once the model was built up and the data were collected,experimenters could utilize the procedure proposed in our research to analyze their multivariate statistics from their simulation runs. This not only saved the operating time and energy but also helped to approch a better decision-making.

In our research, we adopted Bonferroni, Roy-Bose, and Hotelling T^2 multivariate joint confidence region, and utilized M/M/1,Tandem Queue, Priority Queue, Central -Server Computer, and (s,S) Inventory models to examine the effect of the proposed technique. Among those evaluating criteria,the most important one is coverage rate. In general, if the confidence coefficient $1-\alpha$ is 0.9 when we repeat building up confidence intervals,then the theoretic coverage rate is 0.9 also.

In each testing model, we also examine system under several conditions. Consequently, there are thirteen conditions tested among these five models. In each condition, we conduct 100 independent experiments, and calculate its coverage rate. In the mean time, we list the average number of observations needed,the average number of regenerative epochs,and the average operating time of each experiment reference. The confidence coefficient we used is 0.9 in all our experiments.

We use Pascal language to write our programs in this research. Some of the programs were operated on CDC cyber 840 macro computers located in National Cheng-Kung University, some were operated on 16-bit PC/AT personal computer. The reason for using two different kinds of computer is because main frame computers are used for the academic research ; most business companies do not have big computing machines. Besides, personal computers have become cheaper and their functions are advancing too. This has make personal computer more and more popular in business companies. Thus, our research also operated on PC with the hope that business companies can also share our finding and utilize our results.

2. MULTIVARIATE REGENERATIVE METHOD

Suppose that the simulation runs for n cycles, and let the data generated for s parameters be $\{(\underline{X}_1, \underline{N}_1), (\underline{X}_2, \underline{N}_2), \ldots, (\underline{X}_n, \underline{N}_n)\}$, where $\underline{X}_i = (X_{i1}, X_{i2}, \ldots, X_{is})'$ is the vector whose j^{th} component is the sum or integral of observations on j^{th} variable in cycle i, and $\underline{N}_i = (N_{i1}, N_{i2}, \ldots, N_{is})'$ is a vector whose j^{th} component is the cycle length for parameter j in cycle i. Since the simulation is regenerative, we have

$$\nu_j = \frac{E(X_{ij})}{E(N_{ij})}, \quad j = 1, 2, \ldots, s .$$

The regenerative estimator for ν_j is

$$\hat{\nu}_j = \frac{\bar{X}_j}{\bar{N}_j},$$

where $\bar{X}_j = \frac{1}{n}\sum_{i=1}^{n} X_{ij}$, $\bar{N}_j = \frac{1}{n}\sum_{i=1}^{n} N_{ij}$.

We will define the multivariate regenerative estimator to be the vector $\underline{\hat{\nu}} = (\hat{\nu}_1, \hat{\nu}_2, \ldots, \hat{\nu}_s)'$. Let $\bar{N}_D$ be the diagonal matrix which has $\bar{N}_j$ as the j^{th} diagonal element. $\underline{\hat{\nu}}$ can be written as

$$\underline{\hat{\nu}} = \bar{N}_D^{-1}\underline{\bar{X}},$$

where $\bar{N}_D = \begin{bmatrix} \bar{N}_1 & & & 0 \\ & \bar{N}_2 & & \\ & & \ddots & \\ 0 & & & \bar{N}_s \end{bmatrix}$, and $\underline{\bar{X}} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_s)'$.

Now, define $R_{ij} = X_{ij} - \nu_j N_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, s$, and let $\underline{R}_i = (R_{i1}, R_{i2}, \ldots, R_{is})'$, then the sequence $\{\underline{R}_1, \underline{R}_2, \ldots, \underline{R}_n\}$ is an i.i.d sequence of random vectors with mean $\underline{0}$. Denote the covariance matrix of $\underline{R}_i$ by $\underline{\Sigma}$.

The following theorem has been proved by Carson.

$$\sqrt{n}\,(\underline{\hat{\nu}} - \underline{\nu}) \rightarrow N(\underline{0}, \underline{\mu}_D^{-1}\underline{\Sigma}\,\underline{\mu}_D^{-1}) \quad \text{as} \quad n \rightarrow \infty ,$$

where $\underline{\mu}_D = E(\underline{\bar{N}}_D)$.

Proof: Since $\{\underline{R}_1, \underline{R}_2, \ldots, \underline{R}_n\}$ is a sequence of i.i.d. random vectors,we have,by the multivariate central limit theorem

$$\sqrt{n}\underline{\bar{R}} \rightarrow N(\underline{0}, \underline{\Sigma}) \quad \text{as} \quad n \rightarrow \infty ,$$

where $\underline{\bar{R}} = (1/n)\sum_{i=1}^{n}\underline{R}_i$, and $\underline{\bar{N}}_D^{-1}\underline{\bar{R}} = \underline{\hat{\nu}} - \underline{\nu}$.

Since $\bar{N}_j \rightarrow E(N_{ij})$ with probability one, as $n \rightarrow \infty$, we have by the continuous mapping theorem, that

$$\sqrt{n}\underline{\bar{N}}_D^{-1}\underline{\bar{R}} = \sqrt{n}(\underline{\hat{\nu}} - \underline{\nu}) \rightarrow N(\underline{0}, \underline{\mu}_D^{-1}\underline{\Sigma}\underline{\mu}_D^{-1}) .$$

This completes the proof.

Applying the theorem above, we can propose the technique for constructing multivariate joint confidence intervals. Now, we show the processes technique, including Hotelling's T^2, Roy-Bose, and Bonferroni.

2.1. Hotelling's T^2 Method :

Let $\underline{S}$ denote the usual estimator of $\underline{\Sigma}$, given by

$$\underline{S} = \frac{1}{n-1}\sum_{i=1}^{n} \underline{R}_i \cdot \underline{R}_i' \qquad (1)$$

Now, define $T^2 = n\underline{\bar{R}}'\underline{S}^{-1}\underline{\bar{R}}$,

where $\underline{\bar{R}} = \frac{1}{n}\sum_{i=1}^{n}\underline{R}_i$.

Since $\underline{\bar{R}}$ is approximately a multivariate normal distribution for large n, we can show, by applying standard multivariate normal sampling theory, that $F = [(n-s)/(n-1)s] * T^2$ is approximately an F distribution with s and n-s degrees of freedom in the numerator and denominator, respectively. However, since $\underline{\bar{R}} = \underline{\bar{N}}_D(\underline{\hat{\nu}} - \underline{\nu})$, we can write T^2 as

$$T^2(\underline{\nu}) = n(\underline{\hat{\nu}} - \underline{\nu})'\underline{\bar{N}}_D\underline{S}^{-1}\underline{\bar{N}}_D(\underline{\hat{\nu}} - \underline{\nu}).$$

We can therefore find the multivariate confidence region for $\underline{\nu}$ as

$$T^2(\underline{\nu}) \leqq \frac{(n-1)s}{n-s} F_{1-\alpha;s,n-s}$$

Since $\underline{\hat{\nu}}$ would converge to multivariate normal distribution only for large n, this confidence interval will work in the same condition.

2.2. Roy-Bose Method :

Since $R_{ij} = X_{ij} - \nu_j N_{ij}$, in practice, $\underline{S}$ given by (1) can be approximated by replacing $\underline{\nu}$ with $\underline{\hat{\nu}}$. This is the same approach that is used in univariate. Now, suppose that simultaneous confidence intervals are to be computed for l linear combinations, $\delta_1, \delta_2, \ldots, \delta_l$, where $\delta_j = \underline{\pi}_j'\underline{\nu}$, j = 1, 2, ..., l, $\underline{\pi}_j$ is an s-dimensional vector. We can define

$$t^2(\underline{\pi}_j) = [\underline{\pi}_j'(\underline{\hat{\nu}} - \underline{\nu})]^2 / \hat{\sigma}_j^2 ,$$

where $\hat{\sigma}_j^2 = \underline{\pi}_j'\underline{\bar{N}}_D^{-1}\underline{S}\underline{\bar{N}}_D^{-1}\underline{\pi}_j/n$. Then

$$P[t^2(\underline{\pi}_j) \leqq \frac{(n-1)s}{n-s} F_{1-\alpha;s,n-s}, \text{for all } j=1,2,\ldots,l] = 1-\alpha$$

where $F_{1-\alpha;s,n-s}$ is the $100(1-\alpha)$ percentage point of the F distribution with s and n-s degrees of freedom. Therefore, the inequalities

$$\underline{\pi}_j'\underline{\hat{\nu}} - \hat{\sigma}_j[\frac{(n-1)s}{n-s} F_{1-\alpha;s,n-s}]^{1/2} \leqq \delta_j$$

$$\underline{\pi}_j'\underline{\hat{\nu}} + \hat{\sigma}_j[\frac{(n-1)s}{n-s} F_{1-\alpha;s,n-s}]^{1/2} \geqq \delta_j$$

hold simultaneously for j = 1, 2, .., l with probability $1-\alpha$.

2.3. Bonferroni Method :

For each j = 1, 2, ..., l , $t(\underline{\pi}_j)$ follows approximately a Student's t distribution with n-1 degrees of freedom. Therefore, for each j = 1, 2, ..., l , the inequality

$$\underline{\pi}_j'\underline{\hat{\nu}} - \hat{\sigma}_j t(1-\alpha_j/2;n-1) \leqq \delta_j \leqq \underline{\pi}_j'\underline{\hat{\nu}} - \hat{\sigma}_j t(1-\alpha_j/2;n-1)$$

holds with probability $1-\alpha_j$. Bonferroni's inequality states that the probability that all of the inequalities for j = 1, 2, ..., l , hold simultaneously is at least $1-\sum_{j=1}^{n}\alpha_j$.

3. OPERATING PROCEDURE

In this research we defined the relative precision γ_0 of joint confidence intervals as the maximun value of the relative precision of individual intervals. That is, the relative precision of every interval would not be bigger than γ_0 . In this section, we will describe the proposed procedure to build up a simultaneous confidence interval. A flow chart of the procedure is shown in Figure 3-1. The procedure is step by step as follows :

(1) First we generate n_0 number of regenerative epochs. The range of n_0 depends on the model used. Normally, n_0 is bigger in Queueing System and smaller in Inventory System. In any case, n_0 has to be at least one more than the number of variables we observed. So that when building up Roy-Bose joint conf-

idence intervals, the degree of freedom will not be smaller than 0.

(2) Use the sample observations from these n_0 cycles to calculate the Bonferroni joint confidence interval and then check if it fit the relative precision requirement. If its relative precision $\gamma \leqq \gamma_0$, then go to step (4), otherwise go to step (3).

(3) Generate one more regenerative epoch, and add the observed values from this epoch to the original data and go back to step (2).

(4) Check if $\underline{\nu}$ is within the Bonferroni joint confidence interval. Then use the $\hat{\underline{\nu}}$ to calculate the Hotelling's T^2 value and check if $\underline{\nu}$ is inside the confidence region.

(5) Calculate the Roy-Bose joint confidence interval, and check if it satisfies the relative precision requirement. If it does then go to step (7), otherwise, go to step (6).

(6) Generate one more regenerative epoch, and add the observed values from this epoch to the original data and go back to step (5).

(7) Check if $\underline{\nu}$ is within the Roy-Bose joint confidence interval. Then use the $\hat{\underline{\nu}}$ to calculate the Hotelling's T^2 value and check if $\underline{\nu}$ is inside the confidence region.

In the proposed procedure, we do not have to keep each observed data, simply keep their sums. Therefore, the computer memory required is kept to the minimum. So the procedure is very efficient and very fast.

4. EMPIRICAL RESULTS

We have tested the procedure on five models : M/M/1, Tandem Queue, Priority Queue, Central--Server Computer System, and (s,S) Inventory models.

In the first three models, we have used traffic intensity ρ = 0.5, 0.7, and 0.9 to test the performance of the procedure. In Central--Server Computer System, we used μ_2 = 0.5, μ_3 = 0.5, and μ_2 = 0.9, μ_3 = 0.1 two different conditions. In (s,S) Inventory model we tested when the average demand μ_Q = 18 and μ_Q = 25 two conditions.

In every condition, we generated 100 independent replications. The joint confidence coeffiecient $1-\alpha$ is 0.9, its relative precision requirement γ_0 is 0.1. The results of these experiments are listed in Table 4-1 to Table 4-5.

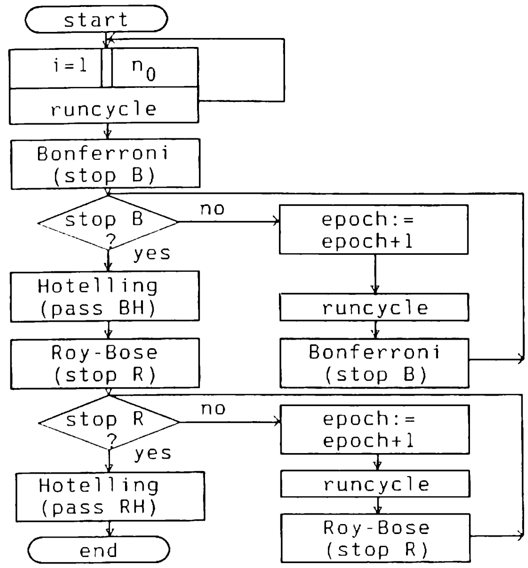

Figure 3-1 : Flow Chart of the Algorithm

In each table, there are two Hotelling T^2 coverage rates, one denoted with [B] and the other with [R]. The one with [B] indicates that when the Bonferroni Joint Interval reaches the relative precision requirement, we use the data to establish a Hotelling T^2 confidence region,and check if it contains the population mean vector to count its coverage rate. And the one with [R] is similarly established except for that it is established after the Roy-Bose simultaneous confidence interval reaches the relative precision requirement.

Among all the coverage rates,the theoretic value of them should be 0.9, since when we build up the joint confidence interval we have used a confidence coefficient of 0.9. While from a practical point of view, we would expect the coverage rate to be higher, we set a hypothesis test here to see if the coverage rate we got were higher than the theoretic value 0.9.

H_0 : $p \geqq 0.9$

H_1 : $p < 0.9$

The decision rule is if $\hat{P} < 0.9 - Z_{0.95}\sqrt{P(1-P)/N}$,

then reject H_0 . After calculation, the critical value of coverage rate P is 0.851 with significance level .05. We put a "*" sign on the upper right hand side of those coverage rates that did not pass the hypothesis test.

The average number of observations and epochs in our experiments are also listed for Bonferroni method and Roy-Bose method. The numbers in the square brackets below them are their standard deviation. We also list the average operating time of PC/AT and CDC for comparison.

Table 4-1 : Empirical Results of M/M/1 Model

Evaluating Criteria ρ	Low (0.50)	Middle (0.70)	High (0.90)
Bonferroni Coverage Rate	0.93	0.93	0.88
[B] Hotelling T^2 Coverage Rate	0.89	0.91	0.85
[B] Average Number of Observations	15311 (2908)	29432 (7602)	184875 (55426)
[B] Average Number of Regenerative Epochs	7649 (1448)	8822 (2268)	18392 (5351)
Roy-Bose Coverage Rate	0.98	0.97	0.95
[R] Hotelling T^2 Coverage Rate	0.91	0.94	0.92
[R] Average Number of Observations	21612 (3716)	42384 (10191)	262727 (67102)
[R] Average Number of Regenerative Epochs	10800 (1849)	12676 (3052)	26168 (6526)
PC/AT Average Operating Time	12'10"	16'30"	65'30"
COD Average Operating Time (CPU seconds)	15.69	22.73	94.28

Table 4-2 : Empirical Results of Tandem Queue Model

Evaluating Criteria ρ_1 ρ_2	0.5 0.5	0.7 0.75	0.9 0.9
Bonferroni Coverage Rate	0.93	0.96	0.92
[B] Hotelling T^2 Coverage Rate	0.92	0.91	0.88
[B] Average Number of Observations	17299 (3476)	44755 (11404)	260223 (86144)
[B] Average Number of Regenerative Epochs	4333 (878)	3346 (843)	2623 (862)
Roy-Bose Coverage Rate	0.97	1.00	0.97
[R] Hotelling T^2 Coverage Rate	0.93	0.95	0.88
[R] Average Number of Observations	29160 (4351)	78938 (16932)	451176 (106287)
[R] Average Number of Regenerative Epochs	7293 (1081)	5921 (1263)	4541 (1057)
PC/AT Average Operating Time	15'10"	29'10"	140'30"
COD Average Operating Time (CPU seconds)	23.70	44.32	207.88

Table 4-3 : Empirical Results of Priority Queue Model

Evaluating Criteria ρ	Low (0.50)	Middle (0.70)	High (0.90)
Bonferroni Coverage Rate	0.93	0.92	0.89
[B] Hotelling T^2 Coverage Rate	0.92	0.90	0.86
[B] Average Number of Observations	6151 (1799)	19728 (5711)	185038 (59913)
[B] Average Number of Regenerative Epochs	3065 (894)	5887 (1707)	18388 (5634)
Roy-Bose Coverage Rate	0.96	0.94	0.96
[R] Hotelling T^2 Coverage Rate	0.88	0.85	0.87
[R] Average Number of Observations	8581 (2268)	26377 (7523)	260668 (90316)
[R] Average Number of Regenerative Epochs	4271 (1124)	7894 (2263)	25956 (8549)
PC/AT Average Operating Time	4'20"	12'0"	62'50"
COD Average Operating Time (CPU seconds)	7.38	18.34	85.66

Table 4-4 : Empirical Results of Central Server Computer Model

Evaluating Criteria μ_2 μ_3	0.5 0.5	0.9 0.1
Bonferroni Coverage Rate	0.84*	0.90
[B] Hotelling T^2 Coverage Rate	0.71*	0.82*
[B] Average Number of Observations	914 (323)	1309 (375)
[B] Average Number of Regenerative Epochs	37 (15)	52 (17)
Roy-Bose Coverage Rate	0.97	0.96
[R] Hotelling T^2 Coverage Rate	0.79*	0.79*
[R] Average Number of Observations	1493 (445)	1628 (506)
[R] Average Number of Regenerative Epochs	61 (18)	66 (20)
PC/AT Average Operating Time	2'50"	5'10"
COD Average Operating Time (CPU seconds)	5.64	8.78

Table 4-5 : Empirical Results of (s,S) Inventory Model

Evaluating Criteria μ_Q	18	25
Bonferroni Coverage Rate	0.94	0.92
[B] Hotelling T^2 Coverage Rate	0.92	0.87
[B] Average Number of Observations	54 (6)	71 (8)
[B] Average Number of Regenerative Epochs	20 (2)	34 (4)
Roy-Bose Coverage Rate	0.98	0.95
[R] Hotelling T^2 Coverage Rate	0.91	0.82*
[R] Average Number of Observations	70 (8)	92 (10)
[R] Average Number of Regenerative Epochs	25 (3)	44 (5)
PC/AT Average Operating Time	50"	50"
CDD Average Operating Time (CPU seconds)	2.27	2.30

5. ANALYSIS AND DISCUSSION

In this research, we tried to overcome the autocorrelation among multivariate simulation output observations. To see if we have accomplished this objective, the coverage rate of confidence intervals becomes the important criterion. As we use 0.9 as the confidence coefficient, the theoretical coverage rate is 0.9.

From our results, the coverage rate of Bonferroni simultaneous confidence intervals are close to 0.9 for most of those five testing models under different conditions, and it is even over 0.95 for Roy-Bose joint intervals. The coverage rates of Hotelling's confidence regions passed the test most of the time except for the central server computer models. So overall, the performance of the technique we proposed is very good.

Since the interval width of Roy-Bose is wider than that of Bonferroni, the coverage rate of Roy-Bose should be higher than the coverage rate of Bonferroni for the same sample data. In our research,to reach the specified relative precision, the Roy-Bose method would require more sample data, so the coverage rate of Roy-Bose is still higher than Bonferroni. The Hotelling's confidence region is an ellipsoid which lies inside the rectangle of Roy-Bose and Bonferroni simultaneous confidence intervals,so its coverage rate is lower than both Bonferroni and Roy-Bose. However, since the interval length of the Roy-Bose simultaneous confidence intervals is designed for the purpose that the probability for all linear combinations of the population parameters to be inside its corresponding confidence intervals is 0.9 , it is wider than for 1 linear combinations. So its theoretical coverage rate should be no less than 0.9. Similarly, since 0.9 is the lower limit for the Bonferroni simultaneous confidence coefficient, the coverage rate for Bonferroni is no less than 0.9 .

In doing the multivariate inferences, we assumed that the sample size is sufficiently large. But in the Computer models and Inventory models the sample size we got when reaching the specified relative precision requirement is still quite small. This might explain why the coverage rate in these two models is low. In the queueing models, the number of observations needed to reach the specified precision requirement is quite large. Therefore the multivariate normality assumption can be satisfied by central limit theorem. And their coverage rates are quite satisfactory.

Our results from PC or CDC Cyber 840 computer are all the same. The only difference is the operating time. The pure machine operating time of CDC is about 40 times faster than that of PC. However, as the main frame computer is shared by many users, sometimes we have to wait for a longer time to get the results than we do on PC . The operating time shown in the tables includes the data generating time and the technique calculation time.

When the relative precision requirement is set close to 0.1, the coverage rate would be close to the theoretical value. But when the relative precision is too small , the number of observations needed will become very large, and the cost will increase tremendously. So we test the performance of the proposed technique for six different precision levels : 0.05, 0.075, 0.1, 0.15, 0.2 and 0.3 at M/M/1 model and Inventory model. The results are indicated in Table 5-1 to 5-5. From our results,the coverage rate of M/M/1 model increased as the precision requirement becomes smaller and most of them are quite satisfactory. And the coverage rate for the Inventory model at different precision levels passed the test too.

Theoretically, the Hotelling confidence region is the ideal inference for a multivariate population parameters. But as it is an ellipsoid, the interpretation is

Table 5-1 : M/M/1, ρ =0.5, Empirical Result under Different Relative Precision

Evaluating Criteria γ_0	0.30	0.20	0.15	0.10	0.075	0.05
Bonferroni Coverage Rate	0.89	0.94	0.92	0.93	0.90	0.91
[B] Hotelling T^2 Coverage Rate	0.78*	0.93	0.86	0.89	0.85	0.88
[B] Average Number of Observations	1383 (583)	3604 (1360)	6556 (2062)	15311 (2908)	26284 (4487)	62390 (7752)
[B] Average Number of Regenerative Epochs	688 (288)	1800 (680)	3277 (1033)	7649 (1448)	13131 (2251)	31159 (3926)
Roy-Bose Coverage Rate	0.92	0.96	0.98	0.98	0.97	0.98
[R] Hotelling T^2 Coverage Rate	0.85	0.88	0.87	0.91	0.90	0.85
[R] Average Number of Observations	1976 (745)	5029 (1879)	9300 (2540)	21612 (3716)	39247 (6704)	85194 (8498)
[R] Average Number of Regenerative Epochs	986 (369)	2514 (944)	4638 (1275)	10800 (1849)	19623 (3330)	42559 (4293)
PC/AT Average Operating Time	1'50"	4'0"	5'0"	12'10"	16'20"	35'30"
COD Average Operating Time (CPU seconds)	3.25	5.21	7.84	15.69	23.14	48.03

Table 5-2 : M/M/1, ρ =0.7, Empirical Result under Different Relative Precision

Evaluating Criteria γ_0	0.30	0.20	0.15	0.10	0.075	0.05
Bonferroni Coverage Rate	0.79*	0.89	0.96	0.93	0.96	0.92
[B] Hotelling T^2 Coverage Rate	0.73*	0.82*	0.81*	0.91	0.93	0.85
[B] Average Number of Observations	2280 (1275)	6670 (3402)	12588 (4693)	29432 (7602)	54223 (11775)	131494 (24070)
[B] Average Number of Regenerative Epochs	675 (368)	1995 (1022)	3760 (1417)	8822 (2268)	16213 (3445)	39402 (7139)
Roy-Bose Coverage Rate	0.91	0.96	0.99	0.97	0.97	0.96
[R] Hotelling T^2 Coverage Rate	0.79*	0.80*	0.90	0.94	0.89	0.80*
[R] Average Number of Observations	3595 (2141)	9325 (4165)	17511 (5651)	42384 (10191)	76994 (13363)	183502 (30734)
[R] Average Number of Regenerative Epochs	1072 (640)	2788 (1256)	5234 (1690)	12676 (3052)	23031 (3894)	55044 (9113)
PC/AT Average Operating Time	3'0"	4'40"	6'40"	16'30"	25'0"	60'20"
COD Average Operating Time (CPU seconds)	3.73	6.55	10.53	22.73	35.31	82.58

Table 5-3 : M/M/1, ρ =0.9, Empirical Result under Different Relative Precision

Evaluating Criteria γ_0	0.30	0.20	0.15	0.10	0.075	0.05
Bonferroni Coverage Rate	0.83*	0.91	0.89	0.88	0.94	0.95
[B] Hotelling T^2 Coverage Rate	0.71*	0.83*	0.79*	0.85	0.86	0.84*
[B] Average Number of Observations	13717 (10738)	37395 (20246)	73374 (30663)	184871 (55426)	361184 (86800)	880256 (252045)
[B] Average Number of Regenerative Epochs	1358 (1036)	3719 (1965)	7304 (2970)	18392 (5351)	35622 (8230)	87782 (24582)
Roy-Bose Coverage Rate	0.87	0.96	0.92	0.95	0.97	0.98
[R] Hotelling T^2 Coverage Rate	0.74*	0.85	0.83*	0.92	0.90	0.87
[R] Average Number of Observations	19940 (13667)	55649 (25288)	105114 (37499)	262727 (67102)	526402 (114019)	1196240 (279950)
[R] Average Number of Regenerative Epochs	1982 (1350)	5519 (2447)	10482 (3633)	26168 (6526)	52756 (10864)	119271 (27422)
PC/AT Average Operating Time	5'50"	12'30"	25'50"	65'30"	110'0"	270'0"
CDD Average Operating Time (CPU seconds)	8.40	19.32	39.55	94.28	156.13	385.36

Table 5-4 : Inventory Model, μ_Q=18, Empirical Result under Different Relative Precision

Evaluating Criteria γ_0	0.30	0.20	0.15	0.10	0.075	0.05
Bonferroni Coverage Rate	0.99	0.91	0.90	0.94	0.90	0.89
[B] Hotelling T^2 Coverage Rate	0.91	0.92	0.89	0.92	0.85	0.88
[B] Average Number of Observations	18 (2)	25 (3)	33 (4)	54 (6)	78 (9)	143 (17)
[B] Average Number of Regenerative Epochs	7 (1)	9 (1)	12 (1)	20 (2)	28 (3)	51 (6)
Roy-Bose Coverage Rate	1.00	0.96	0.97	0.98	0.93	0.95
[R] Hotelling T^2 Coverage Rate	0.88	0.88	0.90	0.91	0.85	0.89
[R] Average Number of Observations	24 (2)	33 (3)	44 (4)	70 (8)	102 (12)	191 (20)
[R] Average Number of Regenerative Epochs	9 (1)	12 (1)	16 (2)	25 (3)	37 (4)	68 (7)
PC/AT Average Operating Time	50"	50"	50"	50"	50"	50"
CDD Average Operating Time (CPU seconds)	2.25	2.25	2.26	2.27	2.29	2.34

Table 5-5 : Inventory Model, μ_Q=25, Empirical Result under Different Relative Precision

Evaluating Criteria γ_0	0.30	0.20	0.15	0.10	0.075	0.05
Bonferroni Coverage Rate	0.95	0.94	0.91	0.92	0.90	0.91
[B] Hotelling T^2 Coverage Rate	0.83*	0.83*	0.81*	0.87	0.80*	0.87
[B] Average Number of Observations	19 (2)	30 (3)	41 (4)	71 (8)	108 (13)	206 (18)
[B] Average Number of Regenerative Epochs	10 (1)	15 (2)	20 (2)	34 (4)	52 (6)	99 (9)
Roy-Bose Coverage Rate	1.00	0.97	0.93	0.95	0.96	0.95
[R] Hotelling T^2 Coverage Rate	0.85	0.80*	0.81*	0.82*	0.85	0.85
[R] Average Number of Observations	26 (2)	38 (4)	54 (6)	92 (10)	143 (16)	274 (23)
[R] Average Number of Regenerative Epochs	13 (1)	19 (2)	26 (3)	44 (5)	69 (8)	132 (12)
PC/AT Average Operating Time	50"	50"	50"	50"	50"	50"
COD Average Operating Time (CPU seconds)	2.25	2.26	2.27	2.30	2.34	2.44

more difficult than the simultaneous confidence interval which is a rectangle. So in practice, people tend to use simultaneous confidence intervals most of the time. In our research we use Hotelling's confidence region as the reference for comparing the coverage rates.

6. CONCLUSION

The application of simulation technology is prevailing in almost every field. With the increase use of simulation technology, the correct inferences from the simulation result is becoming more important. And as the number of factors considered for decision-making is increasing, there is a need for multivariate inferences. In this paper we present a multivariate inference technique using the regenerative method to establish simultaneous confidence intervals and keep their relative precisions to a specified level. This technique can be applied to any steady-state simulation which is regenerative and has more than one variable to be analyzed. The results indicate that the performance of this technique is quite satisfactory. And as the program is written in PASCAL language, it is operatable in almost everywhere on both main frame computer and personal computer.

The only limitation of this technique is that when a simulation model is not regenerative or its regenerative period is too long then this technique is not applicable. Also, when the regenerative points are not easy to identified the application of this technique might be difficult.

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