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Confidence intervals and orthonormally weighted standardized time series

Robert D. Foley
School of ISyE
Georgia Tech
Atlanta, Georgia 30332

David Goldsman
School of ISyE
Georgia Tech
Atlanta, Georgia 30332

Abstract

We find new confidence intervals for the mean of a stationary process. The new intervals are based on orthonormally weighted standardized time series and asymptotically have smaller half-length mean and variance than their predecessors.

1 Introduction

In this paper, we find confidence intervals for the mean μ of a stationary process $Y_1, Y_2, \ldots$. We generalize the standardized time series area and weighted area confidence intervals developed by Schruben [4] and Goldsman and Schruben [3]. An expanded version of this paper containing proofs and further properties is [1].

The standardized time series of the process $Y_1, Y_2, \ldots, Y_n$ is

$$T_n(t) \equiv \frac{\lfloor nt \rfloor (\overline{Y}_n - \overline{Y}_{\lfloor nt \rfloor})}{\sigma\sqrt{n}} \text{ for } 0 \le t \le 1,$$

where the variance parameter $\sigma^2 \equiv \lim_{n\to\infty} n \operatorname{Var}(\overline{Y}_n)$, $\overline{Y}_j \equiv \sum_{i=1}^{j} Y_i/j$ for $j = 1, 2, \ldots$, and $\lfloor \cdot \rfloor$ is the greatest integer function. We assume that μ is finite, σ^2 is well-defined, and $0 < \sigma^2 < \infty$. We also assume that

$$T_n \xrightarrow{\mathcal{D}} \mathcal{B},$$

where $\mathcal{B}(\cdot)$ is a standard Brownian bridge process and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution, and that $T_n(\cdot)$ and $n\overline{Y}_n$ are asymptotically independent (cf. Schruben [4] and Glynn and Iglehart [2]). The remainder of the paper is organized as follows. Section 2 describes the new confidence intervals. Analytical and empirical examples appear in Section 3.

2 The New Confidence Intervals

In this section, we give the confidence intervals. Let

$$A_i(n) \equiv \frac{\sum_{k=1}^{n} w_i(\frac{k}{n})\sigma T_n(\frac{k}{n})}{n}$$

and

$$A_i \equiv \int_0^1 w_i(t)\sigma\mathcal{B}(t)\,dt,$$

for continuous weighting functions $w_i(\cdot)$, $i = 1, 2, \ldots$ which are not identically zero. The next theorem says that the vector of $A_i(n)$'s behaves like the vector of the A_i's as n becomes large.

Theorem 1 *If $w_i(\cdot)$ is continuous on $[0,1]$ for $i = 1, \ldots, d$, then as $n \to \infty$*

$$(A_1(n), \ldots, A_d(n)) \xrightarrow{\mathcal{D}} (A_1, \ldots, A_d)\ .$$

The following theorem gives conditions which the weighting functions must satisfy so that $A_1, \ldots, A_d$ are independent, normal $(0, \sigma^2)$ random variables. First, we need a definition.

Condition O The functions $w_1, \ldots, w_d$ satisfy Condition O if they are orthonormal with respect to $r(s,t) \equiv (s \wedge t)[1-(s \vee t)]$ over the unit square where $\wedge$ denotes minimum and $\vee$ denotes maximum.

Theorem 2 *$A_1, \ldots, A_d$ are independent, normal $(0, \sigma^2)$ random variables iff $w_1, \ldots, w_d$ satisfy Condition O.*

This result has only given us conditions which the weighting functions must satisfy. One method of obtaining orthonormal weighting functions is to take any set of linearly independent functions $v_1, \ldots, v_d$ and use the Gram-Schmidt procedure to orthonormalize them.

Example 1 *Suppose we let $v_i(t) = t^{i-1}$, $i = 1, 2, 3, 4$. Note that the v's are linearly independent. Applying the Gram-Schmidt procedure with respect to $r(s,t)$ yields the orthonormal weighting functions*

$$\begin{aligned} w_1(t) &= \sqrt{12} \\ w_2(t) &= \sqrt{720}(t - 1/2) \\ w_3(t) &= \sqrt{25200}(t^2 - t + 1/5) \\ w_4(t) &= 60(14t^3 - 21t^2 + 9t - 1). \end{aligned}$$

Once we have orthonormal weighting functions, we can compute $(A_1(n), \ldots, A_d(n))$ and construct confidence intervals for μ as follows. Since $(A_1, \ldots, A_d)$ is a vector of independent, normal $(0, \sigma^2)$ random variables, we immediately have that $\sum_{i=1}^{d} A_i^2$ is σ^2 times a chi-squared random variable with d degrees of freedom. Since $(\overline{Y}_n - \mu)/\sqrt{\sigma^2/n}$ is asymptotically normal (0,1) and independent of $(A_1, \ldots, A_d)$ (cf. [4]), we know that

$$\frac{\sqrt{n}(\overline{Y}_n - \mu)}{\sqrt{\sum_{i=1}^{d} A_i^2/d}}$$

asymptotically has a t distribution with d degrees of freedom. Hence, an approximate $100(1-\alpha)\%$ confidence interval for μ is given by

$$\overline{Y}_n \pm t_{d,1-\alpha/2}\sqrt{\frac{\sum_{i=1}^{d} A_i^2(n)}{nd}},$$

where $t_{d,1-\alpha/2}$ is the $1-\alpha/2$ quantile from the t distribution with d degrees of freedom. We call

$$V_W \equiv \frac{\sum_{i=1}^{d} A_i^2(n)}{d}$$

the weighted area estimator for σ^2. The area confidence interval estimator from [4] corresponds to our case with $d=1$ and $w_1(t) = \sqrt{12}$ while the weighted area estimator of [3] corresponds to $d=1$.

3 Examples

From the above, it is clear that we would like the weighting functions to satisfy Condition O. In an expanded version of this paper [1], we define another condition for the weights.

Condition U The weighting function w_i is said to satisfy Condition U if $W_i(1) = \overline{W}_i(1) = 0$ where

$$W_i(t) \equiv \int_0^t w_i(s)ds, \text{ and}$$

$$\overline{W}_i(t) \equiv \frac{1}{t}\int_0^t W_i(s)ds.$$

Conditions O and U together yield several desirable properties for V_W. One such property, *first-order unbiased*, is that the bias of V_W as an estimator of σ^2 is of order $o(1/n)$.

Example 2 *The following are two polynomial weighting functions which satisfy Conditions O and U:*

$$w_1(t) = \sqrt{574560}\left(\frac{-2}{57} + \frac{31t}{76} - t^2 + \frac{25t^3}{38}\right)$$

$$w_2(t) = \sqrt{\frac{63000}{19}}\left(\frac{1}{5} - \frac{9t}{10} + t^3\right).$$

For some simple stochastic processes and low-order polynomial weighting functions, it is possible to carry out the algebra to exactly compute the expected value of V_W.

Example 3 *Consider an MA(1) process $Y_{i+1} = \theta\epsilon_i + \epsilon_{i+1}, i = 1,2,\ldots$ where the ϵ's are i.i.d. normal $(0,1)$ random variables. In [1], we calculate the $E[V_W]$ for both of the weighting functions of Example 2. The result is*

$$E[V_W] = (1+\theta)^2 + O(1/n^2) = \sigma^2 + O(1/n^2).$$

Thus, V_W is first-order unbiased as claimed.

Example 4 *We can also give an infinite sequence of weighting functions which satisfy both conditions by letting $w_i(t) = \sqrt{8}\pi i cos(2\pi i t)$, $i = 1,2,\ldots$.*

Now we give preliminary empirical results comparing several methods of confidence interval estimation. Consider the AR(1) process, $Y_{i+1} = \phi Y_i + \epsilon_{i+1}$, $i = 1,2,\ldots$ where the ϵ's are i.i.d. normal $(0, 1-\phi^2)$ random variables and Y_1 is a standard normal random variable. Let $\phi = 0.9$; such a process is stationary. We compared the performance of five confidence interval methods: batch means with 2 batches (V_B), the area estimator from [4] (V_A), and the weighted area estimators from Example 4 using weighting functions $w_1,\ldots,w_d$ with $d = 1,2$, and 3 (V_{W1}, V_{W2}, V_{W3}, resp.). The results are presented in the following two tables. These results are for estimated coverages and expected half-lengths and are based on 2000 independent replications of the appropriate simulation experiments. Coverage of 90% was desired; hence, the standard error of the coverage estimates is about 0.007. The first table shows that V_B and V_{W1} have better coverage for small sample sizes, but all methods attain the desired coverage as n increases. Also note that for very small sample sizes the coverages of the weighted area confidence intervals with $d > 1$ are worse than those of the other methods, but all methods seem to reach the desired coverage at roughly the same sample size.

The second table shows that the expected half-lengths for V_{W2} and V_{W3} are much smaller than those of the other methods. Similar results hold for the variances of the half-lengths.

In another paper [1], we give additional analytical and empirical results for AR(1), MA(1), and M/M/1 processes. In a sense, our weighted area estimator is at the opposite end of the spectrum from those in [3,4]; they have one estimator ($d=1$) with multiple batches, while we have one batch with several estimators ($d > 1$). The "best" procedure most likely lies between the two ends of the spectrum.

n	V_B	V_A	V_{W1}	V_{W2}	V_{W3}
32	.856	.856	.853	.711	.645
64	.888	.870	.884	.805	.760
128	.888	.876	.898	.882	.836
256	.893	.890	.891	.892	.876
512	.891	.904	.903	.901	.903

Table 1: Coverage Results for the AR(1) Example

n	V_B	V_A	V_{W1}	V_{W2}	V_{W3}
32	2.20	2.34	2.23	0.89	0.65
64	2.07	2.05	2.15	0.92	0.68
128	1.73	1.64	1.76	0.84	0.64
256	1.27	1.24	1.33	0.66	0.54
512	0.92	0.94	0.95	0.50	0.41

Table 2: Expected Half-Lengths—AR(1) Example

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ROBERT D. FOLEY is an Associate Professor in the School of ISyE at Georgia Institute of Technology. He received his Ph.D. from The University of Michigan. His research interests are in applied probability.

School of ISyE
Georgia Institute of Technology
Atlanta, GA 30332
(404) 894-2309
rfoley@gtri01.bitnet

DAVID M. GOLDSMAN is an Assistant Professor in the School of ISyE at Georgia Institute of Technology. He received his Ph.D. from Cornell University. His research interests are in simulation output analysis.

School of ISyE
Georgia Institute of Technology
Atlanta, GA 30332
(404) 894-2365
dgoldsma@gtri01.bitnet