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THE MONTE CARLO ESTIMATION OF FUNCTION VARIATION

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INTRODUCTION

Although one customarily uses the data generated in a Monte Carlo experiment with a fixed input to estimate a parameter of the model under study, it has long been known that these same data can provide estimates of how this parameter varies in response to variation in input values which have not been used in the sample experiment. This ability to study functional variation on a single experiment is implicit in the description of importance sampling in Kahn (1950). The principal contribution of the present paper is to describe the application of this concept to the estimation of variation in system reliability as a function of variation in component reliabilities. The proposed method merges another importance sampling technique, described in Fishman (1986), into the function estimation in a way that considerably reduces the variances of the resulting estimators.

1. CHARACTERIZATIONS OF SYSTEM RELIABILITY

Consider the network $G = (\mathcal{V},\mathcal{E})$ with node set $\mathcal{V}$ and arc set $\mathcal{E}$. For convenience of exposition, assume that nodes represent components that function perfectly and that arcs represent components that fail randomly and independently. To characterize G more completely, we define:

r = number of distinct types of components
q_i = probability that a component of type i functions for i=1,...,r
$\mathbf{q} = (q_1,...,q_r)$
$\mathcal{E}_i$ = set of arcs that use components of type i
$(\mathcal{E}_i \cap \mathcal{E}_j = \emptyset \quad i \neq j \quad \mathcal{E} = \bigcup_{i=1}^{r} \mathcal{E}_i)$
$k_i = |\mathcal{E}_i|$ = number of components of type i
$\mathbf{k} = (k_1,...,k_r)$
e_{ij} = jth arc in $\mathcal{E}_i$
$x_{ij} = 1$ if arc e_{ij} functions, = 0 otherwise
$x_i = \sum_{j=1}^{k_i} x_{ij}$ = number of arcs of type i that function
$\mathbf{x} = (x_{11},...,x_{1k_1};x_{21},...,x_{2k_r};...;x_{r1},...,x_{rk_r})$
$\mathcal{X}$ = set of all arc states x

$$P(\mathbf{x},\mathbf{k},\mathbf{q}) = \prod_{i=1}^{r} [q_i^{x_i}(1-q_i)^{k_i-x_i}] \quad \mathbf{x}\in\mathcal{X} \qquad (1)$$
= probability mass function of states in $\mathcal{X}$

$\phi(\mathbf{x}) = 1$ if the system functions,
= 0 otherwise (2)

$g(\mathbf{q}) = \sum_{\mathbf{x}\in\mathcal{X}} \phi(\mathbf{x})\, P(\mathbf{x},\mathbf{k},\mathbf{q})$ = probability that the system functions. (3)

The interpretation of the system reliability $g(\mathbf{q})$ varies with the type of system under study. For a binary–state system with binary–state components, we consider the $\mathcal{T}$–connectedness problem. Let $\mathcal{T}$ denote a subset of $\mathcal{V}$ and let

$\phi(\mathbf{x}) = 1$ if all nodes in $\mathcal{T}$ are connected when state x occurs
$= 0$ otherwise.

Then

$g(\mathbf{q})$ = probability that all nodes in $\mathcal{T}$ are connected. (4)

When $\mathcal{T} = \{s,t\} \subset \mathcal{V}$, this is called the s–t connectedness problem. When $\mathcal{T} = \mathcal{V}$, it is called the all–terminal connectedness problem. These representations are useful in studying the vulnerability of communication systems.

One can also use the network representation to study the reliability of multi–state systems with binary–state components. Suppose G is a directed acyclic flow network with source node s and terminal node t and that each components of type i has flow capacities zero with probability $1-q_i$ and $b_i>0$ with probability q_i. Let

$x_{ij} = 1$ if arc e_{ij} in $\mathcal{E}_i$ has flow capacity $b_i>0$
$= 0$ if arc flow capacity is zero
Γ = set of all minimal s–t cutsets in G

$$\Lambda = \min_{\mathcal{C}\in\Gamma} \sum_{i=1}^{r} \sum_{j\in\mathcal{E}_i\cap\mathcal{C}} b_i x_{ij}$$
= maximal s–t flow in G

$\phi(\mathbf{x},z) = 1$ if $\Lambda > z$
$= 0$ otherwise.

Then

$$g(q) = \sum_{x\in\mathscr{X}} \phi(x,z)\ P(x,k,q) \quad z\geq 0 \qquad (5)$$
$$= \text{probability that the maximal s–t flow } \Lambda \text{ exceeds } z.$$

2. COMPUTATION USING THE MONTE CARLO METHOD

In its simplest form, the problem of system reliability computation concerns the numerical evaluation of g(q) in (3). A related problem concerns the evaluation of the reliability function $\{g(q),\ q\in\mathscr{Q}\}$ where $\mathscr{Q}$ denotes a set of component reliabilities of interest. Unfortunately, the exact computation of g(q), for fixed q in (3) and, more generally, the exact computation of the reliability measures (4) and (5) all belong to the NP–hard class of problems (Valiant 1979), implying that no polynomial time algorithm exists at present for effecting their exact computation. Also, it is known (Ball 1979 and Buzacott 1983) that computing g(q) at all $q\in\mathscr{Q}$ takes $|\mathscr{Q}|$ times as long as computing g(q) at a single point.

To overcome this limitation to exact computation, one can resort to a Monte Carlo sampling experiment wherein one approximates g(q) by an estimate whose error of approximation decreases as the number of independent trials, on which the estimate is based, increases. Many alternative sampling plans exist and, in principle, one prefers a plan that exploits prior knowledge about the system of interest to achieve an error of specified size at less cost than alternative methods allow.

We now describe the sampling plan described in Fishman (1986) that applies with small modifications to the estimation of diverse reliability measures such as (4) and (5). Later, Section 4 extends this method to the estimation of the function $\{g(q),\ q\in\mathscr{Q}\}$.

3. USING PRIOR INFORMATION

Assume that sufficient prior information exists about the system under study to enable one to identify two binary functions $\{\phi_L(x),\ x\in\mathscr{X}\}$ and $\{\phi_U(x),\ x\in\mathscr{X}\}$ with the properties

$$\phi_L(x) \leq \phi(x) \leq \phi_U(x) \qquad (6)$$

where $\phi(x)$ is defined in (2). Let

$$g_i(q) = \sum_{x\in\mathscr{X}} \phi_i(x,k,q) \qquad i=L,U$$

so that

$$g_L(q) \leq g(q) \leq g_U(q). \qquad (7)$$

Now let

$$Q(x,k,q) = \frac{\phi_U(x)-\phi_L(x)}{g_U(q)-g_L(q)} P(x,k,q) \quad x\in\mathscr{X} \qquad (8)$$

which is a probability mass function. Suppose that one can compute $g_L(q)$ and $g_U(q)$ exactly and at reasonable cost as a function of the size of G. Then the ordering relationships (6) and (7) together with the probability mass function $\{Q(x,k,q),\ x\in\mathscr{X}\}$ in (8) allow one to derive benefits in sampling, as described next.

3.1 Importance Sampling

Here one concentrates sampling in the region of the state space $\mathscr{X} = \{x\in\mathscr{X} |\ \phi_L(x)=0 \text{ and } \phi_U(x)=1\}$ as follows:

1. Set S←0.
2. On each of K independent trials:
   2a. Sample x from $\{Q(x,k,q),\ x\in\mathscr{X}\}$.
   2b. Compute $\phi(x)$.
   2c. Set S←S+$\phi(x)$.
3. Compute summary statistics
   3a. $\hat{g}_K(q) = g_L(q) + [g_U(q)-g_L(q)]S/K$.
   3b. $V(\hat{g}_K(q)) = [g_U(q)-\hat{g}_K(q)][\hat{g}_K(q)-g_L(q)]/(K-1)$.

Here $\hat{g}_K(q)$ is an unbiased estimator of g(q) with

$$\text{var}\ \hat{g}_K(q) = [g_U(q)-g(q)][g(q)-g_L(q)]/K \qquad (9)$$

and $V(\hat{g}_K(q))$ is an unbiased estimator of var $\hat{g}_K(q)$.

One way to assess the benefit of this sampling plan is to compare it with the results for a crude Monte Carlo sampling estimate $\bar{g}_K(q)$ of g, using $\{P(x,k,q)\}$. This too is unbiased but with

$$\text{var}\ \bar{g}_K(q) = g(q)[1-g(q)]/K. \qquad (10)$$

The ratio of variances is then

$$\begin{aligned} R &= \text{var}\ \bar{g}_K(q)/\text{var}\ \hat{g}_K(q) \\ &\geq 1/\{g_U(q)-g_L(q) - 2[g_L(q)g_U(q) \\ &\quad \times [1-g_L(q)][1-g_U(q)]]^{\frac{1}{2}}\}. \qquad (11) \\ &\geq 1. \end{aligned}$$

Fishman (1986) gives timing considerations, confidence intervals that hold for finite K and an example that illustrates the technique in practice.

4. FUNCTION ESTIMATION

In addition to estimating the reliability $g(\mathbf{q})$ at a fixed $\mathbf{q}$, the Monte Carlo method provides a way, at small marginal cost, of estimating the reliability function $\{g(\mathbf{q}), \mathbf{q}\in\mathcal{Q}\}$ where $\mathcal{Q}$ is a set of component reliability vectors of interest. Here q_i for a component of type i may be a function of time (i.e. $q_i=F_i(\tau)$ = probability that a component of type i functions at time τ) or it may reflect potential component reliability improvements the effect of which on system reliability is of interest.

Suppose that one elects to perform K independent replications with component reliability $\mathbf{p}$ using $\{Q(\mathbf{x},\mathbf{k},\mathbf{p})\}$ (importance sampling). Let $\mathbf{x}^{(j)}$ denote the sample component state vector on replication j. The estimators

$$\hat{g}_{aK}(\mathbf{q}) = g_L(\mathbf{q}) + [g_U(\mathbf{p})-g_L(\mathbf{p})] \times \frac{1}{K}\sum_{j=1}^{K} \phi(\mathbf{x}^{(j)})\frac{P(\mathbf{x}^{(j)},\mathbf{k},\mathbf{q})}{P(\mathbf{x}^{(j)},\mathbf{k},\mathbf{p})} \qquad (12)$$

and

$$\hat{g}_{bK}(\mathbf{q}) = g_U(\mathbf{q}) - [g_U(\mathbf{p})-g_L(\mathbf{p})] \times \frac{1}{K}\sum_{j=1}^{K} [1-\phi(\mathbf{x}^{(j)})]\frac{P(\mathbf{x}^{(j)},\mathbf{k},\mathbf{q})}{P(\mathbf{x}^{(j)},\mathbf{k},\mathbf{p})} \qquad (13)$$

are unbiased estimators of $g(\mathbf{q})$ with

$$\begin{aligned} Kv_a(\mathbf{q}) &= K \text{ var } \hat{g}_{aK}(\mathbf{q}) \\ &= \{c[g_U(\mathbf{p})-g_L(\mathbf{p})][g(\mathbf{q}^*)-g_L(\mathbf{q}^*)] \\ &\quad - [g(\mathbf{q})-g_L(\mathbf{q})][g_U(\mathbf{q})-g_L(\mathbf{q})]\} \\ &\quad + [g_U(\mathbf{q})-g(\mathbf{q})][g(\mathbf{q})-g_L(\mathbf{q})] \end{aligned} \qquad (14)$$

$$\begin{aligned} Kv_b(\mathbf{q}) &= K \text{ var } \hat{g}_{bK}(\mathbf{q}) \\ &= \{c[g_U(\mathbf{p})-g_L(\mathbf{p})]\,[g(\mathbf{q}^*)-g_U(\mathbf{q}^*)] \\ &\quad - [g_U(\mathbf{q})-g(\mathbf{q})][g_U(\mathbf{q})-g_L(\mathbf{q})]\} \\ &\quad + [g_U(\mathbf{q})-g(\mathbf{q})][g(\mathbf{q})-g_L(\mathbf{q})] \end{aligned} \qquad (15)$$

$$\begin{aligned} Kc_{ab}(\mathbf{q}) &= K \text{ cov}[\hat{g}_{aK}(\mathbf{q}),\hat{g}_{bK}(\mathbf{q})] \\ &= [g_U(\mathbf{q})-g(\mathbf{q})][g(\mathbf{q})-g_L(\mathbf{q})] \end{aligned} \qquad (16)$$

where

$$c = \prod_{i=1}^{r} [q_i^2/p_i+(1-q_i)^2/((1-p_i)]^{k_i},$$

$$\mathbf{q}^* = (q_1^*,\ldots,q_r^*)$$

and

$$q_i^* = 1/[1+p_i(1-q_i)^2/(1-p_i)q_i^2] \qquad i=1,\ldots,r.$$

In principle, (12) and (13) enable one to estimate $g(\mathbf{q})$ for all $\mathbf{q}\in\mathcal{Q}$ from the single set of replications obtained by sampling with component reliabilities $\mathbf{p}$. Since $[g_U(\mathbf{q})-g(\mathbf{q})][g(\mathbf{q})-g_L(\mathbf{q})]$ in (14) and (15) is the variance of the point estimator $\hat{g}_K(\mathbf{q})$ based on sampling with component reliabilities $\mathbf{q}$, the quantities in curly brackets are the incremental changes in variances that result from sampling with $\mathbf{p}$ instead of $\mathbf{q}$. Of most importance, it is entirely possible for one of these quantities to be negative, implying that an estimate of $g(\mathbf{q})$ with smaller variance than $\hat{g}_K(\mathbf{q})$ is possible.

To put this last observation in perspective, consider the estimator

$$\bar{g}_K(\mathbf{q}) = \theta\hat{g}_{aK}(\mathbf{q})+(1-\theta)\hat{g}_{bK}(\mathbf{q}) \quad 0\le\theta\le1 \qquad (17)$$

which achieves minimal variance by choosing

$$\theta^* = \frac{v_b(\mathbf{q})-c_{ab}(\mathbf{q})}{v_a(\mathbf{q})+v_b(\mathbf{q})-2c_{ab}(\mathbf{q})} \qquad (18)$$

Observe that $v_a(\mathbf{q})\le c_{ab}(\mathbf{q})$ implies that $\hat{g}_{aK}(\mathbf{q})$ in (12) has smaller variance than $\hat{g}_K(\mathbf{q})$ in (9) whereas $v_b(\mathbf{q})\le c_{ab}(\mathbf{q})$ implies that $\hat{g}_{bK}(\mathbf{q})$ in (13) has smaller variance than $\hat{g}_K(\mathbf{q})$ in (9).

Experience with this method of estimation indicates that for moderately high component reliabilities $v_b(\mathbf{q})\le c_{ab}(\mathbf{q})$ is often satisfied with substantial reductions in variance at each $\mathbf{q}$ in $\mathcal{Q}$, as compared to the results from importance sampling at a point in Section 3.1. Moreover, all these estimates follow from just one sampling experiment at $\mathbf{p}$. The presentation will also describe a procedure for choosing the $\mathbf{p}$ for sampling optimally, a method of computing confidence intervals and an example.

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George Fishman is professor and chairman of the Department of Operations Research at the University of North Carolina at Chapel Hill. His principal interest is the development of statistical methodology applicable to the analysis of output from discrete event digital simulation models. He is the author of Concepts and Methods in Discrete Event Digital Simulation published by Wiley in 1973 and of Principles of Discrete Event Simulation published by Wiley in 1978. He is a frequent contributor to the operations research and statistical literature on this topic. At present, he is working on variance reducing methods for network reliability estimation and on the influence of concurrent processing on simulation program structure. Professor Fishman is simulation departmental editor for Management Science and is a member of the Operations Research Society of America, the Institute of Management Science and the American Statistical Association.

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