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ESTIMATION PROCEDURES BASED ON CONTROL VARIATES WITH KNOWN COVARIANCE MATRIX

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ABSTRACT

This paper describes a new procedure for using control variates in multiresponse simulation when the covariance matrix of the controls is known. Assuming that the responses and the controls are jointly normal, we develop a new unbiased control-variates point estimator for the mean simulation response. We also compute the covariance matrix of this point estimator in order to construct an approximate confidence-region estimator for the mean response. If the covariances between the responses and the controls are unknown so that the optimal control coefficients must be estimated, then some of the potential efficiency improvement is lost. This loss is quantified in a new variance ratio. We summarize the results of an extensive experimental study in which we apply the proposed estimation procedure to closed queueing networks and stochastic activity networks.

1. INTRODUCTION

In recent years a large research effort has been focused on the control-variates method as a variance reduction technique for Monte Carlo simulation. This is primarily because of its potential for effective use in a wide variety of discrete event simulation models, usually at a computational cost that is negligible relative to the cost of the simulation itself. Lavenberg and Welch (1981) survey the development of this method for the case of a univariate response. Subsequently specialized controls have been developed for queueing networks (Lavenberg, Moeller and Welch 1982, Wilson and Pritsker 1984), stochastic activity networks (Grant and Solberg 1983, Venkatraman and Wilson 1985), and stochastic flow networks (Fishman 1987). Much of the theoretical development has been concentrated on generalizing the control-variates method to a multivariate simulation response with a nontrivial correlation structure. Assuming joint multivariate normality between the responses and the controls, Rubinstein and Marcus (1985) provide a procedure for applying multiple controls to point and confidence-region estimators for the mean of a multivariate response. When the optimal control coefficients must be estimated, some of the potential efficiency improvement is lost. Venkatraman and Wilson (1986) quantify this efficiency loss.

A key assumption in all of the above work is that the dependency structure of the controls is unknown and therefore must be estimated as an intermediate result in the estimation of the optimal control coefficients. However, in a wide variety of applications, the covariance matrix of the controls is known a priori; and it is only reasonable to expect that incorporating this additional information into control-variates estimation procedures would mitigate the potential efficiency loss that occurs when the optimal control coefficients must be estimated. This consideration motivated the study described below.

This paper is organized as follows. In Section 2 we establish the necessary nomenclature and summarize some standard statistical results for the usual situation in which the dependency structure of the controls is unknown. In Section 3 we examine the situation in which the controls have a known covariance matrix, and we exploit this extra information to develop a new unbiased point estimator for the mean simulation response. In Section 3 we also present an approximate confidence region centered at the point estimator, and we derive an appropriate variance ratio that quantifies the efficiency loss due to estimation of the covariance between the responses and the controls. In Section 4 we describe two classes of simulation models used in our experimental study; and in each case, we emphasize the development of control variates with a known covariance structure. Results of the experimental study are presented in Section 5. A summary of the main findings of this research is given in Section 6.

2. STATISTICAL FRAMEWORK

Suppose that we have a column vector $Y = [Y_1, ..., Y_p]'$ of p simulation responses and we seek an efficient estimator of the mean $\mu_Y \equiv E(Y)$ that incorporates a column vector $C = [C_1, ..., C_q]'$ of q control variates having the known mean $\mu_C = E(C)$. By subtracting from Y an appropriate linear transformation of the known deviation $C - \mu_C$, we have the control-variates estimator $Y(a) = Y - a(C - \mu_C)$. The controlled response $Y(a)$ is an unbiased estimator of μ_Y for a fixed $(p \times q)$ matrix of control coefficients a. Let Σ_Y and Σ_C denote the covariance matrices of Y and C respectively, and let Σ_{YC} denote the covariance matrix between Y and C:

$$\Sigma_Y \equiv \mathrm{Cov}(Y) = E[(Y - \mu_Y)(Y - \mu_Y)'],$$

$$\Sigma_C \equiv \mathrm{Cov}(C) = E[(C - \mu_C)(C - \mu_C)'],$$

$$\Sigma_{YC} \equiv \mathrm{Cov}(Y, C) = E[(Y - \mu_Y)(C - \mu_C)'].$$

The generalized variance of $Y(a)$ is minimized by the optimal matrix of control coefficients

$$\beta = \Sigma_{YC} \Sigma_C^{-1}, \tag{1}$$

and the minimum generalized variance is

$$\det\{\mathrm{Cov}[Y(\beta)]\} = \det(\Sigma_Y - \Sigma_{YC}\Sigma_C^{-1}\Sigma_{CY})$$
$$= \det(\Sigma_Y)\left[\prod_{j=1}^{r}(1 - \rho_j^2)\right], \tag{2}$$

where $\Sigma_{CY} = \Sigma_{YC}'$, $r = \mathrm{rank}(\Sigma_{YC})$ and $\{\rho_j: 1 \le j \le r\}$ are the canonical correlations between Y and C.

Since both Σ_C and Σ_{YC} are frequently unknown in practice, β has to be estimated. Let (Y_j, C_j) denote the result observed on the jth independent replication of the simulation for $1 \le j \le n$. In terms of the statistics

$$\bar{Y} = n^{-1}\sum_{j=1}^{n} Y_j, \quad S_Y = (n-1)^{-1}\sum_{j=1}^{n}(Y_j - \bar{Y})(Y_j - \bar{Y})',$$

$$\bar{C} = n^{-1}\sum_{j=1}^{n} C_j, \quad S_C = (n-1)^{-1}\sum_{j=1}^{n}(C_j - \bar{C})(C_j - \bar{C})',$$

$$S_{YC} = (n-1)^{-1}\sum_{j=1}^{n}(Y_j - \bar{Y})(C_j - \bar{C})', \quad S_{CY} = S_{YC}',$$

the sample analog of (1) is

$$\hat{\beta} = S_{YC} S_C^{-1}, \tag{3}$$

and a controlled point estimator of μ_Y is

$$\bar{Y}(\hat{\beta}) = \bar{Y} - \hat{\beta}(\bar{C} - \mu_C).$$

To construct a confidence region for μ_Y, we assume that Y and C have the joint multivariate normal distribution

$$\begin{bmatrix} Y \\ C \end{bmatrix} \sim N_{p+q}\left(\begin{bmatrix} \mu_Y \\ \mu_C \end{bmatrix}, \begin{bmatrix} \Sigma_Y & \Sigma_{YC} \\ \Sigma_{CY} & \Sigma_C \end{bmatrix}\right). \tag{4}$$

Under this assumption $\bar{Y}(\hat{\beta})$ is an unbiased estimator of μ_Y, and an exact $100(1-\alpha)\%$ confidence ellipsoid for μ_Y is given by

$$[\bar{Y}(\hat{\beta}) - \mu_Y]' S_{Y.C}^{-1} [\bar{Y}(\hat{\beta}) - \mu_Y]$$
$$\le \frac{p(n-q-1)}{n-p-q} D \cdot F(1-\alpha; p, n-p-q),$$

where $S_{Y.C} = (n-1)(n-q-1)^{-1}[S_Y - S_{YC}S_C^{-1}S_{CY}]$, $D = n^{-1} + [(n-1)^{-1}(\bar{C} - \mu_C)' S_C^{-1}(\bar{C} - \mu_C)]$, and $F(1-\alpha; m_1, m_2)$ is the $100(1-\alpha)$ percentile of the F-distribution with m_1 and m_2 degrees of freedom. Using the estimated $\hat{\beta}$ in place of the unknown β results in an efficiency loss which is quantified by the *loss factor*

$$\frac{\det\{\mathrm{Cov}[\bar{Y}(\hat{\beta})]\}}{\det\{\mathrm{Cov}[\bar{Y}(\beta)]\}} = \left(\frac{n-2}{n-q-2}\right)^p,$$

so that the net efficiency of the control-variate technique is given by the *variance ratio*

$$\frac{\det\{\mathrm{Cov}[\bar{Y}(\hat{\beta})]\}}{\det\{\mathrm{Cov}(\bar{Y})\}} = \left(\frac{n-2}{n-q-2}\right)^p \left[\prod_{j=1}^{r}(1 - \rho_j^2)\right]$$

(see Venkatraman and Wilson 1986).

3. A NEW CONTROL-VARIATES ESTIMATOR

In Section 2 we used an unbiased estimator S_C of Σ_C to construct $\hat{\beta}$ as an estimator of the optimal control coefficient matrix β. However in a wide variety of real-world applications, we can devise control variables with a known dispersion matrix Σ_C as well as a known mean vector μ_C. Examples of such applications are given in Section 4. In this section we exploit this additional information to develop a new control-variates estimator for μ_Y.

An alternative estimator of β is obtained by replacing S_C in (3) with Σ_C:

$$\ddot{\beta} = S_{YC}\Sigma_C^{-1}.$$

The construction of the new control-variates estimator is immediately apparent:

$$\bar{Y}(\ddot{\beta}) = \bar{Y} - \ddot{\beta}(\bar{C} - \mu_C).$$

With the assumption of joint multivariate normality in (4), Bauer (1987) proved that $\bar{Y}(\ddot{\beta})$ is unbiased for μ_Y:

$$E[\bar{Y}(\ddot{\beta})] = \mu_Y.$$

Furthermore the covariance matrix of $\bar{Y}(\ddot{\beta})$ is given by

$$\ddot{\Sigma} \equiv \mathrm{Cov}[\bar{Y}(\ddot{\beta})] = \frac{n-2}{n(n-1)}\Sigma_{Y \cdot C} + \frac{q+1}{n(n-1)}\Sigma_Y, \quad (5)$$

where $\Sigma_{Y \cdot C} = \Sigma_Y - \Sigma_{YC}\Sigma_C^{-1}\Sigma_{CY}$. Since Σ_{YC} is still estimated by S_{YC}, the minimum generalized variance in (2) is not achieved. The net efficiency of the control-variates procedure using $\bar{Y}(\ddot{\beta})$ is given by the variance ratio

$$\frac{\det\{\mathrm{Cov}[\bar{Y}(\ddot{\beta})]\}}{\det\{\mathrm{Cov}(\bar{Y})\}} = \left(\frac{n+q-1}{n-1}\right)^p \times \left[\prod_{j=1}^{r}\left(1 - \frac{n-2}{n+q-1}\rho_j^2\right)\right].$$

Let $\ddot{S}$ denote an estimate of $\ddot{\Sigma}$, obtained by replacing $\Sigma_{Y \cdot C}$ and Σ_Y in (5) with the corresponding sample quantities $S_{Y \cdot C}$ and S_Y. To construct a confidence ellipsoid for μ_Y, we make the following assumptions: (1) $\bar{Y}(\ddot{\beta}) \sim N_p(\mu_Y, \ddot{\Sigma})$; (2) $(n-q-1)\ddot{S} \sim W_p(n-q-1, \ddot{\Sigma})$, where $W_p(m_1, \ddot{\Sigma})$ is a $(p \times p)$ random matrix having a Wishart distribution with m_1 degrees of freedom and expected matrix $\ddot{\Sigma}$; (3) $\bar{Y}(\ddot{\beta})$ is independent of $\ddot{S}$. An approximate $100(1-\alpha)\%$ confidence ellipsoid for μ_Y is then given by

$$[\bar{Y}(\ddot{\beta}) - \mu_Y]' \ddot{S}^{-1} [\bar{Y}(\ddot{\beta}) - \mu_Y] \le \frac{p(n-q-1)}{n-p-q} F(1-\alpha; p, n-p-q).$$

Bauer (1987) provides more details on the development of this confidence ellipsoid.

4. EXAMPLE APPLICATIONS

In this section we describe two applications, and in each example we devote special attention to the construction of control variables with a known covariance structure.

4.1 Closed Queueing Networks

In the last few years there has been extensive use of queueing simulation models to assess performance characteristics of various kinds of production systems, telecommunications systems, and computer systems. Queueing networks are commonly used to model such systems; and although there is a large class of queueing networks for which steady state results are available, many queueing networks with complex real-world features are analytically intractable. Such systems usually lend themselves well to study via simulation. However, such simulations frequently consume excessive amounts of computing time to yield steady state results with acceptable precision. It is imperative therefore that such simulations are carried out efficiently.

In the first part of our experimental study, we modeled a simple interactive computer system as a closed queueing network. Figure 1 depicts the basic form of the simulated network. A total of M customers circulate indefinitely among the g service centers. Center 1 can be thought of as a room with M computer terminals, so that users never have to wait for service at center 1. Centers 2, 3, ..., g are single-server queues with FIFO service discipline. Center 2 is the CPU while centers 3, ..., g are peripheral devices available to the CPU as I/O devices or secondary storage. Service time at center 1 is a user's 'think' time. A user's request for CPU service may or may not involve a peripheral device. This means that requests leaving center 1 always enter center 2, while requests departing center 2 reach center j with probability $p(j)$, $j = 1, 3, 4, ..., g$. Requests leaving centers 3, 4, ..., g always return to center 2 for further processing.

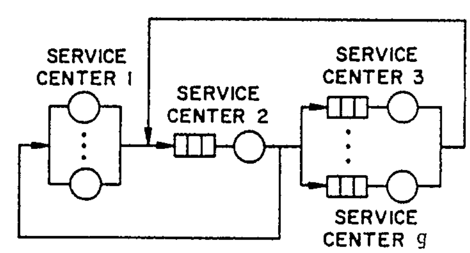

Figure 1. Form of simulated networks with no subnetwork capacity constraint.

A second class of networks that we studied has a subnetwork capacity constraint at the CPU based on the level of multiprogramming allowed in the computer system. Figure 2 represents a network of this class. Here station 2 serves merely as a queue to allow at most M' customers into the subnetwork consisting of the CPU and the peripheral devices, where $M' < M$. In this case requests departing center 3 reach center j with probability $p(j)$, $j = 1, 4, ..., g$ and requests leaving centers 4, ..., g always return to center 3 for further processing.

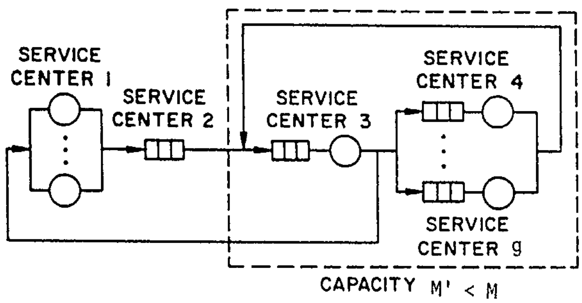

Figure 2. Form of simulated networks with subnetwork capacity constraint.

For both types of networks we assume:

1. Markovian routing so that the next center visited by a customer only depends on his current location;

2. Service times at center j are independent and identically distributed (IID) with mean μ_j and variance σ_j^2;

3. Service time sequences and visitation sequences are mutually independent.

We now develop two sets of control variables both of which have known mean and covariance matrix.

Standardized Work Variables. Suppose the service time process at center j is the IID sequence $\{U_i(j): i \geq 1\}$, $j = 1, ..., g$. Let f_j be the number of service times that are finished at center j in the period $[0, t]$. A standardized work variable for center j is then defined as

$$W_j \equiv \frac{\sqrt{f_j}}{f\,\omega_j} \sum_{i=1}^{f_j} \frac{U_i(j) - \mu_j}{\sigma_j}$$

(Wilson and Pritsker 1984). Here ω_j is the relative frequency with which a customer visits center j and $f = \sum_{j=1}^{g} f_j$. In this case, as the simulation run length increases, $\boldsymbol{W} \equiv [W_1, ..., W_g]'$ converges in distribution to a multivariate normal distribution with mean $\mathbf{0}$ and identity covariance matrix:

$$\boldsymbol{W} \xrightarrow{D} N_g(\mathbf{0}, \boldsymbol{I}_g) \quad \text{as } t \to \infty.$$

Standardized Routing Variables. Noting that all of the routing in the queueing network takes place at the CPU, we define an indicator variable as follows:

$$I_i(j) = \begin{cases} 1 & \text{if the } i\text{th departing request goes to center } j, \\ 0 & \text{otherwise.} \end{cases}$$

A standardized routing variable for center j is then defined as

$$R_j = \sum_{i=1}^{N(t)} \frac{I_i(j) - p(j)}{\{N(t)\,[1-p(j)]\,p(j)\}^{1/2}}, \quad j = 1, ..., g,$$

where $N(t)$ (assumed to be greater than zero) is the number of service requests that depart the CPU in the time interval $[0, t]$. If we let $h = g - 1$, then conditional on the event $\{N(t) > 0\}$, the random vector $\boldsymbol{R} = [R_1, ..., R_h]'$ has expected value $\mathbf{0}$ and nonsingular covariance matrix $\underset{\sim}{\Sigma}_R$, where

$$(\underset{\sim}{\Sigma}_R)_{jk} = \begin{cases} 1 & \text{for } j = k, \\ -\left\{\dfrac{p(j)p(k)}{[1-p(j)][1-p(k)]}\right\}^{1/2} & \text{for } j \neq k. \end{cases}$$

Furthermore, $\boldsymbol{R}$ converges in distribution to a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\underset{\sim}{\Sigma}_R$ as the simulation run length t increases:

$$\boldsymbol{R} \xrightarrow{D} N_h(\mathbf{0}, \underset{\sim}{\Sigma}_R) \quad \text{as } t \to \infty.$$

Finally, it can be proved that $\boldsymbol{W}$ and $\boldsymbol{R}$ are asymptotically independent. Thus the combined vector of standardized work variables and standardized routing variables yields

$$\boldsymbol{C} \equiv \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{R} \end{bmatrix} \xrightarrow{D} N_{g+h}\left\{\mathbf{0}, \begin{bmatrix} \boldsymbol{I}_g & \mathbf{0} \\ \mathbf{0} & \underset{\sim}{\Sigma}_R \end{bmatrix}\right\} \quad \text{as } t \to \infty.$$

See Bauer (1987) for further details.

4.2 Stochastic Activity Networks

In the simulation of a stochastic activity network (SAN), the usual objective is to obtain point and confidence interval estimators of the mean completion time μ_Y of the network. Here we demonstrate the derivation of the known covariance structure for *path control variates* to estimate μ_Y.

A given SAN is completely described by the pair (G, A), where G is the set of all nodes and $A \equiv \{1, 2, ..., c\}$ is the set of all arcs in the network. Corresponding to activity i is the duration A_i with mean μ_i and variance σ_i^2. We assume that the random variables $\{A_i\}$ are mutually independent. Let l denote the number of paths from the source node to the sink node. Corresponding to path j is the arc-set $Q(j) \equiv \{i\colon \text{arc } i \text{ is on path } j\}$, $1 \le j \le l$. The duration of path j is the random variable $D_j \equiv \sum_{i \in Q(j)} A_i$ with mean and variance

$$E(D_j) = \sum_{i \in Q(j)} \mu_i, \quad \mathrm{Var}(D_j) = \sum_{i \in Q(j)} \sigma_i^2, \tag{6}$$

respectively. Note also that for $1 \le j, k \le l$, the covariance between path durations D_j and D_k is

$$\mathrm{Cov}(D_j, D_k) = \sum_{i \in Q(j) \cap Q(k)} \sigma_i^2. \tag{7}$$

The univariate response of interest is the project completion time $Y \equiv \max\{D_1, ..., D_l\}$, and the target estimand is the scalar $\mu_Y = E[Y]$.

Ranking the l complete paths in descending order of expected duration, we choose the first q paths in the list as control paths. The corresponding $(q \times 1)$ random vector C of path durations constitutes our set of path control variates. Note that C has both a known mean $\underset{\sim}{\mu}_C$ and a known dispersion matrix $\underset{\sim}{\Sigma}_C$ with elements computed as shown in (6) and (7).

5. EXPERIMENTAL RESULTS

We conducted an extensive experimental study to evaluate the performance of the two control-variate estimators $\bar{Y}(\hat{\beta})$ (estimator 1) and $\bar{Y}(\ddot{\beta})$ (estimator 2) relative to the direct simulation estimator $\bar{Y}$ (estimator 0). The following two performance measures were used: (a) percentage reduction in the volume of a nominal 90% confidence ellipsoid; and (b) actual coverage probability of a nominal 90% confidence ellipsoid. The experimental procedure involved conducting a *metaexperiment* consisting of m independent simulation experiments, where each experiment involved n independent replications of the basic simulation model. For estimator k, experiment j, let $\hat{V}_j(k)$ be the computed volume and let

$$\hat{P}_j(k) = \begin{cases} 1 & \text{if the computed confidence ellipsoid contains the true mean } \underset{\sim}{\mu}_Y, \\ 0 & \text{otherwise,} \end{cases}$$

for $k = 0, 1, 2,$ and $j = 1, ..., m.$ Across the metaexperiment we computed the averages

$$\hat{V}(k) = (1/m) \sum_{j=1}^{m} \hat{V}_j(k) \quad \text{and} \quad \hat{P}(k) = (1/m) \sum_{j=1}^{m} \hat{P}_j(k).$$

Then the final performance measures for the two control-variates estimators are: (a) the percentage volume reduction $100[\hat{V}(0) - \hat{V}(k)]/\hat{V}(0)$, $k = 1, 2$; and (b) the observed confidence ellipsoid coverage $100\hat{P}(k)$.

5.1 Results for Queueing Networks

In this case the two steady state characteristics estimated were mean response time and CPU utilization, so that $p = 2$. Response time is the elapsed time between the departure of a request from center 1 and its subsequent return to center 1. These characteristics are of interest because the user measures system performance by the mean response time; the system administrator, by CPU utilization. Note that for the first type of queueing network, it is possible to obtain these two quantities analytically (Solberg 1980).

We chose two networks of the first type and two of the second type (with subnetwork capacity constraints) for the experimental study. Various network parameters are presented in Tables 1, 2, and 3. The metaexperiment consisted of $m = 50$ independent simulation experiments, and each experiment involved $n = 20$ replications. At the beginning of each run, all M customers began service at center 1. For type 1 networks the control variables chosen were the four standardized work variables collected at each of the four centers and the three standardized routing variables. For type 2 networks the controls were the three standardized work variables at the CPU and the two busiest disk drives, and four standardized routing variables, excluding routing to less frequented disk drives. Thus $q = 7$ for both types of networks. The simulated time period was 20,000 time units and statistics were cleared at time 2,000 to minimize the effects of initialization bias.

In computing coverage probabilities for the analytically tractable models (systems 1 and 2), we used the exact value of the steady-state mean $\underset{\sim}{\mu}_Y$. To compute coverage probabilities for systems 3 and 4, we estimated μ_Y by $\bar{\bar{Y}}$, the grand mean of the response vector taken across all 1,000 replications of the simulation model. Results of this study are presented in Tables 4 and 5. Note that in system 2, the loss of coverage is much worse for the conventional control-variates estimator $\bar{Y}(\hat{\beta})$ than for the new estimator $\bar{Y}(\ddot{\beta})$.

Table 1: Queueing Network Configurations			
Network	Number of Customers M	Subnetwork Capacity M'	Number of Stations
1	25	25	4
2	15	15	4
3	25	5	7
4	25	10	7

Table 2: Mean Service Times for Queueing Networks							
Network	Service Center						
	1	2	3	4	5	6	7
1	100	1	0.694	6.25	—	—	—
2	100	1	2.78	25	—	—	—
3	100	—	1	2.78	2.78	25	25
4	100	—	1	2.78	2.78	25	25

Table 3: Branching Probabilities for Queueing Networks							
Network	Probability of Branching from CPU to Center j						
	1	2	3	4	5	6	7
1	0.2	0	0.72	0.08	—	—	—
2	0.2	0	0.72	0.08	—	—	—
3	0.2	0	0	0.36	0.36	0.04	0.04
4	0.2	0	0	0.36	0.36	0.04	0.04

Table 4: Percentage Reductions in Volume for Nominal 90% Confidence Regions for the Queueing Network Simulations		
Network	Volume Reduction	
	$\bar{Y}(\hat{\beta})$	$\bar{Y}(\ddot{\beta})$
1	73	45
2	83	52
3	61	43
4	46	34

Table 5: Coverage Probabilities for Nominal 90% Confidence Regions for the Queueing Network Simulations				
Network	With Respect to μ_Y		With Respect to $\bar{\bar{Y}}$	
	$\bar{Y}(\hat{\beta})$	$\bar{Y}(\ddot{\beta})$	$\bar{Y}(\hat{\beta})$	$\bar{Y}(\ddot{\beta})$
1	78	80	86	86
2	28	64	78	80
3	—	—	82	90
4	—	—	83	84

5.2 Results for Stochastic Activity Networks

This part of the experimental study involved the simulation of a set of five SANs in which the following characteristics were systematically varied: (a) the size of the network (the number of nodes and activities); (b) the topology of the network; and (c) the relative dominance (criticality index) of the critical path. *Relative dominance* of a path is defined as the probability that the path is critical in a single realization of the network. For each network, we manipulated the duration of activities on the path with longest expected duration to achieve the following three levels of relative dominance for that path: {50%–60%, 70%–80%, 90%–100%}. In each network 50% of all activities, randomly selected, had exponentially distributed durations; the other activity durations were normally distributed with standard deviation equal to 25% of the mean. Table 6 shows the range in the number of nodes and activities for the five chosen networks. The quantity estimated was the mean completion time μ_Y so that $p = 1$.

Table 6: Stochastic Activity Network Configurations			
Network	Nodes	Activities	Reference†
1	10	14	E, p. 275
2	23	42	A, p. 190
3	30	49	E, p. 318
4	41	56	E, p. 218
5	51	65	M, p. 324

†Note: E = Elmaghraby (1977),
A = Antill and Woodhead (1982), and
M = McKenney and Rosenbloom (1969).

For a given level of relative dominance, we conducted a metaexperiment composed of $m = 32$ independent simulation experiments. Each experiment involved $n = 32$ replications with $q = 3$ controls. In this case the performance measures were: (a) percentage reduction in half-length of a nominal 90% confidence interval; and (b) actual coverage probability of a nominal 90% confidence interval. In computing coverages, we took the overall mean response $\bar{\bar{Y}}$ across the 1,024 replications as the true value of μ_Y.

For the controlled confidence-region estimation procedures, Tables 7 and 8 respectively show the percentage reductions in half-length and the actual coverage probabilities as a function of relative dominance of the path with the largest mean. The percentage reduction in half-length generally increased with increasing levels of dominance. This is to be expected: if the path with the largest mean tends to be the critical path with greater frequency, then the correlation between the overall completion time and the corresponding path control variable becomes progressively larger. The estimator $\bar{Y}(\hat{\beta})$ yields the largest half-length reductions; however, it fails to achieve nominal coverage in several cases. The estimator $\bar{Y}(\ddot{\beta})$, which incorporates the known covariance matrix of controls, yields significant half-length reductions. More importantly, $\bar{Y}(\ddot{\beta})$ is very robust in that it achieves nominal coverage. The estimator $\bar{Y}(\hat{\beta})$ based on the estimated covariance matrix of controls S_C seems to be more sensitive to departures from normality than the estimator $\bar{Y}(\ddot{\beta})$ based on the known matrix Σ_C.

Table 7: Percentage Reductions in Half-Length for Nominal 90% Confidence Intervals for the Stochastic Activity Networks

Model	Relative Dominance	$\bar{Y}(\hat{\beta})$	$\bar{Y}(\ddot{\beta})$
1	50%–60%	49.6	40.2
	70%–80%	60.3	46.6
	90%–100%	82.3	54.3
2	50%–60%	37.2	32.4
	70%–80%	50.4	38.6
	90%–100%	69.6	50.4
3	50%–60%	69.3	53.0
	70%–80%	75.1	56.3
	90%–100%	95.4	63.1
4	50%–60%	34.7	28.2
	70%–80%	65.4	50.8
	90%–100%	90.3	62.2
5	50%–60%	43.5	36.0
	70%–80%	53.5	43.1
	90%–100%	80.3	58.9

Table 8: Coverage Probabilities for Nominal 90% Confidence Intervals for the Stochastic Activity Networks

Model	Relative Dominance	$\bar{Y}$	$\bar{Y}(\hat{\beta})$	$\bar{Y}(\ddot{\beta})$
1	50%–60%	78.1	81.2	90.6
	70%–80%	87.5	81.2	93.8
	90%–100%	84.4	31.3*	84.4
2	50%–60%	87.5	84.4	81.2
	70%–80%	87.5	81.2	84.4
	90%–100%	84.4	68.8*	87.5
3	50%–60%	87.5	68.8*	84.4
	70%–80%	87.5	75.0*	87.5
	90%–100%	87.5	65.6*	90.6
4	50%–60%	84.4	84.4	90.6
	70%–80%	84.4	93.7	87.5
	90%–100%	87.5	31.3*	87.5
5	50%–60%	78.1	81.2	75.0*
	70%–80%	90.6	68.7*	78.1
	90%–100%	87.5	18.8*	84.4

*Significant coverage degradation at the 5% significance level.

6. CONCLUSIONS

The control-variates estimation procedures described in this paper provide an unbiased point estimator and an approximate confidence-region estimator for the mean of a multivariate normal simulation response. In comparison to the standard confidence-region estimator based on direct simulation, the approximation proposed in this paper appears to yield a substantial volume reduction with no significant loss of coverage probability. In comparison to the usual controlled confidence-region estimator, the proposed procedure appears to yield a confidence region which is somewhat larger but which has substantially more reliable coverage properties. The proposed confidence-region estimator seems to be more robust against departures from normality that the usual controlled estimator.

We are currently attempting to develop a more refined estimator of the covariance matrix $\text{Cov}[\bar{Y}(\ddot{\beta})]$. In connection with this work, we are also seeking to develop an improved confidence-region estimator for μ_Y centered at $\bar{Y}(\ddot{\beta})$. We hope that this line of investigation will yield results on the method of control variates that are valuable from both a theoretical and practical standpoint.

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