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SOME METHODS FOR SIMULATING RANDOM FIELDS

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ABSTRACT

Several methods exist for simulating random fields. This paper reviews some of the methods used to simulate stationary random fields in R^n, $n \geq 1$. A recent study comparing the traditional turning bands method to a method called the random impact method is given. The results of this study indicate that the random impact model is comparable to the turning bands method in terms of execution time, and in terms of reproducibility of the covariance functions.

1. INTRODUCTION

Stationary random fields (also called stationary stochastic processes) occur naturally in several areas of science, and the theoretical study of stationary processes has been an active field of research for several years. In the fields of geostatistics, hydrology, mining and others, several methods exist for simulating such processes. One traditional method for simulating stationary processes is called the turning bands method. In this method simulations in higher-dimensional space (usually three dimensions) are reduced to several simulations in one-space. We shall give a brief outline of the method here; greater detail is given in, for example, Journel and Huijbregts (1975).

Another method for simulating stationary random fields is called the random impact method. This method differs sharply from the turning bands method, and bears close resemblance to the method of generating coverage models; see Ahuja and Schacter (1983). In this method spheres of random radii are arranged at random in the plane, and points in the sphere are assigned random values according to the desired stationary distribution.

Comparisons between the turning bands method and the random impact method are made with respect to several factors, and the results of a recent computer study are discussed. In addition, some other methods for simulating stationary processes are mentioned.

2. STATIONARY STOCHASTIC PROCESSES: DEFINITIONS AND PROPERTIES

There are two types of stationary stochastic processes:

2.1. Second order (or wide sense) stationary processes:

The process $\{X(x), x \epsilon R^n\}$ is second order stationary if for all $x, h \epsilon R^n$,

$$E|X(x)|^2 < \infty, EX(x) \equiv m, \text{ and}$$

$$Cov(X(x), X(x+h)) = EX(x)X(x+h) - m^2 = c(h),$$

i.e. is independent of x.

In the one-dimensional case such a process has a spectral representation of the form

$$X(x) = \int_{-\infty}^{\infty} e^{i\lambda x} d\zeta(\lambda), \text{where } \zeta \text{ is a process}$$

with orthogonal increments, and the covariance function $C(h)$ is given by

$$C(h) = \int_{-\infty}^{\infty} e^{i\lambda x} dF(\lambda), \text{ where} F \text{ is a}$$

function of λ only.

The spectral representation above is an L^2 (second order) representation; see Doob (1953).

2.2. Strictly stationary processes:

The process $\{X(x), x \epsilon R^n\}$ is strictly stationary if for all $k \geq 1$ and points $x_1, \ldots, x_k$, $h \epsilon R^n$ the distribution of

$$(X(x_1), \ldots, X(x_k))$$

is the same as that of $(X(x_1 + h), \ldots, X(x_k + h))$. It is easy to see that if $\{X(x), x \epsilon R^n\}$ is strictly stationary with a finite second moment $E|X(x)|^2 < \infty$, then it is also second-order stationary. The converse is not necessarily true, but in the important case where the process is Gaussian (normal) it is true, since the joint distributions of a Gaussian process are completely determined by the first two moments of the process.

Further properties of stationary processes are given in Doob (1953) and Yaglom (1962).

3. SIMULATION OF STOCHASTIC PROCESSES

Some phenomena represented by functions defined in 2, 3, or $n > 3$ dimensions are too complex or insufficiently well understood to be given in analytic form. Such difficulties are well-known in geosciences, hydrology, atmospheric physics and soil physics, to name only a few areas. In such cases there may be functional relationships known between variables at some scales of observation, but in general it is only possible to collect data at a finite number of spatial locations. Often the data collected is a sample of only one realization. Simulation of the random function can provide additional realizations. For example, if rainfall is recorded at a number of gauges for a given event in a region where rain if an infrequent event, it may be desired to predict the best locations for catchments for the next event. If it is assumed that rainfall at a location is represented by a random spatial process with a known covariance then planning requires examining a number of realizations of that random function. The relevance of simulated realizations for planning has been recognized in the exploitation of one deposits, the management of aquifers, and the use of low sulfur coal purchase contracts in conjunction with mine mouth power plants in order to meet air pollution requirements. A number of algorithms have been proposed and utilized, but several questions remain. In particular, there does not exist a coherent set of standards for comparing existing algorithms.

In simulating stationary stochastic processes, it is often the case in practice that the first two moments of the process (the mean function and covariance function) are all that are known or to be estimated. In this case, a second order stationary process will be simulated. However, for both theoretical reasons and ease of simulation, the process is often assumed to be Gaussian.

Several procedures exist for simulating one-dimensional realizations. Extensions to multi-dimensional space are usually inextricable or costly in terms of computer time. In the following we discuss two procedures for simulating multi-dimensional second order stationary processes.

3.1. Turning Bands Method

The turning bands method is a method which reduces n-dimensional simulations (usually $n = 3$) to several one-dimensional simulations, which are rotated in R^n. The one-dimensional simulations use a moving average technique. A brief description of the method is given here. For more details see Journel and Huijbregts (1978).

Given $Y(t)$, a second order stationary process on $-\infty < t < \infty$, s a unit vector in R^n, and x a point in R^n. Without loss of generality, assume $EY(t) = 0$.

If we set $X_s(x) = Y(\langle x, s\rangle)$, where $\langle x, s\rangle$ is the inner product of x and s, it is easy to see that X_s is a second order stationary, zero mean process, with an n-dimensional covariance

$$E\{X_s(x)X_s(x+h)\} = E\{Y(\langle x,s\rangle)Y(\langle x+h,s\rangle)\} = C^{(1)}(h).$$

Now consider unit vectors $s_1, s_2, \ldots, s_k$ uniformly distributed on the unit sphere. This gives processes $X_{s_1}(x), \ldots, X_{s_k}(x)$. If we set $X_k(x) = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_{s_i}(x)$, we see that $X_k(x)$ is a zero mean, second order stationary process. Setting $E\{X_k(x)X_k(x+h)\} = C(h)$, we get $C(h) \underset{k\to\infty}{\longrightarrow} C(r) = \frac{1}{2\pi}\int C^{(1)}(\langle h,s\rangle)dP(s)$, where P is a probability measure on the unit sphere which is invariant under rotations, and $r = |h|$. In the case of three dimensions, it follows easily that $C(r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} C^{(1)}(|r\cos\phi|)\sin\phi d\phi = \frac{1}{r}\int_0^r C^{(1)}(t)dt$, or $C^{(1)}(r) = \frac{\partial}{\partial r} rC(r)$.

In practice, one is given $C(r)$ and wishes to use $C^{(1)}(r)$ in the simulation procedure. The above relation shows that this is easily done.

Summarizing: Given the covariance function $C(r)$, one sets $C^{(1)}(r) = \frac{\partial}{\partial r} rC(r)$, simulates processes $X_{s_1}(x), \ldots, X_{s_k}(x)$ as described above, and sets $X_k(x) = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_{s_i}(x)$. The simulated process $X_k(x)$ then serves as the desired realization of a stationary process with covariance $C(r)$. In practice k, the number of turning band "lines" used, is often set to 15, due to geometric considerations. In this case the differences between $C(h)$ and $C(r)$ will be small, and can be theoretically corrected. For details, see Journel and Huijbregts (1978).

In practice, the following are often used:

1. Spherical Model

$$C(r) = \begin{cases} K[1 - \frac{3}{2}\frac{r}{a} + \frac{1}{2}\frac{r^3}{a^3}], & 0 \le r \le a \\ 0, & r > a \end{cases}$$

$$C^{(1)}(r) = \begin{cases} K[1 - \frac{3r}{a} + \frac{2r^3}{a^3}], & 0 \le r \le a \\ 0, & r > a \end{cases}$$

2. Exponential Model

$$C(r) = Ke^{-\lambda r}, r \ge 0$$

$$C^{(1)}(r) = K[1 - \lambda r]e^{-\lambda r}, r \ge 0$$

3. Gaussian Model

$$C(r) = Ke^{-r^2/a^2}, r \ge 0$$

$$C^{(1)}(r) = K[1 - \frac{2r^2}{a^2}]e^{-r^2/a^2}, r \ge 0$$

When the given process is defined in two-dimensional space, the problem becomes more difficult. In this case there is not such a simple relationship between the two-dimensional and three-dimensional covariances. Some alternatives have been considered. Mejia and Rodriguez-Iturbe (1974) and Rodriquez-Iturbe and Mejia (1974) utilized the radial spectral density and a form of a random cosine series to construct the simulation.

Each of these techniques is only intended to reproduce the first two moments and perhaps the stationary distribution.

Some of the advantages and disadvantages of the tuning bands method are the following:

1. It is fairly easy to understand

2. It is readily available. Software packages are commercially available, and Journel and Huijbregts, (1978) and Carr and Myers (1985) contain listings of computer programs.

3. The turning bands method does not extend easily to higher dimensions.

4. Since the simulated process is the sum of several independent simulated processes, it is approximately normally distributed. Hence it is difficult to use this method to simulate non-normal processes.

Mantoglou (1987) extends the turning bands method to simulate multivariate and anisotropic two-and three-dimensional stochastic processes. Current research by the author and others using a different type of extension of the turning bands methods is being conducted. In this version, both the multivariate and non-isotropic cases are being considered.

3.2. Random Impact Model

The random impact model has its roots in the area of geometrical probability. Several authors have studied similar models in various settings, for example see Ahuja and Schacter (1983), and especially Dagan (1981).

One use of the model was to utilize permeability duties for simulating leakage paths of material through porous rocks; see Silliman (1985). That study led to the present research being conducted by the author and others.

Here we give an illustrative example in the simple case where the $X(i)$'s form a stationary process on the positive integers. Consider $X(i)$, a second order stationary process, with distribution function F, covariance function $C(h)$, and assume $C(h) > 0$ with $C(h) \downarrow 0$ as $h \to \infty$. We also assume $C(h) = 0$ for $h > h_0$. In order to simulate $X(1), \ldots, X(N)$ the following are generated:

1. π: a random permutation of $-h_0 + 1, \ldots, N$.

2. $Y(i), i = -h_0+1, \ldots, N$: independent random variables having the distribution F.

3. $\beta(i), i = -h_0 + 1, \ldots, N$: independent uniform $(0,1)$ random variables.

4. $X^*(i), i = -h_0 + 1, \ldots, N$: random variables which will be assigned values according to the following procedure:

At the ith stage of the simulation procedure, the value $\beta(i)$ is used in conjunction with a table to determine a "block size" h, and those random variables $X^*(\pi(i)), \ldots, X^*(\pi(i) + h - 1)$ which have not previously been assigned values are given the value $Y(i)$. Once $X^*(-h_0 + 1), \ldots, X^*(N)$ have been assigned values, $X(i)$ is set equal to $X^*(i), i = 1, \ldots, N$. The stationary distribution is preserved by the $Y(i)$. The given covariance $C(h)$ determines the table used by the $\beta(i), i = -h_0 + 1, \ldots, N$.

In this example, $C(h) = 0, h > 3$, and it is desired to simulate $X(1), X(2), \ldots, X(5)$.

Suppose the following was obtained:

Stage	π	$\beta(i)$	$Y(i)$
1	0	.4372	.9432
2	2	.7132	.2143
3	-1	.9774	.5111
4	3	.7983	.2222
5	-2	.5137	.7745
6	4	.8001	.8188
7	1	.1162	.3736
8	5	.3719	.6817

Also assume the table giving block sizes for given value of β was as follows:

$\beta(i)$	Block Size
0 - .50	1
.50 - .70	2
.70 - .80	3
.80 - 1.00	4

During stage one, $\pi = 0$ and $Y(1) = .9432$ yield $X^*(0) = .9432$, and $\beta = .4372$ gives a block size of 1. Hence at the end of stage one $X^*(0) = .9432$. During stage two, $\pi = 2$ and $Y(2) = .2143$ gives $X^*(2) = .2143$. Also, $\beta = .7132$ gives a block size of 3. Hence at the end of stage 2, $X^*(2) = X^*(3) = X^*(4) = .2143$.

During stage three, $\pi = -1$ and $Y(3) = .5111$ gives $X^*(-1) = .5111$. Since $\beta = .9774$, the block size is 4. Hence one will set $X^*(i) = .5111$ for each $i, -1 \le i \le 2$ for which $X^*(i)$ has not previously been assigned a value. Since $X^*(0) = .9432$, from stage 1, and $X^*(2) = .2143$, from stage 2, the following will be produced:

$$X^*(-1) = X^*(1) = .5111.$$

Continue in this fashion until $X^*(i)$, $i = -2, \ldots, 5$ have been assigned values. Then set $X(i) = X^*(i)$, $i = 1, \ldots, 5$. Thus in this example, $X(1) = .5111$, $X(2) = X(3) = X(4) = .2143$, $X(5) = .2222$. It is easy to see that $Cov(X(i), X(j)) =$

$$P[X^*(i) = X^*(j)]C(0) + P[X^*(i) \neq X^*(j)]0 = C(0)P[X^*(i) = X^*(j)].$$

The fact that the $X(i)$ process is strictly stationary and has the stationary distribution F is easy to show. The joint distributions of the $X(i)$ are also easy to calculate. For example.

$$P[X(i) \le x_i, X(j) \le x_j] = F(\min(x_i, x_j))P[X^*(i) = X^*(j)] + F(x_i)F(x_j)P[X^*(i) \neq X^*(j)].$$

Given $C(h)$, one must construct a table for block sizes that will reproduce the covariance $C(h)$, i.e. we need $Cov(X(i), X(j)) = C(|j - i|)$. In many practical situations this is easy to do, but it cannot be done in general.

This method is being extended to a very general case in Newman and Wright (1987) and is called the random impact model. This extended model simulates wide-sense stationary processes in any dimension, and with any stationary distribution for a wide class of covariance functions $C(h)$. The following is a description of this model.

Given a stationary covariance function $\sigma(h)$ and stationary distribution F, the process may be defined in two parts: first the n-dimensional space is randomly partitioned into regions V_α in a manner determined by the desired covariance as will be described below. Second, $Z(x)$ is defined by taking independent, identically distributed random variables W_α with distribution F, and assigning W_α to each x in V_α.

To give a picturesque description of the random partition of space, imagine that at random times T_α, explosive projectiles land in space at random locations x_α, and irradiate all points with a random distance R_α of the impact center x_α. The region V_α associated with the impact at x_α is the subset of the affected region which was not affected by any previous impact.

Mathematically, the (x_α, T_α)'s form a stationary point process in $R^n \times [0, \infty)$ and the R_α's are independent, identically distributed, positive valued random variables with common distribution G. If we define S_α to be the sphere of radius R_α centered at x_α, then V_α is the subset of S_α not contained in any $S_{\alpha'}$ with $T_{\alpha'} < T_\alpha$.

Again, it is easy to see that the $Z(x)$ process thus constructed is a strictly stationary process having the desired stationary distribution F. The following result, which is proved in Newman and Wright (1987) gives sufficient conditions on the covariance function $\sigma(h)$ to produce a distribution G as described above:

Theorem: Given a positive isotropic covariance function $C(r)$ defined on R^3, with $\lim_{r\to\infty} \sigma(r) = 0$. To produce a distribution G that will enable the random impact model to simulate a stationary random process with covariance $\sigma(h)$ it is sufficient that the following hold:

$$G_c(u) = \frac{1}{u}\frac{d^2}{du^2}\left[\frac{\sigma(2u)}{\sigma(0) + \sigma(2u)}\right] \text{ is defined}$$

with $G_c(u)$ non-negative, decreasing, and $G_c(0) < \infty$, $G_c(\infty) = 0$.

It is easy to verify that this condition is met in the exponential case, for which $\sigma(r) = Ke^{-\lambda r}$.

In the case where R is a constant random variable, $n = 2$, and F is a discrete random variable, the random impact model is a special case of the coverage models used in image processing; see Ahuja and Schacter (1983). Advantages and disadvantages of the random impact model are the following:

1. The method is extremely easy to understand.

2. It works for any $n = 1, 2, \ldots$

3. It can simulate values at all points in any given finite region.

4. The process simulated is strictly stationary, but its joint distributions are rather unusual. This could be an advantage or disadvantage, depending on the particular physical aspects of the process.

5. At the present time, in order to simulate values of the random variables $Z(x_1), \ldots, Z(x_k)$ the process must continue generating spheres S_α until all the points $x_1, \ldots, x_k$ are covered. This can be time-consuming, and depends on the distribution G.

3.3. A Computer Study Comparing the Two Methods

A recent study was conducted to compare the two methods; see Silliman and Wright (1987). In this study discrete three-dimensional versions of the turning bands and random impact methods were used. The three-dimensional grids varied in size from $5 \times 5 \times 5$ to $20 \times 20 \times 10$. The covariance simulated was an exponential covariance with parameter $\lambda = 3.3$.

The turning bands code used may be found in Thompson, Abubou and Gelhar (1987). The number of turning band lines used by this code varied between 10 and 100. The computer used was a Prime 9955.

The following table gives some of the results of that study in the case where the number of lines used was 15.

Table 1: Comparative CPU Time in Seconds per Simulation Point

Grid Size	Random Impact	Turning Bands
$5 \times 5 \times 5$	0.0013	0.0457
$10 \times 10 \times 10$	0.0011	0.0108
$20 \times 20 \times 10$	0.0017	0.0047

As can be seen from Table 1, the random impact method was faster for smaller grid sizes. But the simulation time per simulation point is independent of grid size for the random impact method, and descreases with grid size for the turning bands method. This implies that for large grids, the turning bands method may be faster. As would be expected, the CPU time per simulation point increases as the number of lines increases.

In all cases there was no significant difference between the average covariance obtained after 100 simulations for the two methods.

3.4. Other Simulation Methods

There are several other methods for simulating random fields. Mejia and Rodriguez-Iturbe (1974) discusses the spectral method, Smith and Freeze (1979) utilize a nearest-neighbor approved, Miles (1969) and (1971) discusses Poisson flats, and Davis (1987) uses an LU decomposition of the covariance matrix. A very good study of the turning bands method is given in Mantoglou and Wilson (1982).

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