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USING PATH CONTROL VARIATES IN ACTIVITY NETWORK SIMULATION

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ABSTRACT

In the simulation of a stochastic activity network (SAN), the usual objective is to obtain point and confidence-interval estimators of the mean completion time for the network. This paper presents a new procedure for using path control variates to improve the efficiency of such estimators. Because each path control is the duration of an associated path in the network, the vector of selected path controls has both a known mean and a known covariance matrix. All of this information is incorporated into point- and interval-estimation procedures for both normal and nonnormal responses. To evaluate the performance of these procedures experimentally, we compare actual versus predicted reductions in point-estimator variance and confidence-interval half-length for a set of SANs in which the following characteristics are systematically varied: (a) the size of the network (number of nodes and activities); (b) the topology of the network; (c) the relative dominance (criticality index) of the critical path; and (d) the percentage of activities with exponentially distributed durations. The experimental results indicate that large variance reductions can be achieved with these estimation procedures in a wide variety of networks.

1. INTRODUCTION

Stochastic activity networks are an important class of simulation models, widely used by corporate management in the scheduling of large projects. Although several approaches for analyzing such networks have been proposed, Monte Carlo simulation frequently is the only feasible analysis technique. Two major reasons for this are: (a) few simplifying assumptions have to be made; and (b) simulation is usually straightforward and, hence, appealing to the practitioner.

Discrete-event simulation of a SAN requires the completion of exactly s events, where s is the number of activities in the SAN. In this context, Monte Carlo simulation of SANs is relatively inexpensive as compared to, for example, the simulation of queueing networks which are notorious for their tremendous computing costs [1]. However, to achieve acceptable precision in estimators based on direct simulation, we typically require a large number of replications of the model. Computing costs can then become prohibitive.

Several variance reduction techniques (VRTs) have been proposed for improving the efficiency of activity network simulations ([2],[3]). Recent work has focused on the *control variate technique* because of its demonstrated potential for effective use in a wide variety of discrete-event simulation models. See [1] and [4] for recent developments concerning this method. The control variate technique is one of the few VRTs that does not require any modification of the structure or operation of the simulation model. Instead it derives its efficiency gains from ancillary information provided by concomitant system variables observed during the course of the simulation. It then employs well-developed regression methods to deliver alternative estimators of system parameters. Consequently it is easier to understand and implement effectively.

2. STATISTICAL FRAMEWORK

Let Y be the random variable representing the project completion time in a SAN. We seek an unbiased estimator for the target parameter $\theta = E(Y)$. *Direct simulation* simply computes the sample mean response $\overline{Y}_n$ from n independent replications of the network to yield an unbiased estimator of θ with $\mathrm{Var}(\overline{Y}_n) = n^{-1}\,\mathrm{Var}(Y)$. Our objective is to derive an alternative estimator $\hat{\theta}_n$ with $E(\hat{\theta}_n) = \theta$ and $\mathrm{Var}(\hat{\theta}_n) < \mathrm{Var}(\overline{Y}_n)$.

To construct a controlled estimator for θ, we must identify a $(q\times1)$ vector of control variables $\mathbf{C}$, having both a known mean μ_C and a strong linear association with Y. In essence we try to predict and counteract the unknown deviation $Y-\theta$ by subtracting from Y an appropriate linear transformation of the known deviations $\mathbf{C}-\mu_C$:

$$Y(\mathbf{b}) = Y - \mathbf{b}'(\mathbf{C} - \mu_C).$$

The controlled response $Y(\mathbf{b})$ is unbiased for any fixed $(q\times1)$ vector of control coefficients. Let

$$\mathrm{Var}(Y) = \sigma_Y^2 = E[(Y-\theta)^2],$$

$$\mathrm{Var}(\mathbf{C}) = \Sigma_C = E[(\mathbf{C}-\mu_C)(\mathbf{C}-\mu_C)'],$$

and

$$\mathrm{Cov}(Y, \mathbf{C}) = \sigma_{YC} = E[(Y-\theta)(\mathbf{C}-\mu_C)].$$

The variance of the controlled response

$$\mathrm{Var}[Y(\mathbf{b})] = \sigma_Y^2 - 2\mathbf{b}'\sigma_{YC} + \mathbf{b}'\Sigma_C\mathbf{b}$$

is minimized by the vector of control coefficients

$$\beta = \Sigma_C^{-1}\,\sigma_{YC}, \tag{1}$$

yielding the minimum variance

$$\mathrm{Var}[\mathrm{Y}(\beta)] = \sigma_Y^2 (1 - R_{YC}^2), \tag{2}$$

where R_{YC} is the multiple correlation coefficient between Y and **C**.

In practice β must be estimated because generally both Σ_C and σ_{YC} will be unknown. Let $\{(Y_j, \mathbf{C}_j): 1 \le j \le n\}$ denote the results observed on n independent replications of the simulation. In terms of the statistics

$$\overline{Y}_n = n^{-1}\sum_{j=1}^{n} Y_j, \quad \overline{\mathbf{C}}_n = n^{-1}\sum_{j=1}^{n} \mathbf{C}_j, \tag{3}$$

$$S_Y^2 = (n-1)^{-1}\sum_{j=1}^{n}(Y_j - \overline{Y}_n)^2, \tag{4}$$

$$\mathbf{S}_C = (n-1)^{-1}\sum_{j=1}^{n} (\mathbf{C}_j - \overline{\mathbf{C}}_n)(\mathbf{C}_j - \overline{\mathbf{C}}_n)', \tag{5}$$

and

$$\mathbf{S}_{YC} = (n-1)^{-1}\sum_{j=1}^{n}(Y_j - \overline{Y}_n)(\mathbf{C}_j - \overline{\mathbf{C}}_n), \tag{6}$$

the sample analog of (1) is

$$\hat{\beta} = \mathbf{S}_C^{-1}\,\mathbf{S}_{YC}. \tag{7}$$

Thus a point estimator of θ is

$$\overline{Y}(\hat{\beta}) = \overline{Y}_n - \hat{\beta}'(\overline{\mathbf{C}}_n - \mu_C). \tag{8}$$

2.1 Analysis Techniques for Normal Outputs

To construct confidence intervals for θ we now consider two situations. First we assume that Y and **C** have the joint multivariate normal distribution

$$\begin{bmatrix} Y \\ \mathbf{C} \end{bmatrix} \sim N_{q+1}\left(\begin{bmatrix} \theta \\ \mu_C \end{bmatrix}, \begin{bmatrix} \sigma_Y^2 & \sigma_{YC}' \\ \sigma_{YC} & \Sigma_C \end{bmatrix} \right).$$

An exact $100(1-\alpha)\%$ confidence interval for θ is then given by

$$\overline{Y}(\hat{\beta}) \pm t(1-\alpha/2;\, n-q-1)\, \hat{\sigma}_{Y\cdot C}\, \Delta, \tag{9}$$

where

$$\hat{\sigma}_{Y\cdot C}^2 = (n-q-1)^{-1}(n-1)\left(S_Y^2 - \mathbf{S}_{YC}'\mathbf{S}_C^{-1}\mathbf{S}_{YC}\right) \tag{10}$$

$$\Delta^2 = n^{-1} + (n-1)^{-1}(\overline{\mathbf{C}}_n - \mu_C)'\mathbf{S}_C^{-1}(\overline{\mathbf{C}}_n - \mu_C), \tag{11}$$

and $t(1-\alpha/2;\, n-q-1)$ is the $100(1-\alpha/2)$ percentile of the t-distribution with $(n-q-1)$ degrees of freedom.

Now the use of $\hat{\beta}$ rather than β means that the minimum variance (2) is not achieved. To measure the efficiency loss due to estimation of the control coefficients, Lavenberg, Moeller and Welch [3] derived the *loss factor*

$$\mathrm{Var}[\overline{Y}(\hat{\beta})] \,/\, \mathrm{Var}[\overline{Y}(\beta)] = (n-2)/(n-q-2). \tag{12}$$

Combining (2) and (12) we have,

$$\mathrm{Var}[\overline{Y}(\hat{\beta})] = \mathrm{Var}(\overline{Y}_n)\left[\left(1 - R_{YC}^2\right)\left(\frac{n-2}{n-q-2}\right)\right]. \tag{13}$$

2.2 Analysis Techniques for Nonnormal Outputs

If Y and **C** are not jointly multivariate normal, then the point estimator $\overline{Y}(\hat{\beta})$ is generally biased because $\hat{\beta}$ and $\overline{\mathbf{C}}_n$ are not independent. To reduce the bias of $\overline{Y}(\hat{\beta})$ and to construct an *asymptotically* valid confidence interval for θ, we use the jackknife statistic (Bratley, Fox and Schrage [5], Section 2.7). Let $\overline{Y}_{(k)}(\hat{\beta})$ denote the estimator computed from (3) through (8) when the k'th observation $(Y_k, \mathbf{C}_k)$ has been deleted from the original data set $\{(Y_j, \mathbf{C}_j): 1 \le j \le n\}$. Using the *pseudovalues*

$$J_k(\hat{\beta}) = n\,\overline{Y}(\hat{\beta}) - (n-1)\,\overline{Y}_{(k)}(\hat{\beta}), \quad 1 \le k \le n,$$

we calculate the jackknife statistic

$$\overline{J}(\hat{\beta}) = n^{-1}\sum_{k=1}^{n} J_k(\hat{\beta})$$

and the associated sample variance

$$S_J^2(\hat{\beta}) = (n-1)^{-1}\sum_{k=1}^{n}[J_k(\hat{\beta}) - \overline{J}(\hat{\beta})]^2.$$

When the joint distribution of Y and **C** satisfies certain mild regularity conditions, the jackknifed point estimator of θ has reduced bias: $E[\overline{J}(\hat{\beta})] = \theta + O(n^{-2})$ (see [6]); and an asymptotically valid $100(1-\alpha)\%$ confidence interval for θ is given by

$$\overline{J}(\hat{\beta}) \pm t(1-\alpha/2;\, n-1)\,(n^{-1/2}\, S_J(\hat{\beta})).$$

3. ESTIMATION WITH PATH CONTROLS

A SAN is completely described by the pair (N, A), where N is the set of all nodes and A is the set of all arcs. Let $s = \#(A)$, the number of arcs in the network. For each $a_i \in A$, $i = 1, \ldots, s$, let A_i be the (nonnegative) random variable representing the corresponding activity duration. We assume that the random variables $\{A_i\}$ are mutually independent. Let $\mu_i \equiv E(A_i)$, and $\sigma_i^2 \equiv \mathrm{Var}(A_i)$. Define p_j, $j = 1, \ldots, m$, as the j'th complete path from the source node to the sink node, and let P_j be the duration of the corresponding path p_j. Then

$$E(P_j) = \sum_{a_i \in p_j} E(A_i), \tag{14}$$

$$\mathrm{Var}(P_j) = \sum_{a_i \in p_j} \mathrm{Var}(A_i), \tag{15}$$

and

$$\mathrm{Cov}(P_j,P_k) = \sum_{a_i \in p_j \cap p_k} \mathrm{Var}(A_i). \qquad (16)$$

The expected project completion time E(Y) is given by E[max { P_1, ..., P_m }].

Ranking the m complete paths in descending order of expected duration, we choose the first q paths in the list as control paths. This (q × 1) vector of path controls has both a known mean μ_C and a known dispersion matrix Σ_C with elements computed as shown above. Clearly these path controls are strongly correlated with Y.

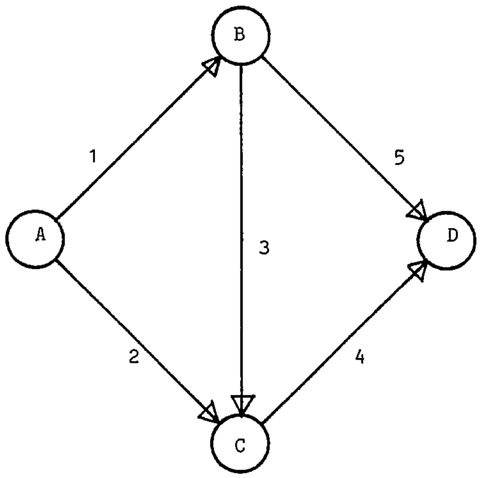

Figure 1.

Consider for example the network shown in Figure 1. Suppose $p_1 = (1,5)$, $p_2 = (1,3,4)$ and $p_3 = (2,4)$. Then $E(P_1) = \mu_1 + \mu_5$, $E(P_2) = \mu_1 + \mu_3 + \mu_4$ and $E(P_3) = \mu_2 + \mu_4$. Choosing these three paths as control variates, we obtain the dispersion matrix

$$\Sigma_C = \begin{bmatrix} \sigma_1^2+\sigma_5^2 & \sigma_1^2 & 0 \\ \sigma_1^2 & \sigma_1^2+\sigma_3^2+\sigma_4^2 & \sigma_4^2 \\ 0 & \sigma_4^2 & \sigma_2^2+\sigma_4^2 \end{bmatrix}.$$

In the case of a SAN, we can therefore use the known Σ_C matrix to compute, analogous to (6), the (q×1) vector of control coefficients

$$\hat\gamma = \Sigma_C^{-1} S_{YC},$$

which yields, analogous to (7), an unbiased point estimator of θ

$$\overline{Y}(\hat\gamma) = \overline{Y}_n - \hat\gamma\,'(\overline{C}_n - \mu_C).$$

Substituting Σ_C^{-1} for S_C^{-1} in (10) and (11), we derive a confidence interval for θ analogous to (9) for the case of normal simulation responses.

To develop the corresponding point and interval estimators for the nonnormal case, we use $\hat\gamma$ in place of $\hat\beta$ in Section 2.2, and thus compute the pseudovalues

$$J_k(\hat\gamma) = n\,\overline{Y}(\hat\gamma) - (n-1)\,\overline{Y}_{(k)}(\hat\gamma), \quad 1 \le k \le n.$$

The jackknifed point estimator of θ is then

$$\overline{J}(\hat\gamma) = n^{-1} \sum_{k=1}^{n} J_k(\hat\gamma),$$

with associated sample variance

$$S_J^2(\hat\gamma) = (n-1)^{-1} \sum_{k=1}^{n} (J_k(\hat\gamma) - \overline{J}(\hat\gamma))^2.$$

An asymptotically valid 100(1−α)% confidence interval for E[Y] is given by

$$\overline{J}(\hat\gamma) \pm t(1-\alpha/2;\, n-1)\,(n^{-1/2}\, S_J(\hat\gamma)).$$

4. EXPERIMENTAL EVALUATION

We conducted an extensive experimental study in order to evaluate the performance of the estimators $\overline{Y}(\hat\beta)$ (estimator 1), $\overline{J}(\hat\beta)$ (estimator 2), $\overline{Y}(\hat\gamma)$ (estimator 3), and $\overline{J}(\hat\gamma)$ (estimator 4). All efficiency gains reported are relative to the direct simulation estimator $\overline{Y}$ (estimator 0). This study involved the simulation of a set of five SANs in which the following characteristics are systematically varied: (a) the size of the network (the number of nodes and activities); (b) the topology of the network; (c) the relative dominance (criticality index) of the critical path; and (d) the percentage of activities with exponentially distributed durations.

We chose the exponential distribution because it has a higher coefficient of variation (equal to 1) than the distributions commonly used in the simulation of SANs, thereby creating the least favorable situation for the control variate estimators. For each of the five networks we varied the percentage of exponentially distributed activity durations over five levels (0%, 25%, 50%, 75%, 100%).

We define relative dominance of a path to be the probability that the path is critical on each realization of the network. For each network and for each level of percentage of exponentially distributed activities, we manipulated the duration of activities on the path with longest expected duration to achieve five levels of relative dominance (50-60%, 60-70%, 70-80%, 80-90%, 90-100%).

The objectives of this experimental investigation were:
1. To track the performance of the control variate technique in yielding significant variance reductions under widely varying conditions;
2. To investigate the effect of utilizing the known dispersion matrix Σ_C in estimating the control coefficients -- in effect, the classic question of expected information versus observed information;
3. To analyze the performance of the control variate technique with jackknifed estimators;
4. To study all of the above as a function of relative dominance of the critical path.

Table 1 shows the range in the number of nodes and activities in the five chosen networks. Figure 2 displays the fourth network. All models were simulated using the simulation language SLAM II [7], and in all cases a discrete-event orientation was employed. A random number generator was used to decide whether or not an activity would have an exponential distribution. The normal distribution with standard deviation equal to 25 percent of the mean was used to model all activities that were not exponentially distributed. The computer programs had a built-in correction to make the sampled durations equal to zero if a negative value was realized, but this was never applied.

Table 1 -- Network Characteristics			
Network	Nodes	Activities	Reference
1	10	14	[8, p.275]
2	23	42	[9, p.190]
3	30	49	[8, p.318]
4	41	56	[8, p.218]
5	51	65	[10, p.324]

For a given configuration of a network (i.e. level of relative dominance and percentage of exponentially distributed activities), we conducted a "meta-experiment" composed of 32 independent simulation experiments. Each experiment involved $n = 32$ replications with $q = 3$ controls; thus the loss factor $(n-2)/(n-q-2)$ was limited to 1.11.

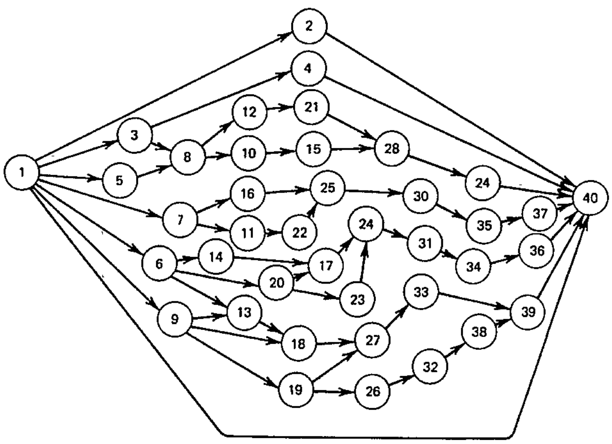

Figure 2. Network number 4.

To provide a fair assessment of the efficiency gains achieved, we calculated three performance measures for each of the four controlled estimators: (a) the percentage reduction in variance; (b) the percentage reduction in the half-length of a nominal 90 percent confidence interval; and (c) the actual coverage probability of a nominal 90 percent confidence interval. For estimator k, experiment j, let

$\hat{V}_j(k)$ = corresponding sample variance estimator,

$\hat{H}_j(k)$ = computed confidence interval half-length

$$\hat{I}_j(k) = \begin{cases} 1 & \text{if the computed confidence interval} \\ & \text{contains the estimand } \theta, \\ 0 & \text{otherwise,} \end{cases}$$

for $k = 0, 1, \ldots, 4$, and $j = 1, \ldots, 32$. In terms of the averages

$$\hat{V}(k) = (1/32) \sum_{j=1}^{32} \hat{V}_j(k),$$

$$\hat{H}(k) = (1/32) \sum_{j=1}^{32} \hat{H}_j(k),$$

$$\hat{I}(k) = (1/32) \sum_{j=1}^{32} \hat{I}_j(k),$$

computed across the entire meta-experiment, we obtained the following performance measures for the k'th estimator ($k = 1,2,3,4$):

$$\text{Variance reduction}(\%) = 100[\hat{V}(0)-\hat{V}(k)] \;/\; \hat{V}(0),$$

$$\text{Half-length reduction}(\%) = 100[\hat{H}(0)-\hat{H}(k)] \;/\; \hat{H}(0),$$

$$\text{Coverage}(\%) = 100\;\hat{I}(k).$$

Coverage probabilities were computed for estimator (0) also. In computing coverages, we used the sample mean of the response across the 1024 replications as the true value of θ.

5. EXPERIMENTAL RESULTS

First we observed that the percentage of exponentially distributed activities is not a significant factor. Tables 2, 3 and 4 show percentage variance reduction, percentage half-length reduction, and coverage percentage, respectively, as a function of the percentage of exponentially distributed activities for one of the five networks (network number 5). Other networks produced similar results. We did not explore the performance of the controlled estimators as a function of this factor any further.

Table 2 -- Percentage Reduction in Variance				
% Exponential Activities	$\bar{Y}(\hat{\gamma})$	$\bar{Y}(\hat{\beta})$	$\bar{J}(\hat{\gamma})$	$\bar{J}(\hat{\beta})$
0%	74.7	90.6	65.4	87.7
25%	63.4	78.6	53.8	75.4
50%	70.4	84.8	58.0	75.7
75%	68.7	81.9	61.0	84.2
100%	72.6	96.7	57.8	95.6

Level of dominance 70 - 80%

Table 3 -- Percentage Reduction in Half-Length				
% Exponential Activities	$\bar{Y}(\hat{\gamma})$	$\bar{Y}(\hat{\beta})$	$\bar{J}(\hat{\gamma})$	$\bar{J}(\hat{\beta})$
0%	49.5	69.2	41.1	64.9
25%	39.3	53.6	32.0	50.4
50%	45.4	60.9	35.2	50.7
75%	43.9	57.3	37.5	60.2
100%	47.5	81.8	35.0	79.0

Level of dominance 70 - 80%

Table 4 -- Coverage Percentages (Nominal Coverage 90%)					
% Exponential Activities	$\bar{Y}$	$\bar{Y}(\hat{\gamma})$	$\bar{Y}(\hat{\beta})$	$\bar{J}(\hat{\gamma})$	$\bar{J}(\hat{\beta})$
0%	90.6	90.6	81.2	96.9	93.7
25%	90.6	96.9	84.4	100.0	93.7
50%	93.7	93.7	84.4	100.0	87.5
75%	96.9	93.7	78.1	100.0	81.2
100%	96.9	93.7	71.9*	100.0	71.9*

Level of dominance 70 - 80%
* -- significantly below nominal coverage

Relative dominance turned out to be quite a significant factor for all three performance measures (Tables 5, 6 and 7). Percentage variance reduction and percentage half-length reduction generally increased with increasing levels of dominance. Coverage tended to worsen with increasing dominance for all estimators; the estimator $\bar{Y}(\hat{\beta})$ especially failed to achieve nominal coverage in several cases.

The two estimators $\bar{Y}(\hat{\gamma})$ and $\bar{J}(\hat{\gamma})$, obtained by incorporating the known dispersion matrix, yielded less variance reduction and half-length reduction. These estimators also generated wider confidence intervals than the estimators using the estimated dispersion matrix S_C. Consequently the coverage was better. This, however, is the most important performance measure for the practitioner, and it seems reasonable to give up some variance reduction for improved coverage.

Table 5 -- Percentage Variance Reduction						
Model	Estimator	Relative Dominance				
		1	2	3	4	5
1	$\bar{Y}(\hat{\gamma})$	82.4	87.8	89.2	90.1	91.0
	$\bar{Y}(\hat{\beta})$	93.4	95.8	97.8	98.4	98.8
	$\bar{J}(\hat{\gamma})$	69.2	74.1	78.2	81.9	83.1
	$\bar{J}(\hat{\beta})$	84.1	88.5	91.2	93.1	93.5
2	$\bar{Y}(\hat{\gamma})$	70.5	74.9	78.1	80.2	81.4
	$\bar{Y}(\hat{\beta})$	88.4	91.7	93.1	95.4	97.0
	$\bar{J}(\hat{\gamma})$	61.9	68.3	75.5	80.4	82.1
	$\bar{J}(\hat{\beta})$	74.2	79.7	84.3	86.1	88.3
3	$\bar{Y}(\hat{\gamma})$	51.1	56.5	62.4	68.7	71.5
	$\bar{Y}(\hat{\beta})$	65.9	68.1	71.3	82.2	94.2
	$\bar{J}(\hat{\gamma})$	39.4	42.5	45.7	54.3	59.2
	$\bar{J}(\hat{\beta})$	48.4	56.5	61.9	69.5	80.1
4	$\bar{Y}(\hat{\gamma})$	40.1	51.9	58.5	65.8	71.3
	$\bar{Y}(\hat{\beta})$	58.2	67.9	77.5	87.5	93.2
	$\bar{J}(\hat{\gamma})$	24.9	41.5	53.6	63.5	69.1
	$\bar{J}(\hat{\beta})$	40.4	49.3	58.1	65.6	70.6
5	$\bar{Y}(\hat{\gamma})$	58.4	66.8	70.4	73.2	75.7
	$\bar{Y}(\hat{\beta})$	74.3	80.1	84.8	88.1	91.3
	$\bar{J}(\hat{\gamma})$	44.9	53.7	58.0	61.4	65.4
	$\bar{J}(\hat{\beta})$	61.8	70.1	75.7	80.3	85.6

Percentage of Exponential Activities fixed at 50%

Table 6 -- Percentage Reduction in Half-Length						
Model	Estimator	Relative Dominance				
		1	2	3	4	5
1	$\bar{Y}(\hat{\gamma})$	57.9	65.0	67.0	68.4	69.9
	$\bar{Y}(\hat{\beta})$	74.2	79.5	85.1	87.3	89.0
	$\bar{J}(\hat{\gamma})$	44.5	49.1	53.3	57.5	58.9
	$\bar{J}(\hat{\beta})$	60.1	66.1	70.3	73.7	74.5
2	$\bar{Y}(\hat{\gamma})$	45.5	49.8	53.0	55.4	56.7
	$\bar{Y}(\hat{\beta})$	65.8	71.1	75.2	78.4	82.6
	$\bar{J}(\hat{\gamma})$	38.3	43.7	50.5	55.7	57.7
	$\bar{J}(\hat{\beta})$	49.2	54.9	60.3	62.7	65.8
3	$\bar{Y}(\hat{\gamma})$	29.9	33.8	38.5	43.9	46.5
	$\bar{Y}(\hat{\beta})$	41.4	43.3	46.3	57.7	75.8
	$\bar{J}(\hat{\gamma})$	22.1	24.2	26.3	32.4	36.1
	$\bar{J}(\hat{\beta})$	28.2	34.0	38.3	44.8	55.4
4	$\bar{Y}(\hat{\gamma})$	22.4	30.4	35.4	41.3	46.3
	$\bar{Y}(\hat{\beta})$	35.2	43.2	52.4	64.5	73.8
	$\bar{J}(\hat{\gamma})$	13.3	23.5	31.9	39.6	44.4
	$\bar{J}(\hat{\beta})$	22.8	28.8	35.3	41.3	45.8
5	$\bar{Y}(\hat{\gamma})$	35.3	42.2	45.4	48.1	50.6
	$\bar{Y}(\hat{\beta})$	49.2	55.3	60.9	65.4	70.4
	$\bar{J}(\hat{\gamma})$	25.8	32.0	35.2	37.9	41.2
	$\bar{J}(\hat{\beta})$	38.2	45.3	50.7	55.6	62.0

Percentage of Exponential Activities Fixed at 50%

Table 7 -- Coverage Percentages (Nominal Coverage 90%)						
Model	Estimator	Relative Dominance				
		1	2	3	4	5
1	$\bar{Y}$	96.9	93.7	93.7	93.7	87.5
	$\bar{Y}(\hat{\gamma})$	81.2	78.1	78.1	78.1	75.0*
	$\bar{Y}(\hat{\beta})$	71.9*	71.9*	71.9*	71.9*	68.7*
	$\bar{J}(\hat{\gamma})$	93.7	93.7	90.6	90.6	87.5
	$\bar{J}(\hat{\beta})$	87.5	84.4	81.2	81.2	78.1
2	$\bar{Y}$	87.5	87.5	90.6	90.6	90.6
	$\bar{Y}(\hat{\gamma})$	78.1	78.1	78.1	75.0*	71.9*
	$\bar{Y}(\hat{\beta})$	78.1	75.0*	71.9*	71.9*	71.9
	$\bar{J}(\hat{\gamma})$	87.5	87.5	84.4	84.4	84.4
	$\bar{J}(\hat{\beta})$	84.4	84.4	84.4	81.2	81.2
3	$\bar{Y}$	90.6	93.7	90.6	87.5	90.6
	$\bar{Y}(\hat{\gamma})$	87.5	87.5	84.4	78.1	78.1
	$\bar{Y}(\hat{\beta})$	84.4	81.2	78.1	75.0*	75.0*
	$\bar{J}(\hat{\gamma})$	90.6	90.6	87.5	87.5	87.5
	$\bar{J}(\hat{\beta})$	87.5	90.6	87.5	84.4	84.4
4	$\bar{Y}$	93.7	93.7	93.7	93.7	93.7
	$\bar{Y}(\hat{\gamma})$	93.7	93.7	87.5	87.5	87.5
	$\bar{Y}(\hat{\beta})$	93.7	90.6	90.6	87.5	84.4
	$\bar{J}(\hat{\gamma})$	100.0	100.0	100.0	100.0	96.9
	$\bar{J}(\hat{\beta})$	100.0	100.0	96.9	93.7	93.7
5	$\bar{Y}$	93.7	93.7	87.5	87.5	84.4
	$\bar{Y}(\hat{\gamma})$	81.2	81.2	78.1	75.0*	71.9*
	$\bar{Y}(\hat{\beta})$	75.0*	71.9*	71.9*	71.9*	71.9*
	$\bar{J}(\hat{\gamma})$	100.0	100.0	87.5	87.5	87.5
	$\bar{J}(\hat{\beta})$	87.5	87.5	81.2	81.2	78.1

Percentage of Exponential Activities fixed at 50%
* -- significantly below nominal coverage

6. CONCLUSIONS

It appears that when the dispersion matrix of controls can be analytically computed, it is better to incorporate this information into the control variate procedure. The procedure yields larger variance reductions with the estimated dispersion matrix, but the statistical reliability of the estimator based on the known dispersion matrix is significantly better.

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