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A RANDOM VARIATE GENERATION METHOD USEFUL IN HYBRID SIMULATION/ANALYTICAL MODELLING

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ABSTRACT

In the hybrid simulation/analytic modelling using uniformization, it is required that the basic random variables involved in such modelling are uniformizable. This requires the samples for random variables of interest to be generated by the uniformization procedure. In this paper we demonstrate the use of uniformization in continuous random variate generation. Using uniformization we represent the continuous random variable of interest as the first passage time of a continuous time stochastic process associated with a Poisson process. An approach using this result is proposed to generate samples for continuous random variables. A dynamic uniformization algorithm and other extensions of the basic uniformization algorithm are also considered.

INTRODUCTION

In the hybrid simulation/analytic modelling using uniformization, it is required that the basic random variables involved in such modelling are uniformizable. This requires the samples for random variables of interest to be generated by the uniformization procedure. The basic concept in the uniformization of a stochastic process $Z \equiv \{Z(t), t \in R_+\}$ is to represent it as a composition of a discrete-time stochastic process $Z \equiv \{Z_n, n \in N_+\}$ and a Poisson process N. Further, if $0 = S_0 < S_1 < S_2 < \ldots$ are the consecutive positions of the points of N on the real line R_+, then $\hat{Z}|(S_n)_0^\infty$ is a Markov chain. The uniformization is a modification of an idea introduced by Jensen (1953) and used by Keilson (1975, 1979), Keilson and Kester (1977), O'Brien (1976), Sonderman (1980) and others. Use of uniformization in the simulation of Markov processes has been explored by Hordijk, Inglehart and Schassberger (1976) and Grassman (1982). In these papers, the authors have made use of the properties of the underlying Poisson process N. As can be seen in Shanthikumar (1983b) one can develop a general approach to simulate a uniformizable point process that makes use of both the properties of the Poisson process N and the Markov property of $\hat{Z}|(S_n)_0^\infty$.

In Section 2 we discuss the conditions for the uniformizability of continuous random variables. The use of uniformization in the computer generation of samples for continuous random variable is discussed and some algorithms for such generation of samples are given in the same section. Non-Poissonian uniformization and its application in the generation of continuous random variables and the simulation of non-homogeneous point processes are discussed in Section 3.

UNIFORMIZATION AND RANDOM VARIATE GENERATION

In this section we will discuss the uniformization of positive continuous random variables and its application in random variate generation. Let $(X_n)_1^\infty$ be a renewal sequence of random variables with the same probability distribution as that of X (i.e., $X_n \stackrel{d}{=} X$). Suppose $Z = \{Z(t), t \in R_+\}$ be the counting process associated with $(X_n)_1^\infty$ and $T_1(Z)$ be its first passage time to level 1. So

$$T_1(Z) = \inf\{t: Z(t) \geq 1, t \in R_+\} \tag{2.1}$$

It is then immediate that

$$T_1(Z) = X_1 \stackrel{d}{=} X \tag{2.2}$$

Therefore obtaining a sample for X is equivalent to obtaining a sample for the first passage time $T_1(Z)$ of the process Z. We will next see how the process Z could be paritally uniformized so that a sample for $T_1(Z)$ could be obtained in a systematic way.

Let $F_X(\cdot)$ be the cumulative distribution function of X. We will assume that the random variable X is continuous and positive so that,

$$f_X(x) = \frac{d}{dx} F_X(x), \quad x > 0 \tag{2.3}$$

and

$$f(x) = \frac{f_X(x)}{\bar{F}_X(x)}, \quad x > 0. \tag{2.3}$$

where $\bar{F}_X(x) = 1 - F_X(x)$, $x > 0$, exist. Then:

$$f_X(x) = r(x) \exp\{-\int_0^x r(t)dt\}, \quad x > 0$$

and

$$\bar{F}_X(x) = \exp\{-\int_0^x f(t)dt\}, \quad x > 0$$

Now define an alternate process $Z = \{Z(t), t \in R_+\}$ such that $Z(0) = 0$, and for some real valued function $g(\cdot)$ defined on R_+ and satisfying $0 \leq g(x) \leq 1$, $x \geq 0$,

$$Z(S_{N(t)}) = \begin{cases} 1 & \text{w.p } g(S_{N(t)}) \\ 0 & \text{w.p } 1-g(S_{N(t)}) \end{cases} \tag{2.5}$$

$$\hat{Z}(t) = Z(S_{N(t)}),\ t > 0. \tag{2.6}$$

for a given sequence $0 = S_0 < S_1 < S_2 < \ldots$ of consecutive positions of points on the real line R_+ of a Poisson process $N(t)$ with rate λ. The process $N = \{N(t),\ t \in R_+\}$ is defined on the same probability space as that of $\hat{Z}$. Let $T_1(\hat{Z})$ be the first passage time of $\hat{Z}$ to reach the level 1. Next we give sufficient conditions under which $T_1(Z) \overset{d}{=} T_1(\hat{Z})$.

Theorem 2.1. Suppose

(i) $r(x) \leq \lambda < \infty,\ x > 0$

and (ii) $g(x) = r(x)/\lambda,\ x > 0.$

Then

$$T_1(Z) \overset{d}{=} T_1(\hat{Z}) \overset{d}{=} X$$

Proof: The proof follows directly from the proof of Theorem 2.1 of Sonderman (1980).

Remark - the construction $\hat{Z}$ as presented above is such that

$$\{Z(t),\ t \leq T_1(Z)\} \overset{st}{=\!=} \{Z(t),\ t \leq T_1(Z)\},$$

where $\overset{st}{=\!=}$ means equality in law.

Theorem 2.1 immediately gives rise to a general approach to generate samples for uniformizable random variables. The algorithm to do this is:

Algorithm 2.1

Step 0. Obtain $\lambda = \sup\{r(x),\ x > 0\}$ and set $t = 0$.
Step 1. Generate a sample y from an exponential distribution with mean $1/\lambda$ and set $t = t + y$.
Step 2. Generate a uniform sample u between 0 and 1.
(i) if $u < r(t)/\lambda$, set $x = t$. STOP
(ii) otherwise go to Step 1.

Note that this is essentially the same as obtaining the first point in the thinning algorithm of Lewis and Shedler (1979a) for the non-homogenous Poisson process. A discrete analogue of this algorithm is developed and modified in Shanthikumar (1983a). There are several uniformizable continuous random variables that are of practical interest. In particular the phase type random variables such as generalized Erlang and hyperexponential random variables, and all random variables with decreasing failure rate (DFR) with $r(0) < \infty$ are uniformizable. Let

$$u = \inf\{r(x),\ x > 0\} \tag{2.7}$$

Then the probability of successfully reaching the level 1 by $\hat{Z}$, at each renewal of N is greater than or equal to $\rho = u/\lambda \leq 1$. So the expected number of exponential samples (or comparisons) needed in Algorithm 2.1 is bounded from above by ρ^{-1}. Naturally, one would prefer to have a high value for ρ. When $\rho = 1$ we need only one observation. However, this case corresponds to the exponential random variable with mean $1/\lambda$. So when a probability distribution function is close to that of an exponential random variable, one may expect this algorithm to perform well. Next we will discuss the application of this algorithm to Hyper-exponential, Logistic and Erlang distributions.

Example 1. Hyper-exponential distributions.

A random variable X has a hyper-exponential distribution if

$$\bar{F}_X(x) = \sum_{n=1}^{K} p_n (e^{-\lambda_n x}) \tag{2.8}$$

for a probability vector $(p_n)_1^K$ and $\lambda_n > 0$, $n = 1,2,\ldots,K$. It is well know that the hazard rate

$$r(x) = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{\sum_{n=1}^{K} p_n \lambda_n e^{-\lambda_n x}}{\sum_{n=1}^{K} p_n e^{-\lambda_n x}},\ x > 0 \tag{2.9}$$

is decreasing and

$$\lambda \triangleq \sup\{r(x),\ x > 0\} = \sum_{n=1}^{K} p_n \lambda_n \tag{2.10}$$

$$\mu \triangleq \inf\{r(x),\ x > 0\} = \inf\{\lambda_n\} \tag{2.11}$$

(e.g., see Barlow and Proschan (1975)).

The average number of comparisons needed for this distribution for different sets of values of $(p_n)_1^K$ and $(\lambda_n)_1^K$ with $K = 10$ are given in Table 2.1. The values of λ_n are selected according to

$$\lambda_1 = 1,$$

$$\lambda_n = \lambda_{n-1} + (\lambda_{max} - 1)/9,\ n = 2,\ldots,10.$$

The probabilities p_n are selected according to

$$p_n = (1-p)p^{n-1}/(1-p^{10}),\ n = 1,2,\ldots,10.$$

Different distributions are obtained using different values for λ_{max} and p.

As one may observe as p and λ_{max} increases the expected number of observations needed in Algorithm 2.1 also increases. This is because the ratio $\lambda/\mu \triangleq \sup\{r(x),\ x > 0\}/\inf\{r(x),\ x > 0\}$ increases as p and λ_{max} increases. As will be seen later a simple modification to Algorithm 2.1 will counteract this effect.

Example 2. Logistic distribution.

A random variable Y has a logistic distribution if

p	λ_{max}	Average number of comparisons	Standard deviation
.9	2	1.0462	0.2330
	4	1.2081	0.6506
	7	1.4570	1.2677
	10	1.6800	1.8198
.7	2	1.0332	0.1977
	4	1.1655	0.5289
	7	1.3818	0.9663
	10	1.6197	1.4848
.5	2	1.0137	0.1188
	4	1.0844	0.3330
	7	1.2246	0.6216
	10	1.3672	0.8820
.3	2	1.0053	0.0740
	4	1.0332	0.1879
	7	1.0994	0.3627
	10	1.1791	0.5245
.1	2	1.0021	0.0458
	4	1.0090	0.0955
	7	1.0276	0.1739
	10	1.0538	0.2464

Table 2.1: Average number of comparisons (and its standard deviation) needed to generate samples for hyper-exponential random variables using Algotithm 2.1. These statistics are taken over 10,000 samples.

$$\bar{F}_Y(y) = e^{-ay}/(1 + e^{-ay}), \quad -\infty < y < \infty \tag{2.12}$$

for $a > 0$. Now consider the conditional random variable

$$X = Y_{|Y>0}$$

Then

$$\bar{F}_X(x) = 2e^{-ax}/(1 + e^{-ax}), \quad x > 0 \tag{2.14}$$

One may now take a sample for X and randomly assign a + or - sign to obtain a sample for Y. So consider

$$r(x) = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{a}{1+e^{-ax}}, \quad x > 0. \tag{2.15}$$

$r(x)$ is increasing,

$$\lambda \triangleq \sup\{r(x), x > 0\} = a \tag{2.16}$$

and

$$\mu \triangleq \inf\{r(x), x > 0\} = a/2. \tag{2.17}$$

In a sample size of 10,000 the average number of comparisons needed to generate a sample for a logistic random variable is observed to be 1.3779 with a standard deviation of 0.6445.

Example 3. Erlang distributions.

A random variable X has an Erlang distribution if

$$F_X(x) = \sum_{r=0}^{K-1} \frac{e^{-\lambda' x}(\lambda' x)^r}{r!}, \quad x > 0 \tag{2.18}$$

for $K \geq 1$ and $\lambda' > 0$. If $K = 1$, the Erlang distribution reduces to an exponential distribution. Now

$$r(x) = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{\frac{e^{-\lambda' x}(\lambda x)^{K-1}\lambda'}{(K-1)!}}{\sum_{r=0}^{K-1} \frac{e^{-\lambda' x}(\lambda' x)^r}{r!}} \tag{2.19}$$

$$= \lambda' \Big/ \{1 + \sum_{r=0}^{K-2} (K-1)! \frac{1}{(\lambda' x)^{K-1-r} r!}\}, \quad x > 0.$$

It is easily verified that $r(x)$ is increasing,

$$\lambda = \sup\{r(x), x > 0\} = \lim_{x\to\infty} r(x) = \lambda' \tag{2.20}$$

and

$$\mu = \inf\{r(x), x > 0\} = r(0) = 0.$$

The average number of comparisons needed for this distribution for different values of K are shown in Table 2.2.

K	Number of comparisons	
	Average	Sample standard deviation
2	1.9994	1.0843
4	4.0268	2.0326
6	6.0506	2.7014
8	8.0187	3.2618
10	10.0068	3.7370

Table 2.2. Average number of comparisons (and its standard deviation) needed to generate samples for Erlang-K random variables using Algorithm 2.1. These statistics are taken over 10,000 samples.

From the above results it is clear that the uniformization technique is not an efficient approach to sample values for Erlang random variables. Even the simple method representing the Erlang-K random variable as a sum of K exponential random variables will perform better than the uniformization algorithm. This clearly indicates that some modifications to the basic algorithm should be made to make it efficient. In the latter part of this section we will consider some modifications to the basic algorithm.

To implement Algorithm 2.1 one needs to compute the values of $r(\cdot)$ whenever it is needed. In some cases this may require considerable computational effort. However, this computational effort may be reduced by an adaptation of the squeeze method to the uniformization technique (e.g., Schmeiser and Lal (1980) for the squeeze method). Let

$$\ell(x) \leq r(x) \leq u(x), \quad x > 0;$$

be some bounds for the hazard rate $r(\cdot)$. Then the following algorithm can be used in place of Algorithm 2.1.

Algorithm 2.2 (squeeze method modification)

Step 0. Obtain $\lambda = \sup\{r(x), x > 0\}$ and set $t = 0$.
Step 1. Generate a sample y from an exponential distribution with mean $1/\lambda$ and set $t = t + y$.
Step 2. Generate a uniform sample u between 0 and 1 and
(i) if $u > u(t)/\lambda$ go to Step 1
(ii) else (iii) if $u < \ell(t)/\lambda$ set $x = t$ and STOP
(iv) else (v) if $u > r(x)/\lambda$ go to Step 1
(vi) else set $x = t$. STOP.

Dynamic Uniformization of Random Variables with Decreasing Hazard Rate

Next we will look at a possible improvement of Algorithm 2.1 for application to distributions with decreasing hazard rates. This improvement is achieved by dynamically changing the uniformization constant λ, at every observation epoch. That is, if a renewal of the underlying point process occurs at time epoch t, the new uniformization constant λ is set equal to $r(t)$, since

$$\lambda \triangleq r(t) \geq r(x), \quad x > t. \tag{2.22}$$

The algorithm incorporating this modification is:

Algorithm 2.3 (dynamic uniformization)

Step 0. Set $\lambda_1 = r(0)$, $n = 1$, and $t = 0$
Step 1. Generate a sample y from an exponential distribution with mean $1/\lambda_n$ and set $t = t + y$
Step 2. Generate a uniform sample u between 0 and 1.
(i) if $u < r(t)/\lambda_n$ set $x = t$. STOP
(ii) else set $n = n+1$, $\lambda_n = r(t)$ and go to Step 1.

Theorem 2.2

The random sample X obtained through algorithm 2.3 has a probability density function

$$f_X(x) = r(x) \exp\{-\int_0^x r(t)dt\}, \quad x > 0$$

Proof:

Consider the conditional probability

$$P\{x < X \leq x + \Delta x | X > x \text{ and the last renewal before time } x \text{ occurred at time epoch } t\} \tag{2.23}$$
$$= \{r(t)\Delta x + 0(\Delta x)\}\left(\frac{r(x)+0(\Delta x)}{r(t)}\right) + 0(\Delta x)$$
$$= r(x)\Delta x + 0(\Delta x), \quad x > t$$

Since the right hand side of (2.23) is independent of t, one has

$$P\{x < X \leq x + \Delta x | X > x\} = \frac{P\{X \leq x+\Delta x\} - P\{X \leq x\}}{P\{X > x\}} \tag{2.24}$$
$$= r(x)\Delta x + 0(\Delta x)$$

Dividing both sides of (2.24) by Δx and taking the limit as $\Delta x \to 0$ one obtains

$$f_X(x) = r(x)\bar{F}_X(x) = r(x)\int_x^\infty f_X(t)dt, \quad x > 0 \tag{2.25}$$

Solving the integral equation (2.25) for $f_X(\cdot)$ with the boundary condition $f_X(0) = r(0)$ one obtains the desired result.

As noted earlier hyper-exponential distributions have decreasing failure (or hazard) rates. Samples from these distributions for various parameter values are generated using Algorithm 2.3. The results of this computation is given in Table 2.3.

The values of $(p_n)_1^K$ and $(\lambda_n)_1^K$ are chosen to be the same as those used in Table 2.1.

p	λ_{max}	Average number of comparisons	Standard deviation
.9	2	1.0432	0.2101
	4	1.1541	0.4069
	7	1.2597	0.5511
	10	1.3376	0.6497
.7	2	1.0311	0.1804
	4	1.1332	0.3788
	7	1.2506	0.5202
	10	1.3352	0.6045
.5	2	1.0135	0.1163
	4	1.0745	0.2778
	7	1.1724	0.4184
	10	1.2565	0.5132
.3	2	1.0052	0.0719
	4	1.0323	0.1802
	7	1.0857	0.2905
	10	1.1406	0.5131
.1	2	1.0021	0.0458
	4	1.0089	0.0939
	7	1.0264	0.2215
	10	1.0504	0.2215

Table 2.3: Average number of comparisons (and its standard deviation) needed to generate samples for hyper-exponential random variables using Algorithm 2.3. These statistics are taken over 10,000 samples.

Comparison of Table 2.3 to Table 2.1 clearly shows that the dynamic uniformization algorithm is a considerable improvement over Algorithm 2.1.

Partitioned Uniformization of Continuous Random Variables with Increasing Hazard Rate

An alternate form of Algorithm 2.3 can be used to obtain samples for uniformizable continuous random variables with increasing hazard rates. Let $[0,t_1), [t_1,t_2), \ldots, [t_{K-1},\infty)$ be K mutually exclusive partition of the support R_+ of the random variable X. Within each of these partitions one may use a uniformization constant suitable for that partition; i.e., the uniformization constant λ_n used for the partition $[t_{n-1},t_n)$ is

$$\lambda_n \triangleq \sup\{r(x), x \in [t_{n-1},t_n)\} = r(t_n)$$

Now the following algorithm can be used to sample a value for a random variable X. Let $t_0 = 0$ and $t_K = +\infty$.

Algorithm 2.4

Step 0. Set $\lambda_1 = r(t_1)$, $n = 1$, and $t = 0$.

Step 1. Sample a value y from an exponential distribution with mean $1/\lambda_n$ and set $t = t + y$.

Step 2. If (i) $t > t_n$ set $n = n + 1$, $\lambda_n = r(t_n)$ and go to Step 1,
(ii) else sample a uniform value u between 0 and 1 and
(iii) if $u \leq r(t)/\lambda_n$, set $x = t$. STOP.
(iv) else go to Step 1.

We will establish the validity of Algorithm 2.3 through the following theorem.

Theorem 2.3

The random sample X obtained through Algorithm 2.4 has a reliability function

$$\bar{F}_X(x) = \exp\{-\int_0^x r(t)dt\}$$

Proof: Consider the conditional probability $P\{X > x|X > t_{n-1}\}$, $x \in [t_{n-1},t_n)$. Since the uniformization constant λ_n is the same in the range $[t_{n-1},t_n)$, from Theorem 2.1, it is clear that

$$P\{X > x|X > t_{n-1}\} = \exp\{-\int_{t_{n-1}}^x r(t)dt\}, \quad x \in [t_{n-1},t_n) \tag{2.26}$$

From (2.26) we note that

$$P\{X > t_k|X > t_{k-1}\} = \exp\{-\int_{t_{k-1}}^{t_k} r(t)dt\}, \quad k = 1,2,\ldots. \tag{2.27}$$

Now it is immediate from (2.26) and (2.27) that

$$P\{X > x\} = P\{X > x|X > t_{n-1}\} \prod_{k=1}^{n-1} P\{X > t_k|X > t_{k-1}\} =$$

$$\exp\{-\int_0^x r(t)dt\}, \quad x \in [t_{n-1},t_n), \quad n = 1,2,\ldots,K.$$

Remark - Algorithm 2.4 can be used to sample a value for a uniformizable continuous random variable, even if the hazard rate does not have the monotonicity property. In such a case one would set

$$\lambda_n = \sup\{r(x), x \in [t_{n-1},t_n)\}, \quad n = 1,2,\ldots,K,$$

and then use Algorithm 2.4 without resetting the value of λ_1 in Step 0 and the value of λ_n in Step 2(i).

So far we have considered continuous random variables that are uniformizable. It should be pointed out, however, that there are several continuous random variables that are not uniformizable. In particular the random variables with finite support are not uniformizable. There are also several random variables with support on the real line R or on the positive half real line R_+ which are not uniformizable. For example, a Gamma random variable with a shape parameter $\alpha < 1$ has a decreasing hazard rate and

$$\lim_{x \to 0} r(x) = +\infty .$$

On the other hand the random variable $X \triangleq Y|_{Y > 0}$, with Y being a unit normal random variable has an increasing hazard rate and

$$\lim_{x \to \infty} r(x) = +\infty .$$

We will briefly outline a modification to the basic algorithm that would allow one to generate samples from some non-uniformizable random variables. Let X be a non-uniformizable random variable with hazard rate infinite in the limit only at either zero or infinity. Now define the random variable

$$X_\varepsilon = \begin{cases} X \text{ if } X \geq \varepsilon \\ \varepsilon \text{ if } X < \varepsilon \end{cases} \quad \text{if } \lim_{x \to 0} r(x) = +\infty \tag{2.28}$$

$$X_\varepsilon = \begin{cases} X & X \leq \varepsilon \\ \varepsilon & X > \varepsilon \end{cases} \quad \text{if } \lim_{x \to \infty} r(x) = +\infty \tag{2.29}$$

Now one can appropriately choose ε such that its continuous part of X_ε is uniformizable. Now one may use a modified partitioned uniformization technique to sample a value for X_ε and use it as an approximation for a sample from X.

NON-POISSONIAN UNIFORMIZATION AND THE SIMULATION OF NON-HOMOGENEOUS POINT PROCESSES

In this section we will consider non-Poissonian uniformization of continuous random variables and its application in the simulation of non-homogeneous point processes. As before let X be a continuous positive random variable with support I. Its cumulative probability distribution function, probability density function and hazard rates are $F_X(\cdot)$, $f_X(\cdot)$ and $r(\cdot)$, respectively. Now let Y be another random variable with the same support I as X and cumulative probability distribution function; probability density function and hazard rate $F_Y(\cdot)$, $f_Y(\cdot)$ and $r_Y(\cdot)$ respectively. Suppose

$$\frac{r(x)}{r_Y(x)} \leq 1, \quad \forall\, x \in I \tag{3.1}$$

Then we say that the random variable X is Y-uniformizable. In the earlier sections we considered the case where Y is an exponential random variable with mean $1/\lambda$. In this case the condition (5.1) reduces to

$$r(x) \leq \lambda < \infty, \quad \forall\, x > 0. \tag{3.2}$$

Now we will consider an altorithm to generate a Y-uniformizable random variable from the samples of Y.

Let

$$p(x) \triangleq r(x)/r_Y(x), \quad \forall\, x \in I.$$

The following algorithm can be used to generate a sample for the random variable X. The validity of this algorithm will be established following the description of the algorithm.

Algorithm 3.1

Step 0. Set t = 0
Step 1. Generate a sample y for the random variable $Y|_{Y>t}$, and set y = t
Step 2. Generate a uniform sample u between 0 and 1.
(i) if u < p(t) set x = t. STOP
(ii) else go to Step 1.

Theorem 3.1

The random sample X obtained through Algorithm 5.1 has a probability density function

$$f_X(x) = r(x)\exp\{-\int_0^x r(t)dt\}, \quad x \in I$$

Proof: Suppose we call each of the values t seen in the above algorithm a renewal epoch. Consider the conditional probability

$$P\{x < X \leq x + \Delta x | X > x \text{ and the last renewal before time } x \text{ occurred at time epoch } t\} \qquad (3.4)$$

$$= \{r_Y(x)\Delta x + O(\Delta x)\}(p(x) + O(\Delta x)), \; x > t$$

$$= r(x)\Delta x + O(\Delta x), \; x > t.$$

Now similar to that in the proof of Theorem 2.3 one has

$$f_X(x) = r(x)\bar{F}_X(x) = r(x)\exp\{-\int_0^x r(t)dt\}, \; x \in I$$

This notion of non-Poissonian uniformization is imbedded in the concept of thinning of non-homogeneous Poisson processes (e.g., Lewis and Shedler (1979a)). It is easily seen that this approach can be extended to simulate non-homogeneous point processes. Next we will illustrate the applicability of the above algorithm through an example.

Example: Consider a random variable X with hazard rate

$$r(x) = \exp\{\alpha_0 + \alpha_1 x + \alpha_2 x^2\}, \; x > 0, \qquad (3.5)$$

$\alpha_1 > 0$ and $\alpha_2 < 0$. Simulation of non-homogeneous Poisson processes with such degree-two exponential polynomial rate function is considered by Lewis and Shedler (1979b). We choose $\alpha_2 < 0$ so that the random variable X could be uniformized by the random variable Y with hazard rate

$$r_Y(x) = \exp\{\alpha_0 + \alpha_1 x\}, \; x > 0 \qquad (3.6)$$

Note that X has a defective distribution. To implement Algorithm 3.1, we first need to device an efficient approach to sample a value for $Y|_{Y>t}$. So consider

$$P\{Y > y | Y > t\} = \exp\{-\int_t^y r_Y(x)dx\}, \; y > t \qquad (3.7)$$

Substituting (3.6) into (3.7) we get after simplification

$$P\{Y > y | Y > t\} = \exp\{-\frac{1}{\alpha_1}[e^{\alpha_1 y} - e^{\alpha_1 t}]e^{\alpha_0})\}, \; y > t \qquad (3.8)$$

Now using the inverse transform method, one has

$$Y|_{Y>t} \overset{d}{=} \frac{1}{\alpha_1} \ell n\{1 - \alpha_1 \exp\{-(\alpha_0 + \alpha_1 t)\}\ell n(U)\} + t, \qquad (3.9)$$

where U is a uniform rnadom variable with support (0,1). Now the algorithm to sample a value for X follows Algorithm 3.1 in a natural way.

Algorithm 3.1a. (for $r(x) = \exp\{\alpha_0 + \alpha_1 x + \alpha_2 x^2\}$, $x > 0$, $\alpha_1 > 0$, $\alpha_2 < 0$)

Step 0. Set t = 0
Step 1. Generate a uniform sample u between 0 and 1 and calculate

$$y = \frac{1}{\alpha_1} \ell n\{1 - \alpha_1 \exp(-(\alpha_0 + \alpha_1 t))\ell n(u)\} + t$$

and set t = y
Step 2. Generate a uniform sample u between 0 and 1 and
(i) if $u < \exp\{\alpha_2 t\}$ set x = t STOP
(ii) else go to Step 1.

Remark - Since X has a defective distribution one should put an upper limit for the value t and use it in Step 2(ii) to terminate the algorithm after finite number of steps.

Similar to the modifications used in Section 2, one may modify Algorithm 3.1 depending on whether p(x) is decreasing or increasing. Further the extension of Algorithm 3.1 to the simulation of a non-homogenous point process is immediate. Hence we will not make any effort to include these modifications here.

CONCLUSION

In this paper we have demonstrated the use of uniformization in continuous random variate generation. In particular we have developed several algorithms to generate samples for uniformizable continuous random variables. A dynamic uniformization and a non-Poissonion uniformization algorithms are also developed. The basic idea of uniformization can be applied in developing hybrid simulation/analytic models of renewal processes (Shanthikumar (1983c)). Shanthikumar and Sargent (1983) discuss the concepts of hybrid simulation/analytic models.

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