VARIANCE REDUCTION TECHNIQUES

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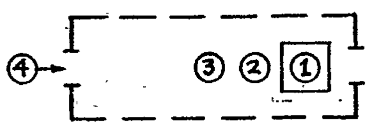
3

THE NEED FOR IMPROVED EFFICIENCY IN DISCRETE-EVENT SIMULATIONS

EXAMPLE: SIMULATING A SINGLE-SERVER QUEUEING SYSTEM TO ESTIMATE THE LONG-RUN AVERAGE WAITING TIME PER CUSTOMER PRIOR TO RECEIVING SERVICE



- EXPONENTIALLY DISTRIBUTED INTERARRIVAL AND SERVICE TIMES
- TRAFFIC INTENSITY $\rho = \frac{\text{ARRIVAL RATE}}{\text{SERVICE RATE}} < 1$
- W_j = WAITING TIME OF jTH CUSTOMER, $j=1,2,\ldots$
- WE WANT TO ESTIMATE

$$\mu_W = \lim_{j\to\infty} E[W_j]$$

TO WITHIN ±5% OF ITS TRUE VALUE USING A 95% CONFIDENCE INTERVAL ESTIMATOR CENTERED ON THE SAMPLE MEAN

$$\overline{W}_n = \frac{1}{n}\sum_{j=1}^{n} W_j$$

4

REQUIRED SAMPLE SIZE WITH DIRECT SIMULATION OF THE PROCESS $\{W_j : j=1,2,\ldots\}$

TRAFFIC INTENSITY ρ	SAMPLE SIZE n
0.01	321,345
0.05	76,283
0.10	46,688
0.20	34,191
0.30	33,106
0.40	36,538
0.50	44,563
0.60	60,442
0.70	94,711
0.80	189,776
0.90	681,073
0.95	2,586,327
0.99	62,084,898

⇒ FREQUENTLY, PRODIGIOUS RUN LENGTHS ARE REQUIRED TO ACHIEVE ACCEPTABLE PRECISION IN SIMULATION-BASED ESTIMATORS

⇒ WE LOOK FOR SUITABLE VARIANCE REDUCTION TECHNIQUES

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5

CORRELATION METHODS

THREE TECHNIQUES TAKE ADVANTAGE OF LINEAR CORRELATION AMONG SIMULATION OUTPUT RESPONSES TO ACHIEVE IMPROVED EFFICIENCY

INDUCED CORRELATION METHODS — REQUIRE THE EXPERIMENTER TO INDUCE POSITIVE OR NEGATIVE CORRELATION AMONG BLOCKS OF SIMULATION RUNS BY MANIPULATING THE RANDOM NUMBER INPUT

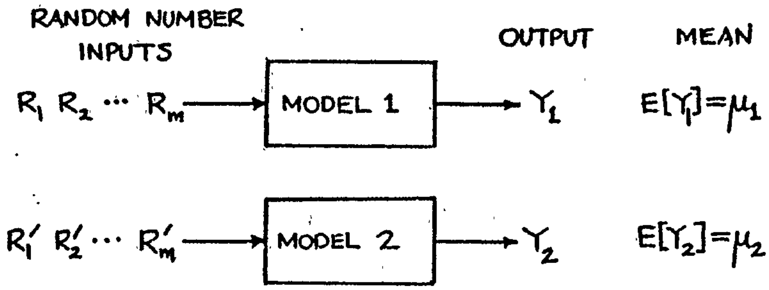
1. COMMON RANDOM NUMBER STREAMS — TO COMPARE 2 OR MORE ALTERNATIVES
2. ANTITHETIC VARIABLES — TO ESTIMATE MEAN RESPONSE OF A SINGLE SYSTEM

CONTROL VARIABLES — THIS METHOD EXPLOITS ANY INHERENT CORRELATION AMONG OUTPUT VARIABLES AND CONCOMITANT SYSTEM VARIABLES

6

COMMON RANDOM NUMBER STREAMS

WE WANT TO COMPARE TWO SYSTEMS BY ESTIMATING THE DIFFERENCE IN THEIR MEANS



USING $Y_1 - Y_2$ TO ESTIMATE $\mu_1 - \mu_2$, WE HAVE

$$E[Y_1 - Y_2] = \mu_1 - \mu_2$$

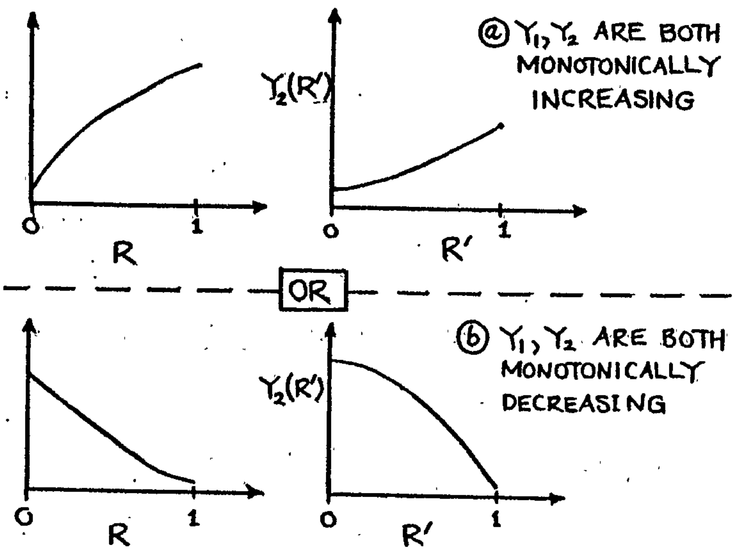
$$Var(Y_1 - Y_2) = Var(Y_1) + Var(Y_2) - 2Cov(Y_1, Y_2)$$

NOTE: IF $Cov(Y_1, Y_2) > 0$, THE ESTIMATOR $Y_1 - Y_2$ HAS A SMALLER VARIANCE THAN THAT OBTAINED WITH TWO INDEPENDENT RUNS

Q: HOW DO WE INDUCE POSITIVE CORRELATION BETWEEN Y_1 AND Y_2?

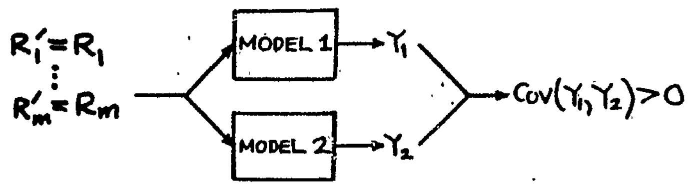
7

IF Y_1, Y_2 RESPOND IN A SIMILAR WAY TO CHANGES IN THE RANDOM NUMBER INPUT



8

THEN IT IS REASONABLE TO EXPECT THAT POSITIVE CORRELATION OF INPUTS ⇒ POSITIVE CORRELATION OF OUTPUTS



1. FOR MODELS WITH COMPLEX, DISSIMILAR RESPONSES LITTLE OR NO EFFICIENCY GAIN MAY RESULT FROM THIS TECHNIQUE
2. MOST WIDELY USED TECHNIQUE IN PRACTICE
3. MULTIPLE COMPARISONS ANALYSIS IS MORE COMPLICATED

REFERENCES

WRIGHT & RAMSAY, "ON THE EFFECTIVENESS OF COMMON RANDOM NUMBERS," MANAGEMENT SCIENCE, VOL. 25 (1979), PP. 649–656.

HEIKES, MONTGOMERY, & RARDIN, "USING COMMON RANDOM NUMBERS IN SIMULATION EXPERIMENTS — AN APPROACH TO STATISTICAL ANALYSIS," SIMULATION, VOL. 25 (1976), PP. 81–85.

9

ANTITHETIC VARIATES

IF Y_1 AND Y_2 ARE REPLICATES OF THE SAME MODEL, WE WANT TO USE $\frac{1}{2}(Y_1+Y_2)$ TO ESTIMATE THE MEAN RESPONSE μ_Y

WE HAVE:

$$E[\tfrac{1}{2}(Y_1+Y_2)] = \mu_Y$$

$$Var[\tfrac{1}{2}(Y_1+Y_2)] = \tfrac{1}{4}\{Var(Y_1)+Var(Y_2)+2Cov(Y_1,Y_2)\}$$

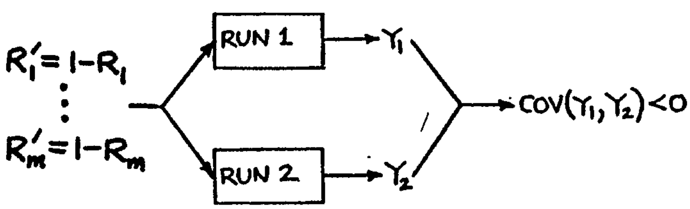
$$= \tfrac{1}{2}Var(Y_1)+\tfrac{1}{2}Cov(Y_1,Y_2)$$

NOTE: IF $Cov(Y_1,Y_2)<0$, THE SAMPLE MEAN $\frac{1}{2}(Y_1+Y_2)$ HAS A SMALLER VARIANCE THAN THAT OBTAINED WITH TWO INDEPENDENT RUNS

HOW DO WE INDUCE NEGATIVE CORRELATION BETWEEN Y_1 AND Y_2?

10

IF $Y_1(R_1, R_2, \ldots, R_m)$ IS A MONOTONE FUNCTION OF EACH OF ITS INPUTS (SEE FOIL 7), THEN IT IS REASONABLE TO EXPECT THAT NEGATIVE CORRELATION OF INPUTS FOR RUNS 1 AND 2 $\Longrightarrow$ NEGATIVE CORRELATION OF OUTPUTS Y_1, Y_2



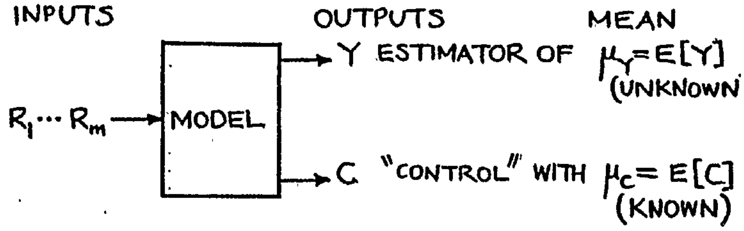
1. METHOD DOES NOT WORK WELL FOR MODELS WITH A COMPLEX RESPONSE FUNCTION
2. SECOND MOST WIDELY USED TECHNIQUE IN PRACTICE

REFERENCES

- SCHRUBEN AND MARGOLIN, "PSEUDORANDOM NUMBER ASSIGNMENT IN STATISTICALLY DESIGNED SIMULATION EXPERIMENTS," J. AMER. STATIST. ASSOC., VOL. 73 (1978), PP. 504–525.
- GEORGE, "VARIANCE REDUCTION FOR A REPLACEMENT PROCESS," SIMULATION, VOL. 29 (1977), PP. 65–74.

11

CONTROL VARIABLES



THE CONTROLLED ESTIMATOR

$$Y(b) = Y - b(C-\mu_C)$$

OF μ_Y HAS

$$E[Y(b)] = \mu_Y$$

$$Var[Y(b)] = Var(Y) - 2\cdot b\cdot Cov(Y,C) + b^2\cdot Var(C)$$

↓

MINIMUM VARIANCE WITH OPTIMAL CONTROL COEFFICIENT

$$\beta = \frac{Cov(Y,C)}{Var(C)}$$

$$\Longrightarrow Var[Y(\beta)] = Var(Y)\cdot(1-\rho_{YC}^2)$$

WITH ρ_{YC} = COEFFICIENT OF LINEAR CORRELATION BETWEEN Y AND C

12

IN PRACTICE, β IS USUALLY UNKNOWN.

⇒ WE NEED A POINT ESTIMATOR $\hat{\beta}$ OF β FROM WHICH WE CAN COMPUTE A CONFIDENCE INTERVAL ESTIMATOR OF μ_Y OVER INDEPENDENT REPLICATIONS OF THE MODEL.

WE ASSUME THE RANDOM VARIABLES Y, C OBSERVED ON EACH RUN HAVE A JOINT NORMAL DISTRIBUTION:

$$\underset{\sim}{Z} = \begin{bmatrix} Y \\ C \end{bmatrix} \sim N(\underset{\sim}{\mu}_Z, \underset{\sim}{\Sigma}_Z)$$

WITH: $\underset{\sim}{\mu}_Z = \begin{bmatrix} \mu_Y \\ \mu_C \end{bmatrix}$ $\qquad \underset{\sim}{\Sigma}_Z = \begin{bmatrix} Var(Y) & Cov(Y,C) \\ Cov(Y,C) & Var(C) \end{bmatrix}$

THIS ENSURES VALIDITY OF THE LINEAR REGRESSION MODEL

$$Y = \mu_Y + \beta(C - \mu_C) + \varepsilon$$

NORMALLY DISTRIBUTED RESIDUAL

$\varepsilon \sim N(0, \sigma_\varepsilon^2)$

WITH MEAN 0 AND VARIANCE

$\sigma_\varepsilon^2 = Var(Y)\cdot(1-\rho_{YC}^2)$

13

ESTIMATION PROCEDURE WITH ONE CONTROL

EXECUTE n INDEPENDENT REPLICATIONS OF THE MODEL TO GENERATE THE DATA SET

$$\left\{ \begin{bmatrix} Y_j \\ C_j \end{bmatrix} : 1 \le j \le n \right\}$$

- COMPUTE THE ORDINARY LEAST-SQUARES ESTIMATE

$$\hat{\beta} = \frac{\sum_{j=1}^{n} (Y_j - \bar{Y}_n)(C_j - \bar{C}_n)}{\sum_{j=1}^{n} (C_j - \bar{C}_n)}$$

ESTIMATE THE INTERCEPT BY AVERAGING THE CONTROLLED VALUES:

$$\hat{\mu}_Y = \bar{Y}_n - \hat{\beta}(\bar{C}_n - \mu_C) = \frac{1}{n}\sum_{j=1}^{n} Y_j(\hat{\beta})$$

COMPUTE THE RESIDUAL MEAN SQUARE

$$\hat{\sigma}_\varepsilon = \frac{1}{(n-2)} \sum_{j=1}^{n} [Y_j(\hat{\beta}) - \hat{\mu}]^2$$

14

5. COMPUTE THE 100(1-α)% CONFIDENCE INTERVAL FOR μ_Y:

$$\hat{\mu}_Y \pm t_{1-\alpha/2}(n-2 \text{ d.f.})\cdot\hat{\sigma}_\varepsilon \cdot \left\{ \frac{\sum_{j=1}^{n}(C_j - \mu_C)^2}{n\cdot\sum_{j=1}^{n}(C_j - \bar{C}_n)^2} \right\}^{\frac{1}{2}}$$

EXTENSION TO q CONTROL VARIABLES:

NOW $\underset{\sim}{C}$ IS A $q\times1$ COLUMN VECTOR WITH KNOWN MEAN

$$\underset{\sim}{\mu}_C = \begin{bmatrix} E(C_1) \\ \vdots \\ E(C_q) \end{bmatrix}$$

AND VARIANCE–COVARIANCE MATRIX

$$\underset{\sim}{\Sigma}_C = [\, Cov(C_r, C_s) \,]$$

15

NOW $\underset{\sim}{b} = [b_1, \ldots, b_q]'$ IS A $q\times1$ COLUMN VECTOR OF CONTROL COEFFICIENTS, AND THE CONTROLLED ESTIMATOR

$$Y(\underset{\sim}{b}) = Y - \underset{\sim}{b}'(\underset{\sim}{C} - \underset{\sim}{\mu}_C)$$

HAS

$$E[Y(\underset{\sim}{b})] = \mu_Y$$

$$Var[Y(\underset{\sim}{b})] = \sigma_Y^2 - 2\underset{\sim}{b}'\underset{\sim}{\sigma}_{YC} + \underset{\sim}{b}'\underset{\sim}{\Sigma}_C \underset{\sim}{b}$$

WHERE

$$\underset{\sim}{\sigma}_{YC} = \begin{bmatrix} Cov(Y, C_1) \\ \vdots \\ Cov(Y, C_q) \end{bmatrix}$$

WITH THE OPTIMAL CONTROL VECTOR

$$\underset{\sim}{\beta} = \underset{\sim}{\Sigma}_C^{-1} \underset{\sim}{\sigma}_{YC}$$

WE HAVE

$$Var[Y(\underset{\sim}{\beta})] = Var[Y]\cdot(1-\rho_{YC}^2)$$

16

IN PRACTICE, β IS USUALLY UNKNOWN

OUR ESTIMATION PROCEDURE FOR μ_Y IS BASED ON THE JOINT NORMALITY ASSUMPTION

$$\underset{\sim}{Z} = \begin{bmatrix} Y \\ \underset{\sim}{C} \end{bmatrix} \sim N(\underset{\sim}{\mu}_Z, \underset{\sim}{\Sigma}_Z)$$

WHERE

$$\underset{\sim}{\mu}_Z = \begin{bmatrix} \mu_Y \\ \underset{\sim}{\mu}_C \end{bmatrix}, \quad \underset{\sim}{\Sigma}_Z = \begin{bmatrix} Var(Y) & \underset{\sim}{\sigma}'_{YC} \\ \underset{\sim}{\sigma}_{YC} & \underset{\sim}{\Sigma}_C \end{bmatrix}$$

THIS ENSURES THE VALIDITY OF THE MULTIPLE LINEAR REGRESSION MODEL

$$Y = \mu_Y + \underset{\sim}{\beta}'(\underset{\sim}{C} - \underset{\sim}{\mu}_C) + \varepsilon$$

$$\varepsilon \sim N(0, \sigma_\varepsilon^2)$$

$$\sigma_\varepsilon^2 = Var(Y) \cdot (1 - \rho_{YC}^2)$$

17

ESTIMATION PROCEDURE WITH q CONTROLS

1. EXECUTE n REPLICATIONS OF THE MODEL TO GENERATE THE DATA SET

$$\left\{ \begin{bmatrix} Y_j \\ \underset{\sim}{C}_j \end{bmatrix} : 1 \le j \le n \right\}$$

2. IN TERMS OF THE QUANTITIES

$$\underset{\sim}{X} = \begin{bmatrix} 1 & (\underset{\sim}{C}_1 - \underset{\sim}{\mu}_C)' \\ \vdots & \\ 1 & (\underset{\sim}{C}_n - \underset{\sim}{\mu}_C)' \end{bmatrix} \quad \text{AND} \quad \underset{\sim}{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

COMPUTE THE ORDINARY LEAST-SQUARES ESTIMATE OF THE CONTROL VECTOR $\underset{\sim}{\beta}$ AND THE INTERCEPT μ_Y

$$\hat{\underset{\sim}{\beta}}^* = \begin{bmatrix} \hat{\mu}_Y \\ \hat{\underset{\sim}{\beta}} \end{bmatrix} = (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Y}$$

3. COMPUTE THE RESIDUAL MEAN SQUARE

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{(n-q-1)} \sum_{j=1}^{n} [Y_j(\hat{\underset{\sim}{\beta}}) - \hat{\mu}_Y]^2$$

18

4. IN TERMS OF D_{11} = ROW 1, COLUMN 1 ENTRY OF $(\underset{\sim}{X}'\underset{\sim}{X})^{-1}$ COMPUTE THE $100(1-\alpha)\%$ CONFIDENCE INTERVAL FOR μ_Y:

$$\hat{\mu}_Y \pm t_{1-\alpha/2}(n-q-1 \text{ d.f.}) \cdot \hat{\sigma}_\varepsilon \cdot \{D_{11}\}^{\frac{1}{2}}$$

EFFICIENCY OF CONTROL VARIATES TECHNIQUE

$$Var[\hat{\mu}_Y(\hat{\underset{\sim}{\beta}})] = \frac{Var(Y)}{n} \cdot (1 - \rho_{YC}^2) \cdot \left(\frac{n-2}{n-q-2}\right)$$

VARIANCE WITH NO CONTROLS

MAX % VARIANCE REDUCTION IF $\underset{\sim}{\beta}$ IS KNOWN

LOSS FACTOR DUE TO ESTIMATION OF $\underset{\sim}{\beta}$

REFERENCE

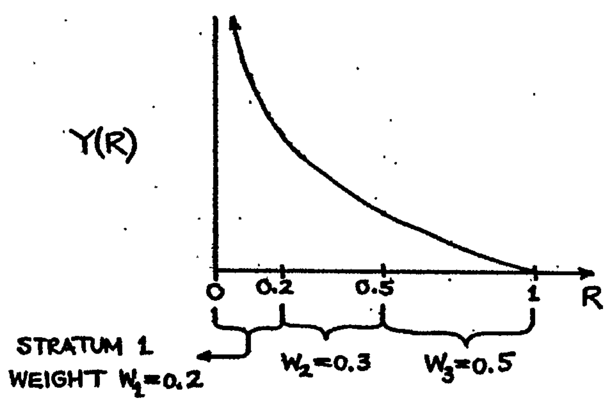
- LAVENBERG AND WELCH, "PERSPECTIVE ON USE OF CONTROL VARIABLES TO INCREASE EFFICIENCY OF MONTE CARLO SIMULATIONS," MANAGEMENT SCIENCE, VOL. 27 (1981), PP. 322-335.

19

STRATIFIED SAMPLING

SUPPOSE OUR "MODEL" HAS A SINGLE RANDOM NUMBER R FOR INPUT, AND THE OUTPUT IS AN EXPONENTIAL VARIATE Y. WE WANT TO ESTIMATE THE MEAN μ_Y

R → MODEL → $Y(R) = -\mu_Y \ln(R)$



OUT OF $n=100$ MODEL RUNS, WE FORCE THE INPUT OF n_h RUNS TO FALL IN STRATUM h, $1 \le h \le L=3$.

20

STRATUM h	SUBSAMPLE $n_h = W_h \cdot n$	STRATUM MEAN $\bar{Y}_h$
1	$Y_{11}\ Y_{12} \cdots\ Y_{1,20}$	$\bar{Y}_1 = \frac{1}{20}\sum_{j=1}^{20} Y_{1j}$
2	$Y_{21}\ Y_{22} \cdots\ Y_{2,30}$	$\bar{Y}_2 = \frac{1}{30}\sum_{j=1}^{30} Y_{2j}$
3	$Y_{31}\ Y_{32} \cdots\ Y_{3,50}$	$\bar{Y}_3 = \frac{1}{50}\sum_{j=1}^{50} Y_{3j}$

THE STRATIFIED MEAN

$$\bar{Y}_{st} = \sum_{h=1}^{L} W_h \bar{Y}_h = 0.2\bar{Y}_1 + 0.3\bar{Y}_2 + 0.5\bar{Y}_3$$

IS A MORE ACCURATE ESTIMATOR OF μ_Y THAN THE MEAN OF A SIMPLE RANDOM SAMPLE OF $n = 100$ RUNS

REFERENCE

• KLEIJNEN, J.P.C., STATISTICAL TECHNIQUES IN SIMULATION, PART I, DEKKER, 1974.

21

IMPORTANCE SAMPLING

IN THE PREVIOUS EXAMPLE, WE WANTED TO ESTIMATE

$$\mu_Y = \int_0^1 Y(r) f(r)\,dr$$

WHERE $f(\cdot)$ IS THE PROBABILITY DENSITY OF A RANDOM NUMBER

$$f(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & \text{OTHERWISE} \end{cases}$$

IF THE INPUT HAS AN ALTERNATIVE DENSITY $h(\cdot)$, WE WILL CALL IT AN "IMPORTANCE NUMBER" U

NOTE: THE RANDOM VARIATE

$$Z = \frac{Y(U)}{h(U)}$$

HAS EXPECTED VALUE

$$E[Z] = \int_0^1 Z(u) h(u)\,du = \int_0^1 \frac{Y(u)}{h(u)} \cdot h(u)\,du$$

$$= \mu_Y$$

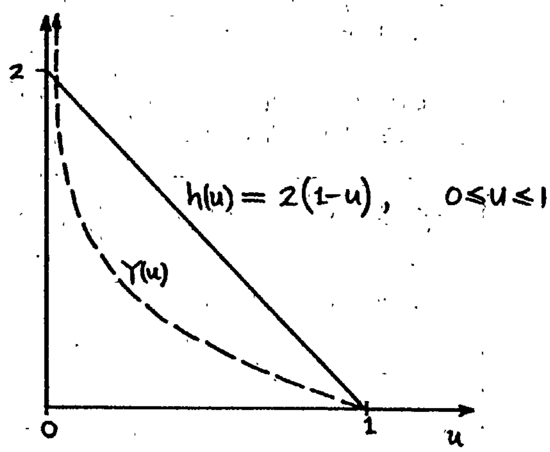
22

IF $h(u)$ CLOSELY MIMICS $Y(u)$, THEN $Z = Y(u)/h(u)$ IS NEARLY CONSTANT

$$\Rightarrow \quad \mathrm{Var}(Z) < \mathrm{Var}(Y)$$

WHEN THE IMPORTANCE DENSITY IS WELL-CHOSEN.

FOR EXAMPLE, WE MIGHT TAKE



23

IMPORTANCE SAMPLING PROCEDURE

1. GENERATE A RANDOM SAMPLE OF SIZE n FROM IMPORTANCE DENSITY $\{U_i : 1 \le i \le n\} \sim h(u) = 2(u-1)$
2. COMPUTE THE RESPONSES $$Z_i = Y(U_i)/h(U_i), \quad 1 \le i \le n$$
3. COMPUTE THE SAMPLE MEAN $$\bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$$

NOTES

1. IF $h(u) = -\ln(u)$ IN THIS EXAMPLE, $Z \equiv \mu_Y$ AND $\mathrm{Var}(Z) = 0$!
2. IF $h(\cdot)$ IS POORLY CHOSEN, WE CAN HAVE $$\mathrm{Var}(Z) > \mathrm{Var}(Y)$$

REFERENCE

• KLEIJNEN, J. P. C., STATISTICAL TECHNIQUES IN SIMULATION, PART I, DEKKER, 1974.

24

CONDITIONAL MONTE CARLO

SUPPOSE THAT WE WANT TO ESTIMATE $E[X]=\mu_X$, AND WE HAVE AN AUXILIARY VARIATE Y SUCH THAT THE CONDITIONAL EXPECTED VALUE

$$E[X \mid Y=y]$$

CAN BE EVALUATED EXACTLY FOR ALL VALUES OF y.

CONDITIONAL MONTE CARLO ESTIMATOR OF μ_X: TAKE A RANDOM SAMPLE OF THE AUXILIARY VARIATE

$$\{Y_j : 1 \le j \le n\}$$

AND COMPUTE THE SAMPLE MEAN

$$\hat{\mu}_X = \frac{1}{n}\sum_{j=1}^{n} E[X \mid Y_j]$$

25

BASIS FOR THIS METHOD:

$$E[X] = E[\, E(X|Y)\,]$$

$$Var[X] = Var[\, E(X|Y)\,] + E[\, Var(X|Y)]$$

$\Rightarrow$ THE RANDOM VARIABLE $Z \equiv E(X|Y)$ HAS

$$E[Z] = \mu_X$$

$$Var[Z] = Var[X] - E[\, Var(X|Y)] \le Var[X]$$

26

EXAMPLE 1. AN OBSERVATION PERIOD T IS EXPONENTIALLY DISTRIBUTED WITH MEAN μ_T. DURING THIS PERIOD SIGNALS ARRIVE ACCORDING TO A POISSON PROCESS WITH RATE λ. IF X IS THE TOTAL NUMBER OF OBSERVED SIGNALS, ESTIMATE $E[X]$.

$$\Rightarrow T \sim f_T(t) = \frac{1}{\mu_T}\exp\left(-\frac{t}{\mu_T}\right), \quad t \ge 0$$

$$X \sim f_X(k) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}, \quad k=0,1,\cdots$$

DIRECT SIMULATION APPROACH:

1. GENERATE A RANDOM SAMPLE $\{T_j : 1 \le j \le n\}$
2. FOR EACH j, GENERATE X_j FROM A POISSON DISTRIBUTION WITH PARAMETER λT_j
3. COMPUTE

$$\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

27

CONDITIONAL MONTE CARLO APPROACH

1. GENERATE A RANDOM SAMPLE $\{T_j : 1 \le j \le n\}$
2. NOTE $$E[X_j \mid T_j] = \lambda T_j, \quad 1 \le j \le n$$
3. COMPUTE $$\hat{\mu}_X = \frac{1}{n}\sum_{j=1}^{n} E[X_j|T_j] = \lambda \bar{T}_n$$

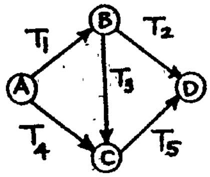
NOTICE ALSO

$$Var(X_j|T_j) = \lambda T_j$$

$$\Rightarrow E[\,Var(X_j|T_j)] = \lambda \cdot E[T_j] = \lambda \mu_T > 0$$

$$\Rightarrow Var(\hat{\mu}_X) = Var(X) - E[\, Var(X|T)] > Var(X)$$

EXAMPLE 2: ESTIMATING DISTRIBUTION OF COMPLETION TIME IN A PERT NETWORK



T_1, T_2, T_3, T_4, T_5 ARE INDEPENDENT RANDOM VARIABLES WITH KNOWN DISTRIBUTIONS F_1, F_2, F_3, F_4, F_5 RESPECTIVELY

PATH TIMES:

$$Z_1 = T_1 + T_2$$
$$Z_2 = T_1 + T_3 + T_5$$
$$Z_3 = T_4 + T_5$$
$$X = \max\{Z_1, Z_2, Z_3\}$$

WE WANT TO ESTIMATE $F_X(t) = Pr\{X \leq t\}$

NOTICE:

$$F_X(t \mid T_1 = t_1, T_5 = t_5)$$
$$= Pr\{T_2 \leq t - t_1\} * Pr\{T_3 \leq t - t_1 - t_5\} * Pr\{T_4 \leq t - t_5\}$$
$$= F_2(t - t_1) * F_3(t - t_1 - t_5) * F_4(t - t_5)$$

SAMPLING PROCEDURE:

1. GENERATE A RANDOM SAMPLE OF n PAIRS $\{[T_{1j}, T_{5j}] : 1 \leq j \leq n\}$
2. AVERAGE THE CONDITIONAL PROBABILITIES

$$\hat{F}_X(t) = \frac{1}{n} \sum_{j=1}^{n} F_X(t \mid T_{1j}, T_{5j})$$

REFERENCES

- CARTER AND IGNALL, "VIRTUAL MEASURES: A VARIANCE REDUCTION TECHNIQUE FOR SIMULATION," MANAGEMENT SCIENCE, VOL. 21 (1975), PP. 607–616.
- M^C GRATH AND IRVING, "APPLICATION OF VARIANCE REDUCTION TO LARGE SCALE SIMULATION PROBLEMS," COMPUTERS & OPER. RES., VOL. 1 (1974), PP. 283–311.