SUPER MANN-WHITNEY SIMULATION OF SYSTEM RELIABILITY*

Laurence L. George

Lawrence Livermore National Laboratory
P. O. Box 808, L-140
Livermore, CA 94550 USA

Abstract

Simulation of system reliability requires an enormous amount of computer time if system reliability is high. If the simulation generates values of component loads and strengths, the values from different runs may be combined to increase the number of runs without generating any more values. The reliability estimator is the proportion of runs without system failure. It is unbiased regardless of the run combining methods. The variance of the reliability estimator can be derived from the variance of Mann-Whitney test statistic or by a jackknife estimator.

Run combining methods mix simulated loads and strengths from different runs. If only loads are combined, the method is called Mann-Whitney simulation. This is because, if the system has one component, the reliability estimator is related to Mann-Whitney test statistic. If all of loads and strengths are combined, the method is superior to the Mann-Whitney simulation because far more samples are generated. This method is called Super Mann-Whitney simulation.

Mann-Whitney and Super Mann-Whitney simulations are practical only if random variable generation is hard.

INTRODUCTION

Component reliability is the probability of component survival. Component survival can be modeled as occurring when load is less than strength (1). System reliability is the probability of survival of some subsets of components. A fault tree specifies which subsets are required for survival. System reliability is modeled this way in the Seismic Safety Margins Research Program (2).

System reliability computation is difficult because:

1. there are many components,
2. there are many subsets whose survival causes system survival, and
3. loads and strengths are dependent among themselves and on each other.

System reliability simulation is an alternative, but simulation requires a large number of simulation runs if component reliabilities are high. (Figure 1 shows system reliability simulation.) Mann-Whitney simulation (3) requires far fewer simulation runs. (Figure 2 shows Mann-Whitney simulation.) It re-uses every one of the n**2 combinations of simulated loads and strengths to estimate system reliability. Basu (4) uses this technique to estimate component reliability. Super Mann-Whitney simulation is the same as Mann-Whitney simulation except that it uses all combinations of load-strength pairs. (Figure 3 shows Super Mann-Whitney simulation.)

For example, suppose there are m=2 components, n=2 loads and strengths for each component, and n**2=4 load-strength pairs for each component. There are n**(2*m)=16 pairs of pairs, and each yields a distinct system state, survival or failure. (Figure 4 lists all combinations.) Check each component to see whether its load X is less than its strength Y, then check to see whether at least one subset of components required for system survival has loads less than strengths. If so, count one more survival.

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Super Mann-Whitney Simulation (continued).

FIGURE 1

Simple Simulation of Mechanical System Reliability

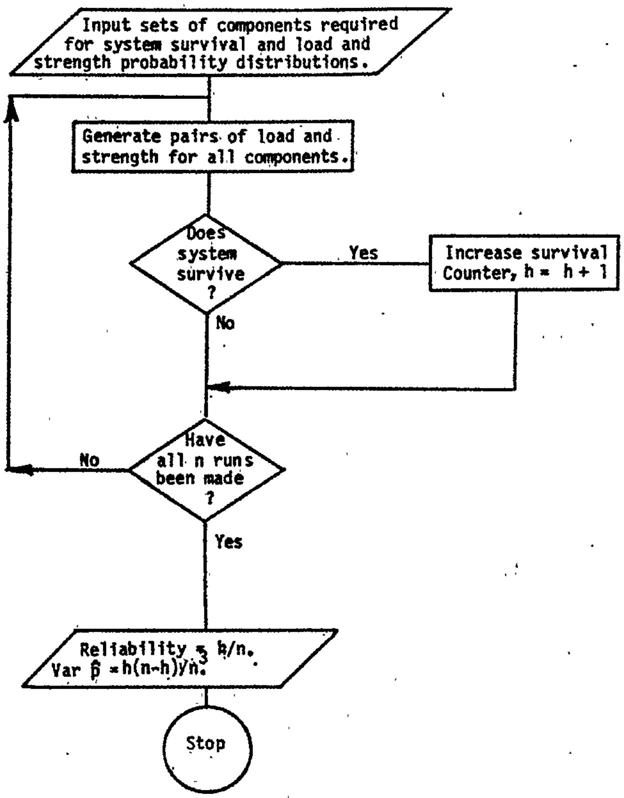

FIGURE 2

Mann-Whitney Simulation of System Reliability

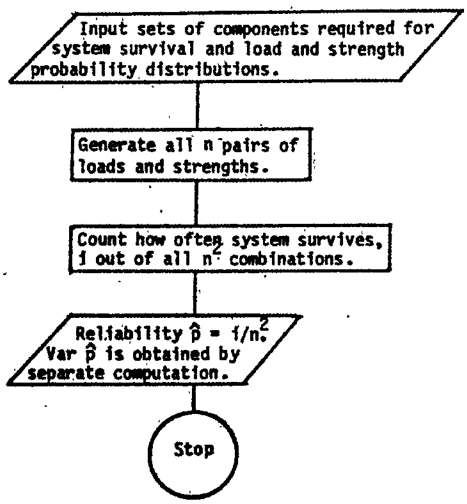

FIGURE 3

Super Mann-Whitney Simulation of System Reliability

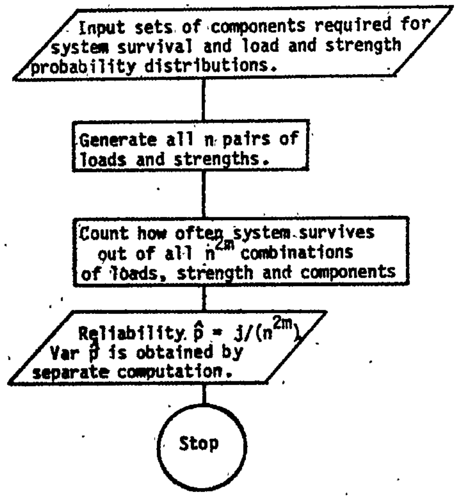

FIGURE 4

Combinations of Loads and Strengths for Two Components and Two Runs

Index	Loads		Strengths		
1	X_{11}*	X_{21}	Y_{11}	Y_{21}	Simple Simulation
2	X_{12}	X_{22}	Y_{12}	Y_{22}	Mann-Whitney Simulation
3	X_{11}	X_{21}	Y_{12}	Y_{22}	
4	X_{12}	X_{22}	Y_{11}	Y_{21}	
5	X_{11}	X_{21}	Y_{11}	Y_{22}	
6	X_{11}	X_{21}	Y_{12}	Y_{21}	
7	X_{11}	X_{22}	Y_{11}	Y_{21}	
8	X_{12}	X_{21}	Y_{11}	Y_{21}	Super Mann-Whitney Simulation
9	X_{11}	X_{22}	Y_{12}	Y_{21}	
10	X_{11}	X_{22}	Y_{11}	Y_{22}	
11	X_{12}	X_{21}	Y_{12}	Y_{21}	
12	X_{12}	X_{21}	Y_{11}	Y_{22}	
13	X_{12}	X_{22}	Y_{12}	Y_{21}	
14	X_{12}	X_{22}	Y_{11}	Y_{22}	
15	X_{12}	X_{21}	Y_{12}	Y_{22}	
16	X_{11}	X_{22}	Y_{12}	Y_{22}	

*The first subscript is the component number and the second is the run number.

SYSTEM RELIABILITY ESTIMATION

The proportions of survivals in all simulation runs are system reliability estimators. For simple simulation of n independent runs with h survivals, the estimator is h/n. For Mann-Whitney simulation of n**2 runs with i survivals, the estimator is i/(n**2). For Super Mann-Whitney simulation of n**(2*m) runs with j survivals, the estimator is j/(n**(2*m)).

Estimators should be accompanied by indications of their accuracy. The proportions are unbiased estimators. Variances of the proportions indicate the accuracy of the proportion estimators. For simple simulation, the variance of the proportion is estimated by h(n-h)/(n**3). For Mann-Whitney simulation, the variance of the proportion is derived from the variance of the Mann-Whitney test statistic (5), n*n*(2n+1)/12, when the loads and strengths have the same distribution functions (except for a location parameter). For Super Mann-Whitney simulation, the variance of the proportion is estimated by a jackknife (6). The jackknife requires computing the proportion estimator many times where each computation is based on all but one run. The variance estimator is a function of the proportion estimators (6).

Confidence intervals on system reliability can be estimated from the distribution function of the proportion of survivals. The exact distribution of the Mann-Whitney test statistic can be computed for small n (7). The asymptotic distribution for large n is normal. For Super Mann-Whitney simulation, the distribution of the proportion depends on the definition of system survival. The distribution can be approximated using extreme value theory of dependent random variables for series or parallel systems (8).

TESTS

The following alternatives were tested:

1. DUMSIM - simple simulation as in Figure 1 using "IF" statements to determine whether X>Y for component failure.
2. EFFSIM - simple simulation of a series system using a vector minimization on the vector Z=X-Y. If Z>0, the system survives.
3. MWSIM - Mann-Whitney simulation of system reliability as in Figure 2.
4. DUMSIMJK - the same as DUMSIM with a jackknife variance estimator.
5. MWSIMJK - Mann-Whitney simulation of system reliability with a jackknife variance estimator.
6. SUPERMANN - Super Mann-Whitney system reliability simulation as in Figure 3.

The numbers of runs were 11, 25, 100. The probabilities of component failures (independent events) were 0.5. All systems were components in series. The numbers of components were 1, 2 and 4 so the correct reliabilities were 0.5, 0.25 and 0.0625.

Table 1 shows the results. EFFSIM was not included because its execution time was worse than DUMSIM. Vector minimization takes more time than several "IF" statements. The reliability estimator is $\hat{p}$. Its variance is estimated by $\hat{\sigma}_{\hat{p}}$ or $\hat{\sigma}_{\hat{p}}$(JK). The run times for random variable generation, $\hat{p}$ computation and the jackknife variance estimation are given in microseconds on a CDC 7600 computer. Core requirement was about 3 percent.

CONCLUSIONS

1. Mann-Whitney simulation of single component reliability reduces variance a little but requires about 10 times the CPU time.
2. Mann-Whitney plus Jackknife reduces variance at a huge cost of CPU time.
3. Jackknife alone does not reduce variance.
4. Super Mann Whitney simulation requires a horrible amount of CPU time.

Therefore, use Mann-Whitney or Super Mann-Whitney simulation <u>only</u> if random variables are hard to generate.

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TABLE 1

Test Results

PROGRAM	NUMBER OF COMPONENTS IN SERIES	NUMBER OF RUNS	$\hat{p}$	$\hat{\sigma}_{\hat{p}}$	$\hat{\sigma}_{\hat{p}}$(JK)	μ_{sec}(RN)	μ_{sec}(P)	μ_{sec}(JK)
DUMSIM	1	25	.48	.09992+	---	4307	175	---
DUMSIM	1	100	.46	.0498	---	16486	527	---
MWSIM	1	25	.4752	.0825	---	4178	2589	---
MWSIMJ	1	100	.5175	.0409		16598	40583	
DUMSIMK	1	25	.48	.0992+	.10199	4182	174	3041
MWSIMJK	1	25	.4752		.01999	4401	2666	2,387,579
DUMSIM	4	25	.04	.0392+	---	16704	301	---
MWSIM	4	25	.0992		---	16372	5993	
DUMSIMK	4	25	.04	.0392+	.04	16413	301	6179
MWSIMJK	1	25	.4752	.0825	.01999	4227	2603	
MWSIMJK	4	25	.0992		.0120	16674	6007	4,499,250
MWSIMJ	4	25	.0992			16303	6089	
DUMSIMK	4	100	.0700	.02551	.02564	64536	993	
SPRMNN	2	25	.2637		---	8095	1550123	
SPRMNN	4	11	.117			7324	1,388,394,095	

* The Mann-Whitney estimate of asymptotic standard deviation of $\hat{p}$ is correct only when the number of components is 1.

\+ For DUMSIM, true $\hat{\sigma}_{\hat{p}} = \sqrt{p(1-p)/25} = .1$ when the number of components = 1 and $\hat{\sigma}_{\hat{p}} = .0484$ when the number of components = 4.