A REGRESSION TECHNIQUE FOR DETERMINING STEADY STATE CONDITIONS IN TIME SERIES SIMULATIONS

Charles W. Beall

Department of Decision Sciences
University of Hawaii
Honolulu, Hawaii 96822

Abstract

In many situations, the simulator finds himself faced with the problem of simulating a time series whose initial state is different from the state of the system after a large number of observations have been made. This is frequently referred to as the problem of the initial bias or the initial tranient problem. In this paper, we borrow from the theory of convergence in distribution to develop a criterion for convergence to a steady state process and then develop estimation techniques based on simple linear regression to determine when the time series under investigation has converged to a steady state process.

1. INTRODUCTION

In simulating a discrete stochastic process, $\{X_n; n=1,2,\ldots\}$, it is frequently desired to obtain estimates of the "steady state" statistics of the process $\{X_n\}$. That this process has values at discrete points in "time," as the index parameter, n, shall be considered, may be either a natural consequence of the process being simulated or may result from sampling a continuous process at discrete points in time. In either case, this type of process represents a large class of problems encountered in digital computer based simulations.

While the theory for obtaining estimates from samples from covariance stationary stochastic processes is relatively well developed, the theory for obtaining estimates for processes with an initial bias is not. The initial state of the process is considered to be X_0. The problem of the initial bias arises from the fact that the statistics of the random variable X_0, as characterized by its cumulative distribution function, are generally different from those exhibited by the random variables X_n as the time index, n, grows arbitrarily large. This problem arises naturally in most simulations of random processes since the simulator must choose to start the process in some particular state, and, in general, the simulator has no prior precise knowledge of the long run behavior of the stochastic process being simulated. In other cases, the initial state of the process arises naturally from the definition of the system being simulated.

Conway (3) was among the first to comment on the problem of the initial bias as it affects the simulator's ability to estimate the long run mean of the stochastic process under investigation. Other pioneering efforts by Fishman (4, 5) and Gordon (9) developed additional methods for the same purpose. The methods developed by these researchers together with some variants of those methods were investigated by Gafarian, Ancker, and Morisaku (8) with the general conclusion that all were lacking in terms of one or more of several performance measures. Other, more recent investigations have been conducted by Kelton (10), Gafarian (7), Law (11), Law and Carson (12), and Schruben (13). Much of this research has been conveniently summarized by Wilson and Pritsker (14, 15). Most of this research has concentrated on the issues surrounding the estimation of the steady state mean of a process with initial bias.

Much of the prior research on the problem of the initial bias has used the fact that the mean of the stochastic process approached some limiting value as the definition of the term "steady state." The notion of a stochastic process approaching covariance stationarity has been used to make inferences regarding estimators for the long run mean. This research specifically embraces the definition of covariance stationarity of defining the term, "steady state." More precisely, a stochastic process will be said to possess a steady state if, as $n \to \infty$, the process approaches covariance stationarity; or:

$$\lim_{n \to \infty} E[X_n] = \mu \qquad (1)$$

$$\lim_{n \to \infty} \mathrm{Cov}(X_n, X_{n+k}) = R(k) \qquad (2)$$

Proceedings of the 1982
Winter Simulation Conference
Highland * Chao * Madrigal, Editors

82CH1844-0/82/0000-0439 $00.75 © 1982 IEEE

The particular definition for steady state is chosen with specific, and fairly obvious, intent. If a stochastic process possesses the quality of being covariance stationary, then the great body of theory that is available for use with covariance stationary processes can be brought to bear on the estimation process. Since (1) and (2) are true only in the limit, the problem of determining when a process with an initial bias is a steady state process may be conveniently summarized as that of finding a value of n such that (1) and (2) are, by some measure, "close enough" to being true to allow all observations which follow time n to be treated as though they come from a covariance stationary process. It is fairly obvious that this is a problem involving convergence concepts. That is, the problem to be solved is that of determining a value of n such that the process converges to a covariance stationary process with sufficient "closeness" to allow spectral theory and other techniques applicable to covariance stationary processes to be applied.

As is usually the case, the first step in this study was the selection of a model from which to develop theoretical results. The criteria used in model selection were: (a) the model must be capable of representing a process or processes which have an initial bias but which tend toward covariance stationarity; (b) the model must be straightforward enough to allow meaningful results to be obtained; and (c) the model must provide sufficient theoretical basis to serve as a guideline in the analysis of more complex models. The selected model was the first-order autoregressive process.

THE FIRST-ORDER AUTOREGRESSIVE PROCESS WITH AN INITIAL BIAS

The first-order autoregressive process provides a convenient model from which to begin the development of the theory for determining when a process is close to steady state. Though the model is a relatively simple one, it may be used to represent several types of processes which are generally of interest to the simulator. The theory for the first-order autoregressive process without a particular starting condition, that is, without an initial bias is very well developed by Box and Jenkins (2). The conditions for which the process is covariance stationary are straightforward; hence, for the process without an initial bias, under these conditions the process will, by (1) and (2) possess a steady state.

The generalized first-order autoregressive process as defined by Box and Jenkins is:

$$\tilde{X}_n = a\,\tilde{X}_{n-1} + E_n;\ n = \ldots, -1, 0, 1, 2, \ldots \quad (3)$$

together with:

$$\tilde{X}_n = X_n - \mu;\ \text{for all } n \quad (4)$$

so that:

$$X_n - \mu = a\,(X_{n-1} - \mu) + E_n$$

or:

$$X_n = aX_{n-1} + (1-a)\mu + E_n \quad (5)$$

Stationarity is ensured if $|a| < 1$. The parameter, μ, is called the level parameter, and, if the process is stationary, then:

$$E[X_n] = \mu \text{ for all } n \quad (6)$$

The process, E_n, is a sequence of independent, identically distributed random variables from the normal distribution with mean zero and variance, σ_E^2, which will be denoted NIID $(0, \sigma_E^2)$. Finally, the autocorrelation function for this process is given by:

$$\text{corr}(X_n, X_{n+k}) = r(k) = a^{|k|} \quad (7)$$

which clearly shows the covariance stationarity property.

In order to obtain a process with an initial bias for further analyses, a modification of the process given in (3), (4), or (5) is considered. This modification consists of two changes. First, the time index, n, is restricted to the positive integers and zero. Second, it is assumed that when $n = 0$, the distribution of the initial state of the process, X_0, is completely known to the simulator. It is this model which will be referred to as the AR(1) in this paper. Since the mean of this process approaches μ as $n \to \infty$, as will be shown later, it seems reasonable to define the deviation from the long run mean at the initial time, $n = 0$, as the initial bias. Thus:

$$\tilde{X}_0 = X_0 - \mu \quad (8)$$

is defined to be the initial bias and it is, in general, a random variable. Except for the change in the time index, relations (3), (4), and (5) still hold since they are simply different ways of expressing the same model. However, the mean value function and autocorrelation function of (6) and (7) respectively will not, in general apply to the AR(1) process as defined here.

Since (3) is the simplest form of the AR(1) model, it is generally easier to analyze the $\{\tilde{X}_n\}$ process

than the $\{X_n\}$ process. This technique will be used first to obtain another version of the AR(1) model which directly gives the relationship between any random variable in the process $\{\tilde{X}_n\}$ and the initial bias, $\tilde{X}_0$. By applying (3) repeatedly, it is possible to obtain by induction that:

$$\tilde{X}_n = a^n X_0 + \sum_{j=0}^{n-1} a^j E_{n-j} \tag{9}$$

Substituting (4) in (9) leads to:

$$X_n = a^n X_0 + (1-a^n)\mu + \sum_{j=1}^{n} a^{n-j} E_j \tag{10}$$

In order to investigate the long run behavior of $\tilde{X}_n$ (and, at the same time, X_n), it is first necessary to determine and distribution functions for each $\tilde{X}_n$ and X_n. We will begin by considering the $\{\tilde{X}_n\}$ process. The characteristic function for $\tilde{X}_n$ will be developed from the characteristic function for $\tilde{X}_0$ and E_n. The characteristic function E_n is:

$$\phi_{E_n}(u) = E[\exp(iuE_n)] \text{ for all } n \tag{11}$$

where $i = \sqrt{-1}$. However, since the sequence (E_n) is NIID$(0, \sigma_E^2)$, we have:

$$\phi_{E_n}(u) = \exp(-\sigma_E^2 u^2/2) \text{ for all } n \tag{12}$$

Similarly, the characteristic function for $\tilde{X}_0$ is:

$$\phi_{\tilde{X}_0}(u) = E[\exp(iu\tilde{X}_0)] \tag{13}$$

which cannot be further specified since no distribution function for $\tilde{X}_0$ has yet been assumed. Then general form of the characteristic function for $\tilde{X}_n$ is:

$$\phi_{\tilde{X}_n}(u) = \phi_{\tilde{X}_0}(a^n u) \prod_{j=1}^{n} \phi_{E_n}(a^{n-j}u) \tag{14}$$

Since $\phi_{E_n}(u)$ is known from (12), if the characteristic function for $\tilde{X}_0$ were known, then the characteristic function for $\tilde{X}_n$ could be determined. Then, in principal, the distribution function for $\tilde{X}_n$ can be found by taking the inverse transform. Thus, before more specific results can be obtained, some assumptions must be made about the distribution of $\tilde{X}_0$. In this paper we will consider only the case where X_0 is known and nonrandom. We let $\tilde{X}_0 = \tilde{x}_0$ with probability 1. Then the characteristic function for $\tilde{X}_0$ is simply:

$$\phi_{\tilde{X}_0}(u) = \exp(iu\tilde{x}_0) \tag{15}$$

Then, deriving the characteristic function for $\tilde{X}_n$:

$$\phi_{\tilde{X}_n}(u) = \exp\left\{ i(a^n\tilde{x}_0)u - \sigma_E^2 u^2 [(1-a^{2n})/(1-a^2)]/2 \right\} \tag{16}$$

where (16) is recognizable as the characteristic function of a normally distributed random variable with mean $\tilde{\mu}_n$ and variance $\tilde{\sigma}_n^2$ where:

$$\tilde{\mu}_n = a^n\tilde{x}_0 \tag{17}$$

$$\tilde{\sigma}_n^2 = \sigma_E^2(1-a^{2n})/(1-a^2) \tag{18}$$

For the original process $\{X_n\}$, if $\tilde{X}_n = X_n + \mu$, then each X_n is normally distributed with mean $\tilde{\mu}_n + \mu$ and variance $\tilde{\sigma}_n^2$.

Finally, we take the limit in (16) and let:

$$\phi_X(u) = \lim_{n \to \infty} \phi_{X_n}(u)$$

$$= \exp\{-\sigma_E^2 u^2/[2(1-a^2)]\} \tag{19}$$

Then $\tilde{X}$ will have the normal distribution with mean zero and variance $\sigma_E^2/(1-a^2)$. Since $\phi_{\tilde{X}_n}$ converges to $\phi_{\tilde{X}}$, we also have that $\tilde{X}_n \to \tilde{X}$ in distribution. And, since we can also define $X = \tilde{X} + \mu$, we will have $X_n \to X$ in distribution where X is normal with mean μ and variance $\sigma_E^2/(1-a^2) = \sigma^2$.

We now proceed to apply directly the steady state definitions of (1) and (2) to the AR (1) model. It is noted for the moment that condition (1) is satisfied if the process converges in distribution as discussed previously. Concentration will now be centered on definition (2) in order to investigate how the autocovariance of the process $\{X_n\}$ might approach the steady state autocovariance function $R(k)$.

The autocovariance of the $\{\tilde{X}_n\}$ process is the same as the autocovariance of the $\{X_n\}$ process. Then:

$$\text{Cov}(X_n, X_{n+k}) = \text{Cov}(\tilde{X}_n, \tilde{X}_{n+k})$$

$$= E[\tilde{X}_n \tilde{X}_{n+k}] - \tilde{\mu}_n \tilde{\mu}_{n+k} \tag{20}$$

where $\tilde{\mu}_n = a^n\tilde{x}_0$ and $\tilde{\mu}_{n+k} = a^{n+k}\tilde{x}_0 = a^n a^k \tilde{x}_0 = a^k \tilde{\mu}_n$. Then, continuing with this derivation.

$$E[\tilde{X}_n \tilde{X}_{n+k}] = a^{2n+k} E[\tilde{X}_0^2] + a^n \sum_{i=1}^{n+k} a^{n+k-i} E[\tilde{X}_0 E_i]$$

$$+ a^{n+k} \sum_{j=1}^{n} a^{n-j} E[\tilde{X}_0 E_j]$$

$$+ \sum_{j=k}^{n} \sum_{i=1}^{n+k} a^{n-j+n+k-i} E[E_j E_i] \qquad (21)$$

where, in (21), we have $E[\tilde{X}_0 E_j] = E[\tilde{X}_0 E_i] = 0$ for all i and j since $\tilde{X}_0$ and any of the noise random variables are independent and since the noise is NIID $(0,\sigma_E^2)$. Furthermore, the term in the last summand, $E[E_j E_i]$ is just σ_E^2 if j=i and is zero otherwise. Therefore:

$$E[\tilde{X}_n, \tilde{X}_{n+k}] = a^{2n+k} E[\tilde{X}_0^2] + a^{2n+k} \sigma_E^2 \sum_{j=1}^{m} a^{-2j} \qquad (22)$$

In (22), m = min(n,n+k), (22) reduces further to:

$$E[\tilde{X}_n, \tilde{X}_{n+k}] = a^{2n+k} \left\{ E[\tilde{X}_0^2] + \sigma_E^2 \cdot [1-a^{2m}]/[a^{2m}(1-a^2)] \right\} \qquad (23)$$

Finally, (23) is substituted into (20):

$$\mathrm{Cov}(\tilde{X}_n, \tilde{X}_{n+k}) = a^{2n+k} \left\{ \mathrm{Var}[X_0] + (a^{-2m}-1)\sigma_E^2/(1-a^2) \right\} \qquad (24)$$

If we substitute for m using the defining relationship m = min(n,n+k), and consider the cases k < 0 and k > 0 separately, ultimately we will obtain:

$$\mathrm{Cov}(\tilde{X}_n, \tilde{X}_{n+k}) = \sigma_E^2 (a^{|k|} - a^{2n+k})/(1-a^2) + a^{2n+k} \mathrm{Var}[\tilde{X}_0] \qquad (25)$$

If we take the limit in (25), we obtain:

$$\lim_{n \to \infty} \mathrm{Cov}(\tilde{X}_n, \tilde{X}_{n+k}) = \sigma_E^2 a^{|k|}/(1-a^2) = \sigma^2 a^{|k|} = \sigma^2 r(k) \qquad (26)$$

where r(k) is the correlation function of the generalized first-order autoregressive process.

Thus, the autovariance function for the first-order autoregressive process with an initial bias does indeed approach that of the generalized first-order autoregressive process.

It is critical to note that the convergence of the autocovariance function as indicated in (25) occurs at a rate which is determined by the factor a^{2n}. The other condition for autocovariance stationarity, as expressed in (1) requires convergence of the first moment, which, as noted in (17) converges at a rate determined by a^n. We further require that $|a| < 1$ for both types of convergence and for autocovariance stationarity. If we compare these rates with the rates of convergence for convergence in distribution we will note that for convergence in distribution, we require that the mean of the normal distribution converge at a rate dependent upon a^n and that its variance converge at a rate dependent upon a^{2n}. Thus, both convergence in distribution and convergence to the autocovariance function will occur at the same rates. However, any practical application of the property that the process converges to its steady state autocovariance function would require some sort of successive computation of the autocovariance function for all possible lags. However, since the process converges in distribution at a comparable rate, it seems reasonable to search for a computational procedure based on the concept of convergence in distribution.

Since the AR(1) process converges in distribution, the notion that is exploited here is that it is possible, given a, μ, and σ_E^2, to find a value of n such that the distribution of X_n is, by some measure, close to the distribution of X, the limiting random variable. If we denote by n* the value of n such that the simulator is satisfied that the process is close enough to steady state, then a test, based on the concept of convergence in distribution is to find n* such that for all $n \geq n^*$, we have:

$$\max |F_n(x;a,\mu,\sigma_E^2) - F(x;a,\mu,\sigma_E^2)| < \varepsilon \qquad (27)$$

where expressing the dependence of both distribution functions on a, μ, and σ_E^2 is for emphasis. The simulator must specify the value of ε, which is, in effect, a "tightness" criterion for the test. However, in the case where the differences are between normal distributions, this should be fairly straightforward-since ε is a determination of closeness in probability. Now, since we have that:

$$|\mu_n - \mu| \to 0 \text{ monotonically} \qquad (28)$$

$$\sigma_n^2 \to \sigma^2 \text{ monotonically} \qquad (29)$$

then $|F_n(x; a, \mu, \sigma_E^2) - F(x; a, \mu, \sigma_E^2)| \downarrow 0$ monotonically for all values of x. Therefore, n* is the first (i.e., lowest value of n such that (27) becomes true. So, the computational procedure is, given a, μ, and σ_E^2, evaluate (27) for successive values of n until the condition is satisfied. This will determine the value of n*

Generally, this would require an evaluation of F_n and F for all (infinitely many) values of x. However, as developed by Beall (1), the maximum of the absolute difference between the two distribution functions must occur at one of two values of x:

$$x_a, x_b = \frac{-b \pm \sqrt{b^2 - 4dc}}{2d} \text{ ; if } d = 0 \quad (30)$$

where

$$d = \sigma_n^2 - \sigma^2 \quad (31)$$

$$b = 2(\mu_n\sigma^2 - \mu\sigma_n^2) \quad (32)$$

$$c = \mu^2\sigma_n^2 - \mu_n^2\sigma^2 - 2\sigma^2\sigma_n^2 \log(\sigma_n/\sigma) \quad (33)$$

and where special cases are discussed in the reference.

It is noted that given x_0, μ, and σ_E^2, n* is a function only of ε, |a| and a term which will be referred to as the bias-to-noise ratio:

$$\text{Bias-to-Noise ratio} = |x_0 - \mu|/\sigma_E \quad (34)$$

Theoretical values of n* for two examples are shown in Figures 1 and 2.

AN ESTIMATION TECHNIQUE FOR DETERMINING STEADY STATE CONDITIONS

In the previous section, a technique was developed for determining an index, n*, given a simulator-specified criterion for tightness, ε, given the distribution of the initial state of the system, X_0, and given the AR(1) model parameters a, μ, and σ_E^2. However, these parameters are, in general, unknown and must be inferred from some sequence(s) of observations of a given AR(1) process. We must then determine some method of estimating these parameters in order to apply the convergence test. Since the process is based on an auto-regressive model, some technique for estimation based on regression analysis would seem to be a logical candidate.

The general form of the regression model is very straightforwardly obtained from the previously given form of the AR (1) model. Recall from (5) that:

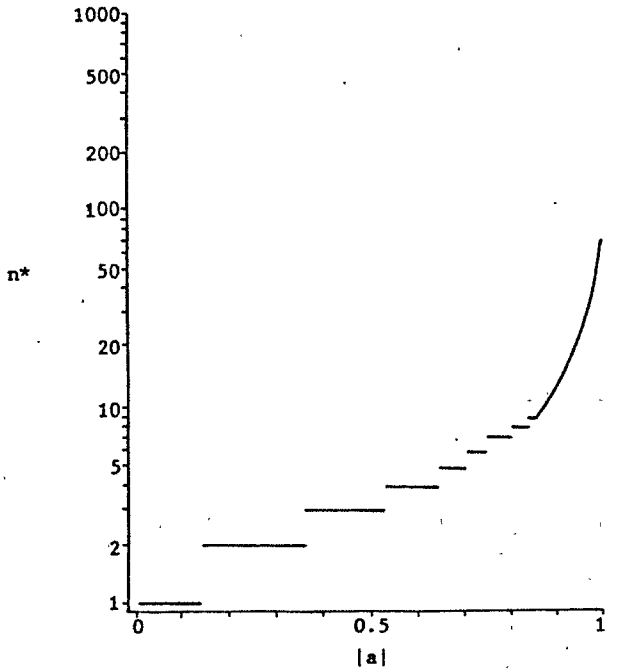


Figure 1. Theoretical values of n* for ε = 0.05 with a bias-to-noise ratio of 1.

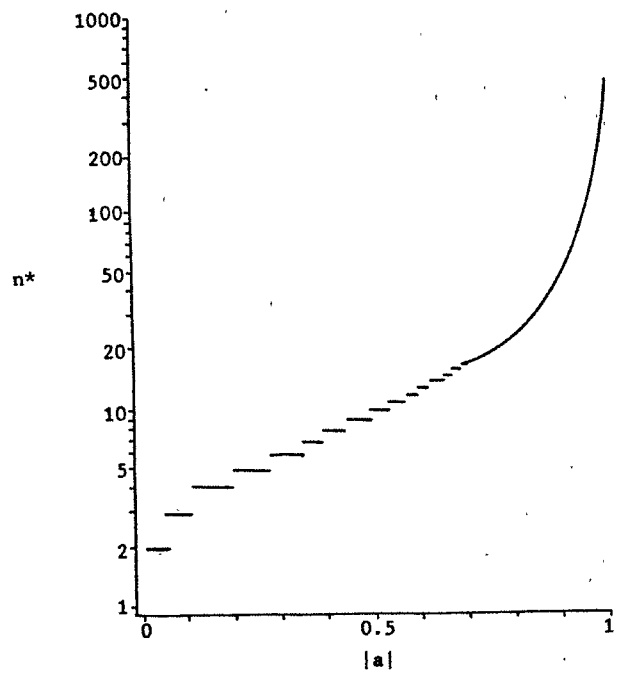


Figure 2. Theoretical values of n* for ε = 0.05 with a bias-to-noise ratio of 100.

$$X_n = aX_{n-1} + (1-a)\mu + E_n \qquad (35)$$

Conditioning each random variable on the previous observation, we obtain:

$$Y_n = X_{n+1} | X_n = x_n$$

$$= ax_n + (1-a)\mu + E_n$$

$$= ax_n + \beta_0 + E_n \qquad (36)$$

where (36) is easily recognizable as a general form of a linear model with slope, $\beta_1 = a$, and with an intercept given by:

$$\beta_0 = (1-a)\mu \qquad (37)$$

Then, using techniques similar to those described in Fuller (6), the model of (36) may be applied directly to obtain the desired estimates.

We will assume henceforth that all estimation in this process is developed from observations taken from a single replication of the AR(1) process. That is, we have a sequence of observations $(x_n; n=1,2,3..., N)$ from the sequence of random variables $(X_n; n=1,2,3..., N)$. Based on (36), we will also have a sequence of observations, $(y_n; n=1,2,3,..., N-1) = (x_n; n=2,3,4,..., N)$ from the random variables $(Y_n; n=1, 2,3,..., N-1)$. Therefore, after N observations have been taken from the AR(1) process, these observations may be grouped into the m=N-1 point pairs: (x_1,y_1), $(x_2,y_2),..., (x_m,y_m)$. This set of pairs of observations is the same as the set: (x_1,x_2), (x_2,x_3), ..., (x_{N-1},x_N). Then, the estimates are the usual set:

$$a = \left[\sum_{i=1}^{m} (x_i-\bar{x})(y_i-y)\right] \Big/ \left[\sum_{i=1}^{m} (x_i-\bar{x})^2\right] \qquad (38)$$

and,

$$\hat{\beta}_0 = \bar{y} - \hat{a}\bar{x} \qquad (39)$$

where (38) $\bar{x}$ and $\bar{y}$ are the sample means.

From (37), (38), and (39), an estimate for μ is simply:

$$\hat{\mu} = \hat{\beta}_0 / (1-\hat{a})$$

The estimate for σ_E^2 is easily developed from the residuals of the regression process as:

$$\sigma_E^2 = (m-2)^{-1} \sum_{i=1}^{m} (y_i-\hat{y}_i)^2 \qquad (41)$$

$$\hat{y}_i = \hat{a} + \hat{\beta}_0 x_i; \; i=1,2,..., m \qquad (42)$$

These estimators are based entirely on the "traditional" estimators obtained using linear regression methods. The reference above to the fact that each Y_i is completely determined by an x_i which is non-random after the i^{th} observation and by a random noise term E_i, is the key to being able to apply these estimation techniques. However, due to the correlation between successive observations, the usual properties of these estimators (unbiasedness, variances and so forth) are no longer straightforward.

A MONTE CARLO EVALUATION OF THE ESTIMATORS

Having established an estimation process for determining when an AR(1) process reaches steady state (or, determining when it does not possess a steady state), it remains to validate the techniques previously described through some experimental process. Further the estimator, $\hat{N}^*$, has an, as yet, undefined performance. In order to perform these validations and assessments, a series of Monte Carlo computer simulation runs were made.

An evaluation of the performance of the estimates $\hat{a}$, $\hat{\mu}$, and $\hat{\sigma}_E^2$ was conducted by performing a single Monte Carlo replication of an AR(1) experiment. Several cases were examined by varing the auto-regressive parameter, a, and the bias-to-noise ratio. Typical results of these trials are shown in Figures 3 through 6. For each case, the results are shown by plotting the estimates $\hat{a}$, $\hat{\mu}$, and $\hat{\sigma}_E^2$ which are obtained using the linear regression methods. Each estimate is shown as it was computed after each new sample point is added to the regression analysis. Finally, each set of figures contains the estimates for n* (i.e., $\hat{n}^*$) based on $\hat{a}$, $\hat{\mu}$, and $\hat{\sigma}_E^2$ at each value of n. Obviously, each set of estimates can be computed only when a sufficient number of observations has been made (three for a and $\hat{\mu}$, four for $\hat{\sigma}_E^2$ and thus for $\hat{N}^*$).

The results tend to verify that the regression method will ultimately produce a fairly good set of estimates of the parameters of the process. In order to compute $\hat{n}^*$ for each of the cases examined, the value of ε=0.05 was used. For Figures 3 and 4, the theoretical value of n* is zero since the bias-to-noise ratio is zero and the process starts in steady state. However, the method given cannot compute an $\hat{n}^*$ less than unity. In each of the cases examined, $\hat{n}^*$ does approach unity.

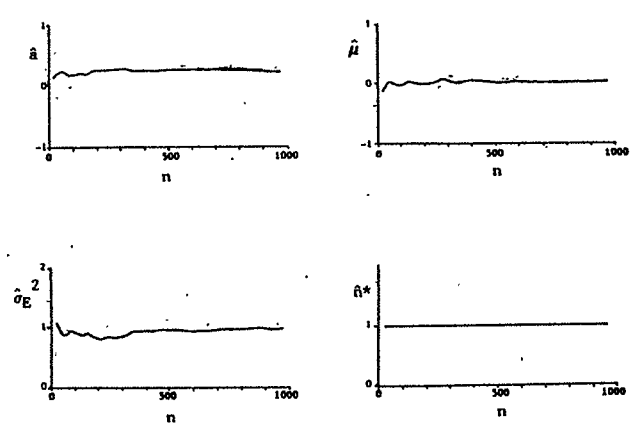


Figure 3. Performance of the Regression Estimators When a=0.2 and the bias-to-noise ratio is 0.

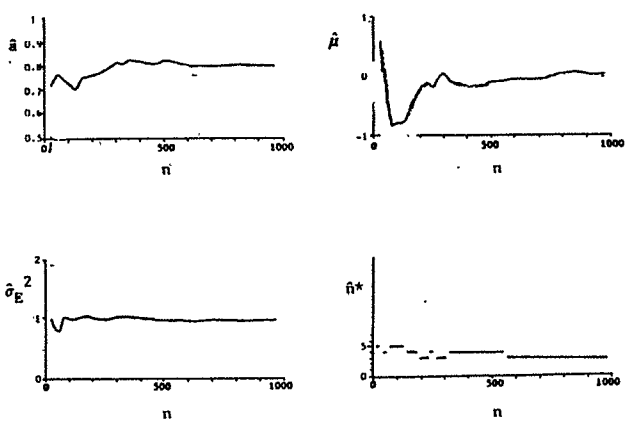


Figure 4. Performance of the Regression Estimators When a=0.8 and the bias-to-noise ratio is 0.

Though a theoretical basis exists for determining the properties of the estimates $\hat{a}$ and $\hat{\sigma}_E^2$ the properties of the estimator $\hat{N}^*$ are more obscure. Therefore, a series of experiments was performed in order to assess the performance of this estimator statistically. The AR(1) process was simulated until a value of n* was obtained such that $n \geq \hat{n}^*$. The value of n for which this was first true was then considered as an observation of n*. Thirty independent replications of this experiment were performed for each case considered. Thus, the t-statistic may be applied to the thirty observations of n* obtained in order to estimate the mean, $E[\hat{N}^*]$, and its variance, from which the 95% confidence intervals for n* may be computed.

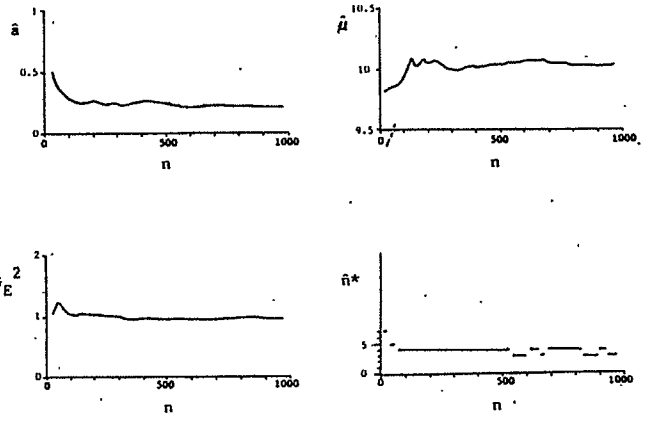


Figure 5. Performance of the Regression Estimators When a=0.2 and the bias-to-noise ratio is 10.

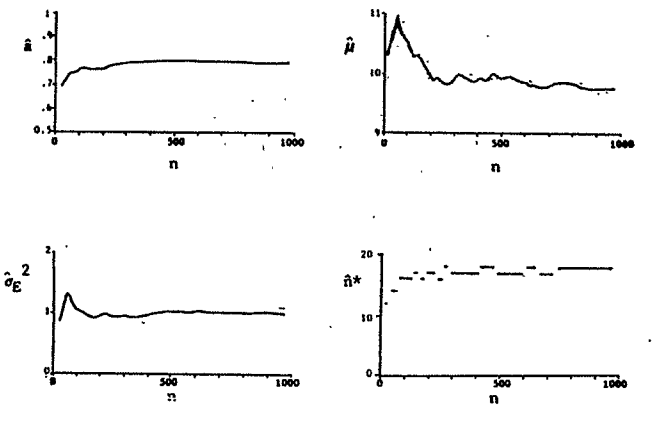
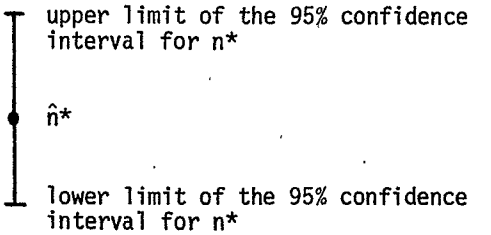


Figure 6. Performance of the Regression Estimators When a=0.8 and the bias-to-noise ratio is 10.

The results of this set of experiments are shown in Figures 7 through 12. In the figures, the experimental results are shown plotted over the theoretical values of n* shown in Figures 1 and 2 as well as for other theoretical values of n* not discussed previously. In Figures 7 through 12 the following convention is used for depicting the experimental results:



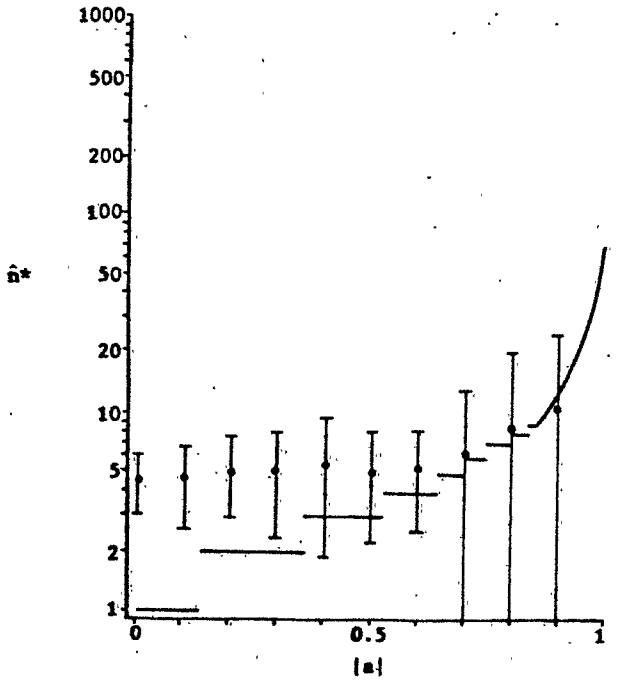


Figure 7. Performance of the Estimator $\hat{N}^*$ when ε=0.05 and the bias-to-noise ratio is 1.

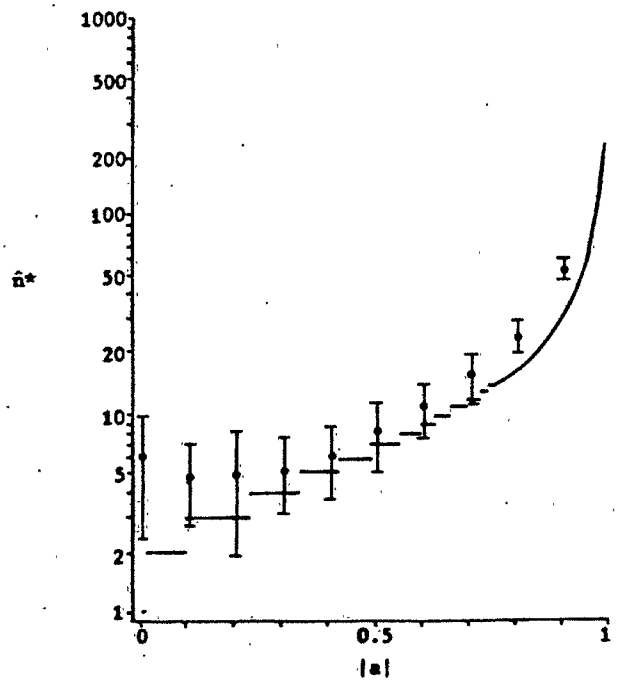


Figure 8. Performance of the Estimator N^* when ε=0.05 and the bias-to-noise ratio is 10.

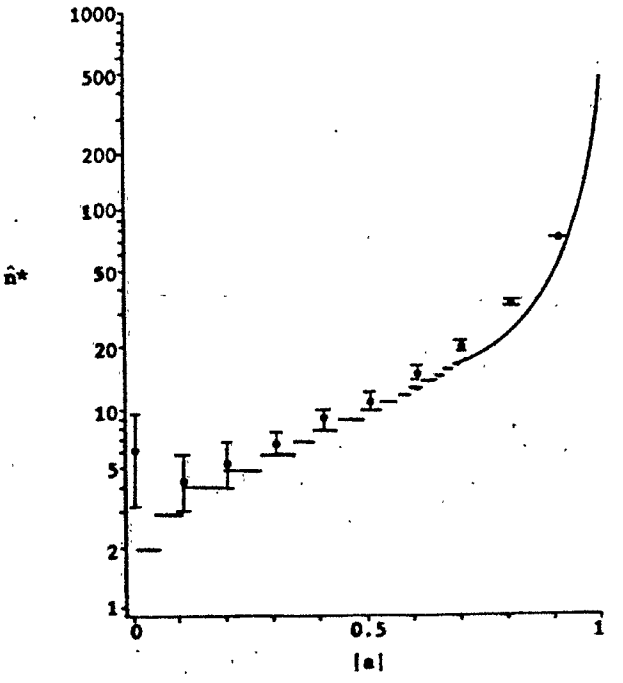


Figure 9. Performance of the Estimator $\hat{N}^*$ when ε=0.05 and the bias-to-noise ratio is 100.

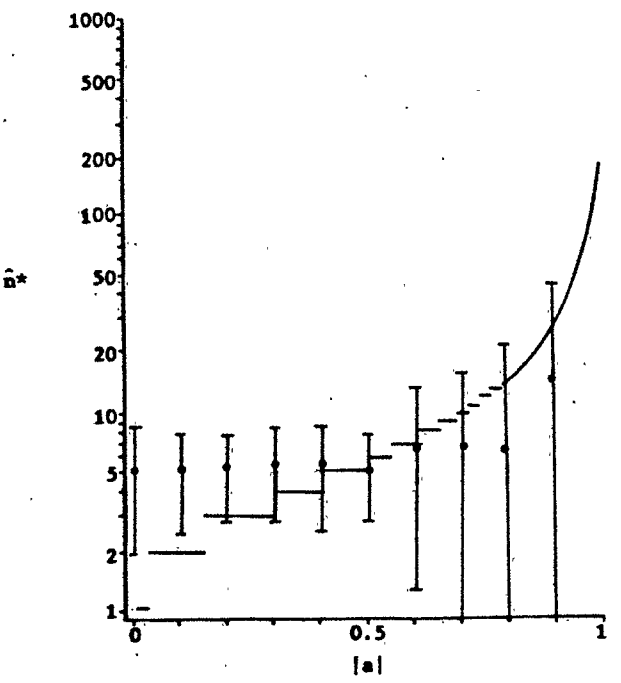


Figure 10. Performance of the Estimator $\hat{N}^*$ when ε=0.01 and the bias-to-noise ratio is 1.

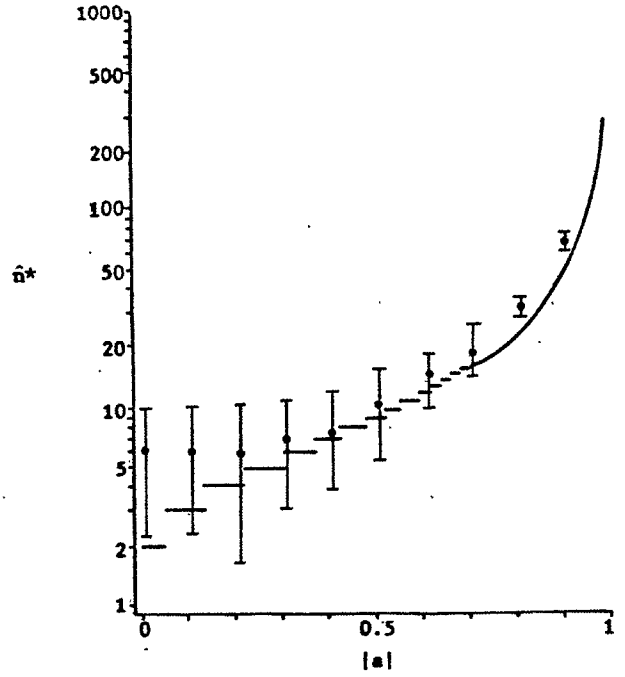


Figure 11. Performance of the Estimator $\hat{N}^*$ when ε=0.01 and the bias-to-noise ratio is 10.

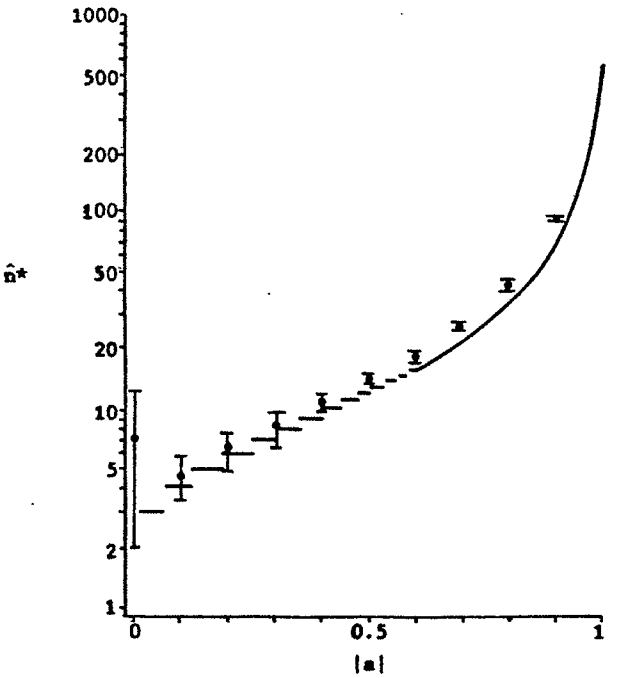


Figure 12. Performance of the Estimator $\hat{N}^*$ when ε=0.01 and the bias-to-noise ratio is 100.

We note, first, that the estimation process is necessarily biased for the lower values of a. The regression requires a minimum of four observations in order to estimate n*, regardless of the theoretical value of n*. Secondly, the confidence intervals, if considered as a percentage of the true value of n*, are wider for low values of the bias-to-noise ratio. Finally, we note that for those values of a, ε, and bias-to-noise ratio for which n* is theoretically greater than four, the estimator $\hat{N}^*$ is generally, though not always, slightly positively biased. In these ranges of a, the behavior of the estimate $\hat{n}^*$ is very close to that of n*.

SUMMARY AND CONCLUSIONS

We have presented a series of experimental results which tend to lend credence to the theory developed. While these experiments were by no means exhaustive, they did cover a wide range of AR (1) processes, and, for the AR(1) model, it is reasonable to conclude that the methods for detecting steady state are, in fact, proper and useful. Of special importance is the discussion of the properties of the estimator $\hat{N}^*$. The proposed method shows every promise of performing well when applied to the AR(1) process with an initial bias.

The total research effort may be summarized by pointing out that for a particular model, the AR(1) process, a method has been developed for determining when the process is, by some preset criterion, very close to being a covariance stationary process, which is, by previous definition, a steady state process.

ACKNOWLEDGEMENTS

I would like to thank Professor A. V. Gafarian of the Department of Industrial and Systems Engineering, University of Southern California, for his aid, comfort, and encouragement during the long course of this research. Also, I wish to express my appreciation to Professors C. J. Ancker, Jr., and T. E. Harris.

BIBLIOGRAPHY

1. Beall, Charles W. 1982. The Effects of Initial Bias in the Simulation of Discrete Time Series, Unpublished Ph. D. Dissertation, Department of Industrial and Systems Engineering, University of Southern California, Los Angeles. (July).

2. Box, George E. P. and Gwilym Jenkins. 1976. Time Series Analysis: Forecasting and Control. Holden-Day, San Francisco.

3. Conway, R. W. 1963. Some Tactical Considerations in Digital Simulation. Management Science. 10, 47-61.

4. Fishman, George S., 1972. Bias Considerations in Simulation Experiments. Operations Research. 20, 785-790.

5. Fishman, George S., 1973. Concepts and Methods in Discrete Event Simulation. Wiley-Interscience, New York.

6. Fuller, Wayne A. 1976. Introduction to Statistical Time Series. Wiley, New York.

7. Gafarian, A. V. 1977. Problems of the Initial Transient in Digital Computer Simulation, Department of Industrial and Systems Engineering, University of Southern California, Los Angeles. (March)

8. Gafarian, A. V., C. J. Ancker, Jr., and T. Morisaku, 1978. Evaluation of Commonly Used Rules for Detecting "Steady State" in Computer Simulation. Naval Research Logistics Quarterly. 25, 511-529.

9. Gordon, G., 1969. System Simulation. Pretice-Hall, Englewood Cliffs, New Jersey.

10. Kelton, William David IV. 1980. The Startup Problem in Discrete Event Simulation, Report No. 80-1, Department of Industrial Engineering, University of Wisconsin, Madison. (April).

11. Law, Averill M., 1980. Statistical Analysis of the Output from Terminating Simulations. Naval Research Logistics Quarterly 27, 131-143.

12. Law, Averill M. and John S. Carson. 1979. A Sequential Procedure for Determining the Length of a Steady-State Simulation. Operations Research. 27, 1011-1025.

13. Schruben, Lee W. 1980. Detecting Initialization Bias in Simulation Output, Technical Report No. 444, School of Operations Research and Industrial Engineering, College of Engineering, Cornell University, Ithaca, New York. (January, Revised December, 1980).

14. Wilson, James R. and A. Alan B. Pritsker. 1978a. Evaluation of Startup Policies in Simulation Experiments. Simulation.

15. Wilson, James R. and A. Alan B. Pritsker. 1978b. A Survey of Research on the Simulation Startup Problem. Simulation. 31, 55-58.