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INITIAL BIAS AND ESTIMATION ERROR IN DISCRETE EVENT SIMULATION

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Abstract

This paper analyses several stochastic systems and obtains expressions for the initial bias. The bias is compared with the estimation error of the same system starting in statistical equilibrium. It turns out that the bias and the estimation error are closely related to each other. Furthermore, it is shown that if the estimation error is small, the bias becomes negligible.

INTRODUCTION

Suppose a certain system is simulated from time 0 to time T, and the problem is to find the expectation of a certain variable H. H may, for instance, be a queue length, the amount of stock in an inventory etc. We assume that H changes randomly through time, but that the expectation of H approaches an equilibrium E(H). The simulation is to provide an estimation $\hat{H}_T$ of E(H).

Since H varies randomly, any estimator $\hat{H}_T$ will also vary, resulting thus in an estimation error. Secondly, the simulation must start under certain initial conditions, and these initial conditions will normally influence H(t), the value of H at a later time t. Since $\hat{H}_T$ depends on H(t), $0 \le t \le T$, the estimator usually becomes biased.

In this paper, we analyze the bias and the estimation error for a number of theoretical models, including the M/D/∞ queue, the M/M/∞ queue and the continuous Markov process. For these models, analytical expressions for the bias and the estimation error can be found. Though these models may or may not be representative for a typical discrete event system simulated, they open at least new prespectives.

It is a well known fact (3) that the variable H of a stochastic system never reaches an equilibrium. However, the probability $\Pi_i(t)$ that H(t) = i will often reach an equilibrium value Π_i for large t. In this case, we say that the system is in a stochastic equilibrium. In this paper, we restrict ourselves to systems that have such a stochastic equilibrium.

It is possible to start a simulation in such a way that it is in a stochastic equilibrium already at time t = 0. For instance, if H(t) describes a Markov process, one can make sure that H(0) is a random variable with the distribution $P[H(0) = i] = \Pi_i$. This method is of course only viable if the equilibrium probabilities are known, and that is not normally the case in simulation. In the models we analyze, however, this can be done quite easily. If a system starts in a stochastic equilibrium, it is easy to find unbiased estimators $\hat{H}_T$ for E(H). We can thus concentrate on the standard derivation of $\hat{H}_T$ which we denote $Std(\hat{H}_T)$. Because in the systems considered, $\hat{H}_T$ will become normally distributed, Std $(\hat{H}_T)$ is a meaningful measure for the estimation error. Here, as well as later, $Std(\hat{H}_T)$ will mean the standard deviation in stochastic equilibrium.

To find an expression for the initial bias, we have to start from a certain initial value H(0).

Initial Bias (continued)

Here, we will always assume that $H(0) = 0$. However, the extension of our theory to other initial states is straightforward.

As a measure for the bias, we then use

$\text{Bias}(\hat{H}_T) = E(\hat{H}_T) - E(H)$

To estimate $E(H)$, one can use several estimators. GPSS, Simscript, and other simulation languages support the continuous time average $\hat{H}_T = \overline{H}(T)$, which is defined as

$$\overline{H}(T) = \frac{1}{T}\int_0^T H(t)dt.$$

Other people (1) have argued that $H(T)$ should only be measured at the times $t = kh$, where $k = 0, 1, 2, \ldots, n$ and $h = T/n$, and the average of these values should be used as an estimator. Thus, one has

$$\hat{H}_T = \overline{H}_h(T) = \frac{1}{n+1}\sum_{k=0}^{n} H(kh), \quad n = T/h.$$

Suprisingly, $\overline{H}(T)$ is not normally the estimator with the lowest standard deviation. Indeed, Grassmann(2) has shown that in the case of the $M/D/\infty$ queue, the best estimator for $E(H)$ is $H_h(T)$, provided h is equal to the service time S.

There are other estimators for $E(H)$ as well. In particular, Halfin (4) uses

$$\hat{H}_T = \overline{H}_{A()}(t) = \int_0^T A(t)H(t)dt \Big/ \int_0^T A(t)dt,$$

where $A()$ is chosen in such a way that $\text{Std}(\overline{H}_{A()}(t))$ is minimal. We will not use this estimator here, however. Indeed, references (2) and (4) show that the difference between reasonable estimators in small. For us, it turns out to be convenient to use $\overline{H}_S(T)$ in the case of the $M/D/\infty$ queue, and to use $\overline{H}(T)$ in all other cases.

THE M/D/∞ QUEUE

Suppose an $M/D/\infty$ queue has an arrival rate of λ and a constant service time of S. Let H be the number of elements in the system, and suppose $E(H)$ is estimated as follows

$$\hat{H}_T = \overline{H}_S(T) = \frac{1}{n+1}\sum_{k=0}^{n} H(kS), \quad T = nS. \tag{1}$$

It can be shown (2) that $\overline{H}_S(T)$ is the minimum variance estimator in statistical equilibrium, and that the variance of $\overline{H}_S(T)$ becomes

$$\text{Var}(\overline{H}_S(T)) = \lambda S^2/(T+S).$$

If the system starts with $H(0) = 0$, equation (1) becomes

$$\overline{H}_S(T) = \frac{1}{n+1}\sum_{k=1}^{n} H(kS).$$

For any $t>S$, $H(t)$ is independent of $H(0)$ (see (5)), and its expectation is λS. Therefore

$$\begin{aligned}\text{Bias}(\overline{H}_S(T)) &= E(\overline{H}_S(T)) - E(H)\\ &= \frac{1}{n+1} n\lambda S - \lambda S\\ &= -\lambda S/(n+1).\end{aligned}$$

Since $n = T/S$, this gives

$$\text{Bias}(\overline{H}_S(T)) = \lambda S^2/(T+S).$$

Consequently, the variance and the bias are equal, except for the sign. Let $\text{Coef}(\overline{H}_S(T))$ be the ratio of the bias with the standard deviation. Then, one has

$$\begin{aligned}\text{Coef}(\overline{H}_S(T)) &= \text{Bias}(\overline{H}_S(T)) / \text{Std}(\overline{H}_S(T))\\ &= -\text{Std}(\overline{H}_S(T)).\end{aligned}$$

This means that the bias is unimportant iff the estimation error is small, and important iff it is large. For instance, if T is chosen such that the standard error is 0.1, the bias is practically irrevelant. However, if the choice of T is such that a standard error of 10 results, the bias becomes predominant.

THE M/M/∞ QUEUE

In this section, we consider the M/M/∞ queue. While doing this, we introduce some basic ideas which will prove important in the next section. The estimator that will be used is

$$\overline{H}(T) = \frac{1}{T}\int_0^T H(t)\ dt.$$

For a stochastic system in equilibrium, the variance of $\overline{H}(T)$ can be found as follows

$$\mathrm{Var}(\overline{H}(T)) = \frac{1}{T^2}\mathrm{Var}(\int_0^T \overline{H}(t)\ dt)$$

$$= \frac{2}{T^2}\int_0^T\int_0^t \mathrm{Cov}(t-x)\ dx\ dt$$

$$= \frac{2}{T^2}\int_0^T\int_0^t \mathrm{Cov}(x)\ dx\ dt. \qquad (2)$$

Here, Cov (x) is the covariance between H(t) and H(t+x).

According to Reynolds (5), one has for the M/M/∞ queueing system

$$\mathrm{Cov}(x) = \frac{\lambda}{\mu}e^{-\mu x}.$$

Hence

$$\mathrm{Var}(\ \overline{H}(T)\ ) = \frac{2}{T^2}\int_0^T\int_0^t \frac{\lambda}{\mu}e^{-\mu x}\ dx\ dt$$

$$= \frac{\lambda}{\mu}\frac{1}{\mu}\frac{2}{T^2}[T - \frac{1}{\mu}(1 - e^{-\mu T})] .$$

For large μT, the term

$$R_1 = \frac{\lambda}{\mu}\frac{2}{\mu^2T^2}(1 - e^{-\mu T})$$

becomes negligible, and one has

$$\mathrm{Var}(\ \overline{H}(T)\ ) = 2\frac{\lambda}{\mu}\frac{1}{\mu T} + R_1. \qquad (3)$$

To find the bias, given H(0) = 0, we define

$$L(t) = E[H(t)].$$

L(t) is thus the expected number of elements in the system at time t. It is well known that

$$L(t) = \frac{\lambda}{\mu} - [\frac{\lambda}{\mu} - L(0)]\ e^{-\mu t}.$$

Since we assumed that H(0) = L(0) = 0, this gives

$$L(t) = \frac{\lambda}{\mu}[1 - e^{-\mu t}].$$

We can now calculate

$$E[\overline{H}(T)] = E[\frac{1}{T}\int_0^T H(t)\ dt]$$

$$= \frac{1}{T}\int_0^T E[H(t)]\ dt = \frac{1}{T}\int_0^T L(t)\ dt.$$

For our case, this gives

$$E[\overline{H}(T)] = \frac{1}{T}\int_0^T L(t)\ dt = \frac{1}{T}\int_0^T \frac{\lambda}{\mu}[1 - e^{-\mu t}]\ dt.$$

$$= \frac{\lambda}{\mu}[1 - \frac{1}{T\mu}(1 - e^{-\mu T})]$$

Because $E(H) = \lambda/\mu$, one finds

$$\mathrm{Bias}(H(T)) = \frac{\lambda}{\mu}[1 - \frac{1}{T\mu}(1 - e^{-\mu T})] - \frac{\lambda}{\mu}$$

$$= -\frac{\lambda}{\mu}\frac{1}{T\mu}(1 - e^{-\mu T}).$$

To find how this bias behaves if μT increases we note that

$$e^{\mu T} > 1 + \mu T > \mu T.$$

Consequently

$$e^{-\mu T} < 1/(\mu T).$$

Initial Bias (continued)

Moreover, $e^{-\mu T} > 0$. Hence

$$\text{Bias} = -\frac{\lambda}{\mu}\frac{1}{T\mu} + R_2 \tag{4}$$

where R_2 is negligible.

$$0 < R_2 = \frac{\lambda}{\mu}\frac{1}{T\mu} e^{-\mu T} < \frac{\lambda}{\mu}\frac{1}{(\mu T)^2}.$$

When comparing (3) to (4) one finds thus that for large μT, the bias of $\overline{H}(T)$ is half as large as the variance of $\overline{H}(T)$. We can also calculate the ratio of the bias to the standard derivation as:

$$\text{Coef}\ (\ \overline{H}(T)\ ) \approx -\sqrt{\frac{\lambda}{\mu}\frac{1}{\mu T}}/\sqrt{2} = -2\ \text{Std}\ (\ \overline{H}(T)\ ).$$

THE BIAS IN A CONTINUOUS MARKOV PROCESS

Let $H(t)$ be a continuous process with a distribution $P[H(t)=i] = \Pi_i\ (t)$. Furthermore, $\Pi_i(t)$ converges towards a unique equilibrium probability Π_i, and we define

$$d_i(t) = \Pi_i(t) - \Pi_i. \tag{5}$$

We now want to find an expression for the bias of the continuous time average $\overline{H}(T)$. One has

$$\begin{aligned}
\text{Bias}(\overline{H}(T)) &= E[\overline{H}(T)] - E(H) \\
&= E[\frac{1}{T}\int_0^T H(t)\ dt] - E(H) \\
&= \frac{1}{T}\int_0^T [E(H(t)) - E(H)]\ dt \\
&= \frac{1}{T}\int_0^T \left[\sum_{i=1}^{N} i\ \Pi_i(t) - \sum_{i=1}^{N} i\ \Pi_i\right] dt \\
&= \frac{1}{T}\int_0^T \sum_{i=1}^{N} i\ d_i(t)dt \\
&= \frac{1}{T}\sum_{i=1}^{N} i \int_0^T d_i(t)dt.
\end{aligned} \tag{6}$$

Thus, as soon as one has an expression for the integral of $d_i(t)$, one has found the bias.

In a continuous Markov process with states 0, 1, ...N, this is easily accomplished. To see this, let $a_{i,j}$ be the rate of going from state i to j with

$$a_{i,\ i} = -\sum_{j=0}^{N} a_{i,j}\ .$$

It is well known (see eg. (3)) that for such a process

$$\Pi_j'(t) = \sum_{i=0}^{N} \Pi_i\ (t)a_{i,j}, \qquad j = 0,1,2, \ldots N$$

Here, the prime denotes the derivative. Also, one has for equilibrium

$$0 = \sum_{i=0}^{N} \Pi_i\ a_{i,j}, \qquad j = 0,1,2,\ldots,N.$$

The difference between these 2 equations gives

$$\Pi_j'(t) = \sum_{i=0}^{N} (\Pi_i\ (t) - \Pi_i)\ a_{i,j} = \sum_{i=0}^{N} d_i(t)\ a_{i,j}$$

Since $d_j' = \Pi_j'(t)$ (see equation (5)), this means

$$d_j'(t) = \sum_{i=0}^{N} d_i(t)\ a_{i,j}\ , \qquad j = 0,1,2,\ldots,N. \tag{7}$$

The $d_i(t)$ have a sum of zero because

$$\sum_{i=0}^{N} d_i(t) = \sum_{i=0}^{N} [\Pi_i\ (t) - \Pi_i] = \sum_{i=0}^{N} \Pi_i(t) - \sum_{i=0}^{N} \Pi_i = 0.$$

Therefore

$$d_0(t) = -\sum_{i=1}^{N} d_i(t).$$

Using this expression, (7) becomes

$$d_j'(t) = \sum_{i=1}^{N} d_i(t)\, a_{i,j} - \sum_{i=1}^{N} d_i(t)\, a_{0,j}$$

$$= \sum_{i=1}^{N} d_i(t)\, (a_{i,j} - a_{0,j}) \;,\; j = 1,2, \ldots, N. \qquad (8)$$

It proves helpful to express the results obtained thus far in matrix form. For this purpose, we define

$$B = [a_{i,j} - a_{0,j}]$$
$$\underline{d}(t) = [d_1(t), d_2(t), \ldots, d_N(t)]$$
$$\underline{d}'(t) = [d_1'(t), d_2'(t), \ldots, d_N'(t)],$$

Note that B is a square matrix of order N x N which has no entries for row zero and column zero. Similarly, the vectors $\underline{d}$ (t) and $\underline{d}'(t)$ do not contain the terms $d_0(t)$, respectively, $d_0'(t)$. Using these conventions, (8) becomes

$$\underline{d}'(t) = \underline{d}(t)\; B.$$

Together with $\underline{d}$ (0), this differential equation uniquely determines $\underline{d}(t)$. The elements of $\underline{d}(0)$ can vary freely within their range, that is, there is no additional equation to be staisfied by the $d_i(0)$. Let $\underline{d} = [d_j]$ be the equilibrium vector $\underline{d}(t)$. Clearly, $\underline{d}$ must satisfy

$$0 = \underline{d}B \qquad (10)$$

Because of the above, this is the only equation determining $\underline{d}$.

It is well known that under these conditions B is invertable provided (10) has only the trivial solution $\underline{d} = [0]$. This is the case here. Indeed, if there is a unique equilibrium,

$$d_j = \Pi_j(t) - \Pi_j = \Pi_j - \Pi_j = 0.$$

This proves that B is invertable.

To find the integral of $\underline{d}(t)$, we integrate equation (9)

$$\underline{d}(t) = \int \underline{d}(t)dt\; B.$$

This gives

$$\int \underline{d}(t)dt = \underline{d}(t)B^{-1}$$

or

$$\int_0^T \underline{d}(t)dt = [\underline{d}(T) - \underline{d}(0)]B^{-1}.$$

This expression can now be substituted into (6) to obtain the bias. To write this final result in nicer form, we introduce the column vector

$$\underline{c} = [1, 2, \ldots, N]^T.$$

Now

$$\text{Bias}(H(T)) = \frac{1}{T}\int_0^T \underline{d}(t)dt\; \underline{c} = \frac{1}{T}\Big[\underline{d}(T) - \underline{d}(0)\Big]\, B^{-1}.$$

As T increases, $\underline{d}(T)$ goes to zero and one has

$$\text{Bias}(\overline{H}(T)) = -\frac{1}{T}\underline{d}(0)B^{-1}\underline{c} + R_1,$$

where R_1 becomes negligible. Moreover, if $H(0) = 1$,

$$d_i(0) = \Pi_i(0) - \Pi_i = -\Pi_i,\; i = 1, 2, \ldots, N,$$

that is

$$\text{Bias}(\overline{H}(T)) = \frac{1}{T}\underline{\Pi}\; B^{-1}\underline{c} + R_1, \qquad (11)$$

where $\Pi = [\Pi_1, \Pi_2, \ldots, \Pi_N]$. The above expression for the bias is surprisingly simple, and it is easily calculated. In the next section, a very similar expression for the variance of $\overline{H}(T)$ in equilibrium will be derived.

THE ESTIMATION ERROR IN AN EQUILIBRIUM MARKOV PROCESS

Equation (2) gives the following expression for the variance of $\bar{H}(T)$:

$$\mathrm{Var}(\bar{H}(T)) = \frac{2}{T^2}\int_0^T\int_0^t \mathrm{Cov}(x)\,dx\,dt.$$

It remains thus to calculate Cov(x). To do this, we define

$$P_{ij}(x) = P(H(t+x) = j \mid H(x) = i).$$

If the process starts in equilibrium, one has

$$P[H(0) = i \cap H(x) = j] = P[H(0) = i]\cdot$$

$$= P[H(t + x) = j \mid H(x) = i)$$

$$= \Pi_i P_{ij}(x).$$

If $\mu = E(H)$, the covariance becomes

$$\mathrm{Cov}(x) = \mathrm{Cov}(H(0), H(x))$$

$$= \sum_{i=0}^{N}\sum_{j=0}^{N} (i-\mu)\Pi_i P_{ij}(x)(j-\mu).$$

This expression can be written in a more convenient form. First we note that $j-\mu$ can be replaced by j. Moreover, we introduce

$$r_j(x) = \sum_{i=0}^{N} (i-\mu)\Pi_i P_{ij}(x). \qquad (12)$$

Then

$$\mathrm{Cov}(x) = \sum_{j=0}^{N}\left[\sum_{i=0}^{N}(i-\mu)\Pi_i P_{ij}(x)\right]j = \sum_{j=1}^{N} j r_j(t). \qquad (13)$$

We now derive a convenient expression to calculate $r_j(t)$. To do this, we use the well known relationship (see e.g. refernece (3)).

$$P_{ij}'(x) = \sum_{k=0}^{N} P_{ik}(x)a_{kj}. \qquad (14)$$

If one differentiates (12) and uses (14) one finds

$$r_j'(t) = \sum_{i=0}^{N}(i-\mu)\Pi_i P_{ij}'(x)$$

$$= \sum_{i=0}^{N}(i-\mu)\Pi_i\sum_{k=0}^{N} P_{ik}(x)a_{kj}$$

$$= \sum_{k=0}^{N}\left[\sum_{i=0}^{N}(i-\mu)\Pi_i P_{ik}(x)\right]a_{kj}$$

$$= \sum_{k=0}^{N} r_k(x)a_{kj}.$$

Thus

$$r_j'(t) = \sum_{i=0}^{N} r_i(x)a_{ij}.$$

This equation has exactly the same structure as the corresponding equation for $d_j(t)$, which is equation (7). Moreover, like the $d_j(t)$, one can show that the $r_j(t)$ have a sum of zero. This is done as follows:

$$\sum_{j=0}^{N} r_j(x) = \sum_{j=0}^{N}\sum_{i=0}^{N}(i-\mu)\Pi_i P_{ij}(x)$$

$$= \sum_{i=0}^{N}(i-\mu)\Pi_i\sum_{j=0}^{N} P_{ij}(x)$$

$$\sum_{i=0}^{N}(i-\mu)\Pi_i = \sum_{i=0}^{N} i\Pi_i - \mu = 0.$$

We conclude therefore that the integral of $r_j(x)$ can be calculated like the one of $d_j(t)$. In other words

$$\int_0^T \underline{r}(x)dx = [\underline{r}(T) - r(0)]B^{-1}, \quad (15)$$

with

$$\underline{r}(x) = [r_1(x), r_2(x), \ldots, r_N(x)].$$

We are now almost done. Using (2), (13), (15), and $\underline{c} = [1, 2, \ldots, N]^T$, one finds:

$$\mathrm{Var}(\overline{H}(t)) = \frac{2}{T^2} \int_0^T\int_0^t \mathrm{Cov}(x)\, dx\, dt$$

$$= \frac{2}{T^2} \int_0^T\int_0^t \sum_{j=1}^{N} j r_j(x)\, dx\, dt$$

$$= \frac{2}{T^2} \sum_{j=1}^{N} j \int_0^T\int_0^t r_j(x)\, dx\, dt$$

$$= \frac{2}{T^2} \int_0^T\int_0^t \underline{r}(x)\, dx\, dt\, \underline{c}$$

$$= \frac{2}{T^2} \int_0^T \left[\underline{r}(t) - \underline{r}(0)\right] B^{-1} dt\, \underline{c}$$

$$= \frac{2}{T^2} \int_0^T \left[\underline{r}(t) - \underline{r}(0)\right] B^{-1}\, dt\, \underline{c}$$

$$= -\frac{2}{T}\underline{r}(0)B^{-1}\underline{c} + \frac{2}{T^2}\left[\underline{r}(T) - \underline{r}(0)\right]B^{-2}\underline{c}.$$

We now consider again the case that $T\to\infty$. If $\underline{r}(T)$ stays bounded, as it will because of (11), the second term of the above expression will be less than a/T^2 for some value a, and it is therefore negligible.

Consequently

$$\mathrm{Var}(\overline{H}(T)) = -\frac{2}{T}\underline{r}(0)B^{-1}c + R, \quad (16)$$

where R becomes negligible for high T. $\underline{r}(0)$ can be found from equation (12), giving

$$r_j(0) = \sum_{i=0}^{N} (i-\mu)\Pi_i P_{ij}(0) = (j-\mu)\Pi_j.$$

Equation (16) and equation (11) are identical except for the initial conditions.

CONCLUSIONS

In this section, we investigated the relevance of our findings in respect to simulations. To do this, it is assumed that the simulation must yield an estimator which does not have a standard error over e, or, what is the same, the variance of the estimator must not exceed e^2. What can be said about the bias under this condition? To answer this question, let us first review the main results of the paper.

In this paper, we used two estimators, namely the discrete time average $\overline{H}_h(T)$ and the continuous time average $\overline{H}(T)$. According to the literature (1, 2, 4), these estimators are hardly different, especially for large T, and we will therefore treat them as identical. In this sense, we will use $\hat{H}_T$ for either estimator.

In all cases considered, we found that the bias and the variance of $\hat{H}_T$ to be given by equations of the following form.

$$\mathrm{Bias}\ (\hat{H}_T) = b/T + R_1$$

$$\mathrm{Variance}\ (\hat{H}_T) = v/T + R_2,$$

Here R_1 and R_2 are terms that are negligible, and they will be ignored from now on.

Since the variance must be e^2, we have

Initial Bias (continued)

$$v/T = e^2 .$$

This gives the following run-length T for the simulation

$$T = v/e^2 .$$

Because the bias is b/T, this means

$$\text{Bias } (\hat{H}_T) = b/T = (b/v)\ e^2.$$

Moreover, it is reasonable to complare this bias against the standard deviation of $\hat{H}_T$. In other words, we should calculate

$$\text{Coef } (\hat{H}_T) = \text{Bias } (\hat{H}_T) \ / \ \text{Std } (\hat{H}_T) = (b/v)\ e.$$

If this coefficient is small, say 0.1 the bias is relatively unimportant, whereas it must be removed by some means if it is 10 or even larger.

In this paper, we have shown how to calculate b and v for a number of cases. In the case of the M/D/∞ queue, we had

$$-b = v = \lambda S^2,$$

which gives a ratio b/v of -1. For the M/M/∞ queue, the values are

$$b = -\lambda/\mu^2$$

$$v = + 2\lambda \ /\mu^2$$

$$b/v = - 1/2.$$

Equations (11) and (16) implicitely give formulae for b and v. Using these equations, we obtained b and v for the M/M/1 queue and the M/M/2 queue with finite waiting room. Some results of these calculations are given in Table 1. One can see that |b/v| is never greater than ½, and usually much smaller. Thus, for any standard error e less than 1, the bias tends to be irrevelant. For high e, of course, the bias can be substantial. We thus conclude that the bias tends to be negligible for simulations that are done at run lengths giving a high precision, and that the bias is relevant if the precision requirements are low.

Table 1. b and v for M/M/1 and M/M/2 - queues.

Number of servers	Max in system	service rate	arrival rate	traffic intensity	-b	v	|b/v|
1	10	1	0.5	0.5	3.8	21.4	0.177
1	10	1	0.9	0.9	41.7	231.5	0.180
1	20	1	0.5	0.5	3.5	24.0	0.146
1	20	1	0.9	0.9	199.2	2368.7	0.084
2	10	0.5	0.5	0.5	5.1	23.5	0.217
2	10	0.5	0.9	0.9	40.2	199.4	0.202
2	20	0.5	0.5	0.5	5.3	26.6	0.199
2	10	0.5	0.9	0.9	194.7	2205.3	0.088

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