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RANDOM VARIATE GENERATION[1]

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ABSTRACT: The presentation begins with a discussion of basic concepts and concludes with a state of the art survey of methods for generating random variates on a digital computer. This paper updates the more than 300 references cited in last year's paper. Copies of the visual aids are given.

1. INTRODUCTION

The references which have appeared during the last year or which were inadvertantly omitted in Schmeiser (1980) are listed in Section 2.

Figures 1-29 in the Appendix are copies of the visual aids, which were requested by several persons last year.

2. ADDITIONAL REFERENCES

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APPENDIX

RANDOM VARIATE GENERATION:

A SURVEY

BRUCE SCHMEISER

PURDUE UNIVERSITY

SUPPORTED BY: OFFICE OF NAVAL RESEARCH
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INPUT MODELLING

SCHMEISER / 2

1. HYPOTHESIZE FAMILY
2. DETERMINE PARAMETERS
   2.1 FROM DATA
   2.2 FROM OPINIONS
3. DIAGNOSTIC CHECKING
4. RANDOM VARIATE GENERATION

THE TRANSFORMATION OF THE BASIC SOURCE OF RANDOMNESS, USUALLY U(0,1), TO THE DISTRIBUTION OF INTEREST.

OUTLINE

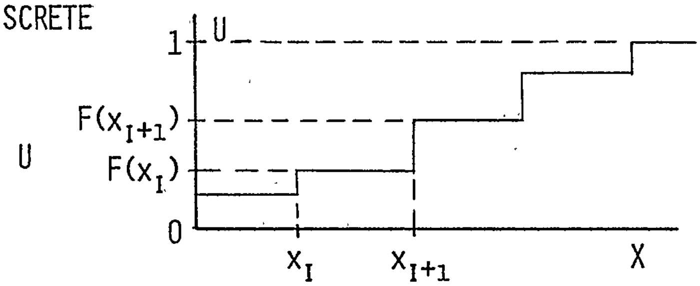
SCHMEISER / 3

1. FUNDAMENTAL CONCEPTS
   1.1 INVERSE TRANSFORMATIONS
   1.2 COMPOSITION
   1.3 ACCEPTANCE/REJECTION
   1.4 SPECIAL PROPERTIES
2. CRITERIA FOR ALGORITHM COMPARISON
3. STATE OF THE ART
   3.1 CONTINUOUS UNIVARIATE
   3.2 DISCRETE UNIVARIATE
   3.3 MULTIVARIATE
      - CONTINUOUS VECTORS
      - DISCRETE VECTORS
      - POINT PROCESSES
      - TIME SERIES
      - ORDER STATISTICS
      - GEOMETRIC PROBLEMS

1.1 INVERSE TRANSFORMATION: $X = F^{-1}(U)$

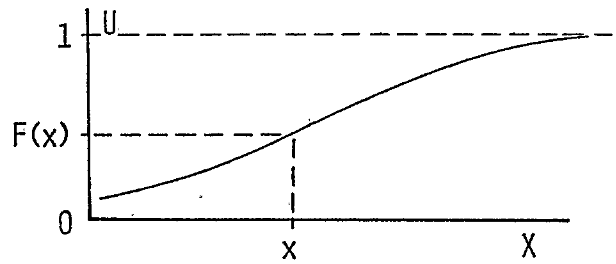
SCHMEISER / 4

DISCRETE



$$P(X = x_{I+1}) = F(x_{I+1}) - F(x_I)$$
$$= P(F(x_I) \le U \le F(x_{I+1}))$$
WHEN $U \sim U(0,1)$

CONTINUOUS



RESULT: $U = F(X) \sim U(0,1)$

NOTE: NUMERICAL INVERSION ROUTINES FOR NORMAL, GAMMA, AND BETA DISTRIBUTIONS ARE AVAILABLE.

ADVANTAGES OF THE INVERSE TRANSFORMATION, $X = F^{-1}(U)$

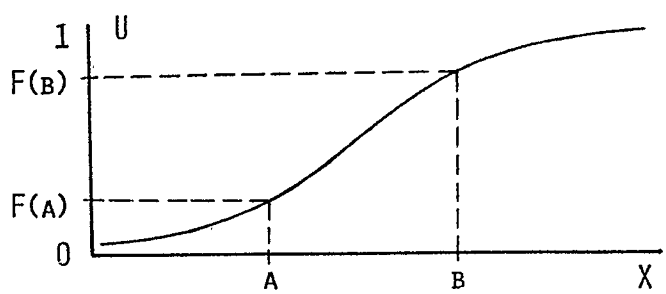
SCHMEISER / 5

A. TRUNCATED DISTRIBUTIONS

-- WANT $A \le X \le B$

-- ALGORITHM

1. $W = F(A) + (F(B) - F(A)) * U$
2. $X = F^{-1}(W)$



ADVANTAGES OF THE INVERSE TRANSFORMATION, $X = F^{-1}(U)$ SCHMEISER / 6

A. TRUNCATED DISTRIBUTIONS

B. ORDER STATISTICS: SCHUCANY (1972) AND LURIE AND HARTLEY (1972)

-- WANT TO GENERATE $X_{(I,N)}$

-- RESULT: $U_{(I,N)} \sim$ BETA $(I, N-I+1)$

-- ALGORITHM: 1. GENERATE $U_{(I,N)}$

2. $X_{(I,N)} = F^{-1}(U_{(I,N)})$

-- SPECIAL CASES:

1. $X_{(N,N)} = F^{-1}(U^{1/N})$
2. $X_{(1,N)} = F^{-1}(1 - (1-U)^{1/N})$
3. $X_{(I,N)} = F^{-1}(1 - (1-U_{(I-1,N)})*U_I^{1/(N-I+1)})$

WHERE $U_I \sim U(0,1)$

ADVANTAGES OF THE INVERSE TRANSFORMATION, $X = F^{-1}(U)$ SCHMEISER / 7

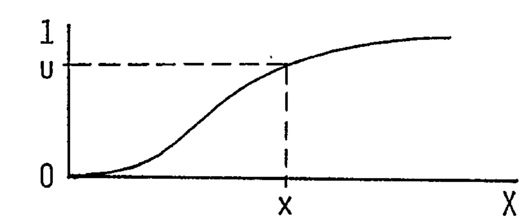
A. TRUNCATED DISTRIBUTIONS

B. ORDER STATISTICS

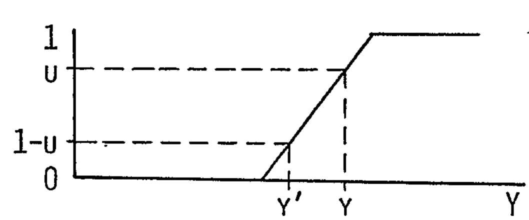
C. VARIANCE REDUCTION

-- FACILITATES CORRELATION INDUCTION

1. EASIER SYNCHRONIZATION
2. OBTAINS MOST EXTREME CORRELATIONS



$X = F^{-1}(U)$



$Y = F^{-1}(U) \Longrightarrow$ MAX CORR(X,Y)

$Y' = F^{-1}(1-U) \Longrightarrow$ MIN CORR(X,Y)

COMPOSITION SCHMEISER / 8

A. DISCRETE MIXING

-- IF DENSITY FUNCTION IS $f(x) = \sum_{I=1}^{N} p_I f_I(x)$

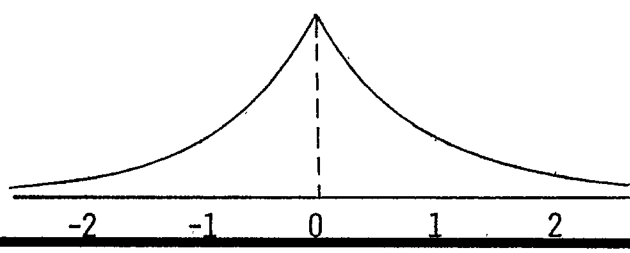
WITH $\sum_{I=1}^{N} p_I = 1$ AND $0 \le p_I$ FOR ALL I,

-- THEN VARIATES MAY BE GENERATED USING

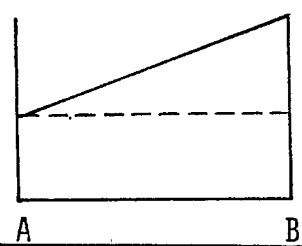
1. SELECT I WITH PROBABILITY p_I
2. GENERATE X FROM $f_I(\cdot)$.

-- EXAMPLES

LAPLACE OR DOUBLE EXPONENTIAL



TRAPEZOIDAL



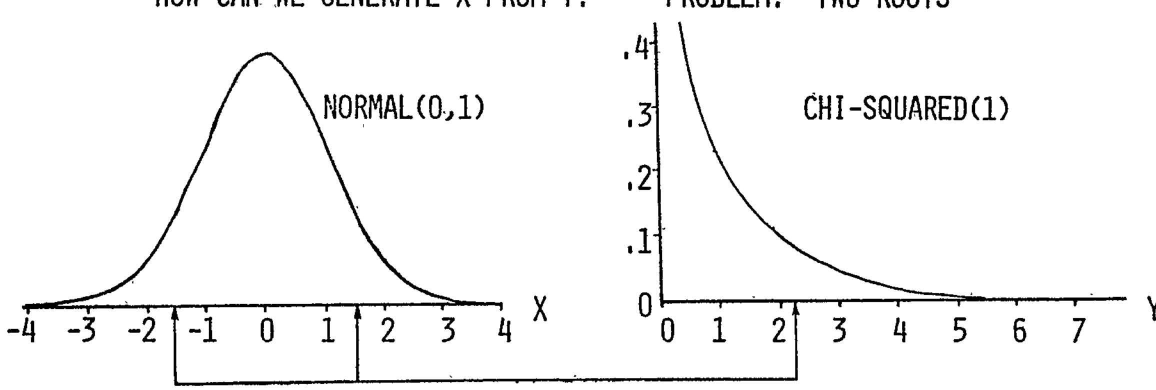
COMPOSITION SCHMEISER / 9

A. DISCRETE MIXING

1. MULTIPLE ROOTS: MICHAEL, SCHUCANY AND HAAS (1976)

-- WE CAN GENERATE $Y \sim$ CHI-SQUARED(1) USING $Y = X^2$
IF $X \sim N(0,1)$

-- HOW CAN WE GENERATE X FROM Y? PROBLEM: TWO ROOTS



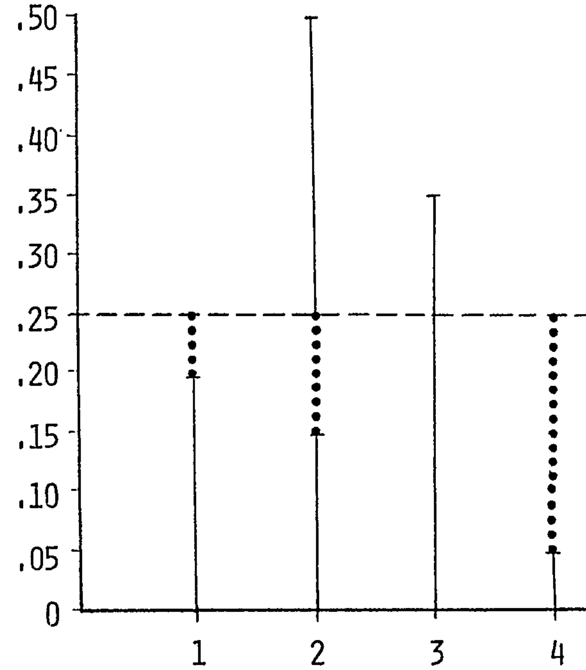
ANOTHER EXAMPLE: $X \sim$ INVERSE GAUSSIAN, $Y \sim$ CHI-SQUARED(1) } $Y = A(X-B)^2/(B^2X)$

COMPOSITION SCHMEISER / 10

A. DISCRETE MIXING

1. MULTIPLE ROOTS
2. ALIAS METHOD: WALKER (1977)

THEOREM: ANY DISCRETE DISTRIBUTION $P_1, P_2, \ldots, P_N$; WHERE N IS FINITE; MAY BE EXPRESSED AS AN EQUIPROBABLE MIXTURE OF N TWO-POINT DISTRIBUTIONS.



	1	2	3	4
P_I	.20	.40	.35	.05
$ALIAS_I$	2	3	---	2
F_I	.8	.6	1.0	.2

COMPOSITION SCHMEISER / 11

A. DISCRETE MIXING

B. CONTINUOUS MIXING

-- IF THE CDF CAN BE WRITTEN AS $F_X(x) = \int F_{X|Y}(x)\, dG(y)$

-- THEN VARIATES MAY BE GENERATED USING

1. GENERATE y FROM $G(\cdot)$
2. GENERATE x FROM $F_{X|y}(\cdot)$

-- EXAMPLE: PEARSON TYPE VI

1. $B \sim$ GAMMA(D,C)
2. $X \sim$ GAMMA(A,1/B)

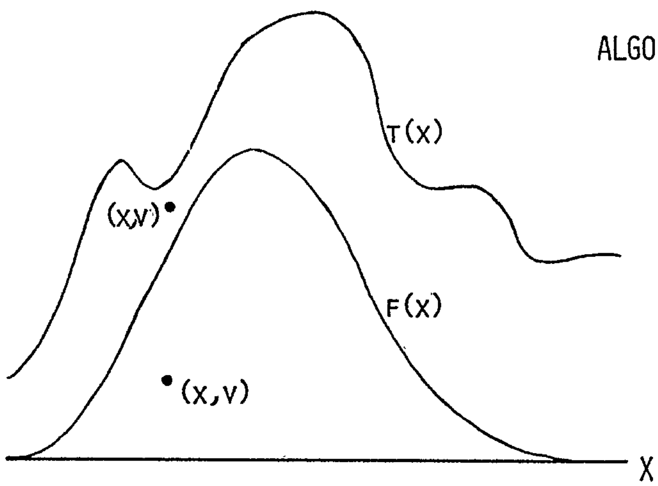
-- EXAMPLE: NEGATIVE BINOMIAL (N,P)

1. $Y \sim$ GAMMA(N,(1-P)/P)
2. $X \sim$ POISSON(Y)

ACCEPTANCE/REJECTION

SCHMEISER / 12

GIVEN: $T(X) \geq F(X)$ FOR ALL X



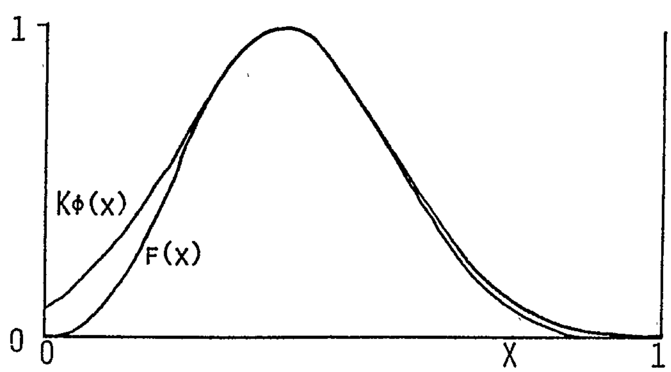
ALGORITHM:

1. GENERATE (X,V) UNIFORMLY OVER THE REGION UNDER $T(\cdot)$
2. IF $V \leq F(X)$, THEN X IS ACCEPTED; OTHERWISE REJECT X BY GOING TO STEP 1.

ACCEPTANCE/REJECTION

SCHMEISER / 13

-- EXAMPLE: BETA DISTRIBUTION, AHRENS AND DIETER (1974)



ALGORITHM

1. GENERATE $X \sim \text{NORMAL}(\mu,\sigma^2)$
2. GENERATE $V \sim U(0, K\,\phi(X))$
3. IF $V \leq F(X)$, ACCEPT X; OTHERWISE GO TO STEP 1

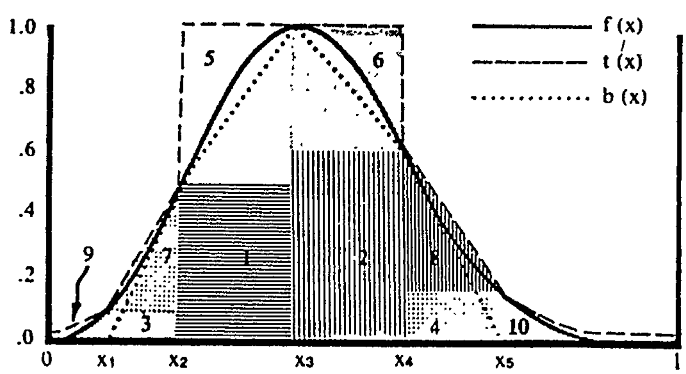
$\phi(X) = \text{EXP}\{-.5((X-\mu)/\sigma)^2\}$

$F(X) = X^P\,(1-X)^Q\;\;I(X)_{(0,1)}$

COMPOSITION -- ACCEPTANCE/REJECTION

SCHMEISER / 14

EXAMPLE: BETA DISTRIBUTION, SCHMEISER AND BABU (1980)



ALGORITHM B4PE

1. SELECT REGION I
2. GENERATE x FROM $T_I(x)$
3. GENERATE $v \sim U(0, T_I(x))$
4. IF $v \leq B(x)$, ACCEPT x
5. IF $v \leq F(x)$, ACCEPT x; OTHERWISE, REJECT x BY GOING TO STEP 1

SPECIAL PROPERTIES

SCHMEISER / 15

A. DIRECT TRANSFORMATIONS

-- ALGORITHM

1. GENERATE $Y_1, Y_2, \ldots, Y_N$ FROM SOME DISTRIBUTIONS (MAYBE IID)
2. CALCULATE X AS A FUNCTION OF THE Y's

-- EXAMPLES

- LOGNORMAL
- BETA AS RATIO OF GAMMA's
- F AS A RATIO OF CHI-SQUARED's
- T FROM NORMAL AND CHI-SQUARED
- BINOMIAL AS A SUM OF BERNOULLI TRIALS
- RATIO OF UNIFORM's, KINDERMAN AND MONAHAN (1977)

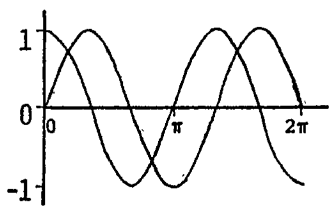
$$X = U_1 / U_2$$

SPECIAL PROPERTIES SCHMEISER / 16

A. DIRECT TRANSFORMATIONS

B. TRIGONOMETRIC FUNCTIONS

-- WANT $\sin(\alpha)$, $\cos(\alpha)$, AND $\tan(\alpha)$, WHERE $\alpha \sim U(0, 2\pi)$

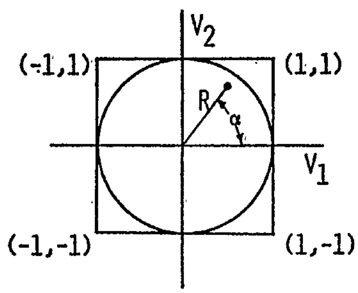


RESULT: $\sin(\alpha) \overset{D}{=} \cos(\alpha)$

RESULT: $\cos(\alpha) \overset{D}{=} \cos(2\alpha)$

$= \cos^2(\alpha) - \sin^2(\alpha)$

$\overset{?}{=} \dfrac{V_1^2 - V_2^2}{V_1^2 + V_2^2}$



$\alpha \sim U(0,2\pi)$

$R = (V_1^2+V_2^2)^{1/2}$

$\sin(\alpha) = V_2/R$

$\cos(\alpha) = V_1/R$

$\tan(\alpha) = V_2/V_1$

2. CRITERIA FOR ALGORITHM COMPARISON SCHMEISER / 17

1. ACCURACY
   - 1.A THEORETICAL
   - 1.B ERROR INDUCED BY U(0,1) NOT BEING RANDOM
   - 1.C ERROR INDUCED BY COMPUTER ARITHMETIC
2. EXECUTION SPEED
   - 2.A SET-UP TIME
   - 2.B MARGINAL TIME
3. EASE OF IMPLEMENTATION
   - 3.A NUMBER OF LINES OF CODE
   - 3.B SUPPORT ROUTINES REQUIRED
   - 3.C BIT MANIPULATION
4. PORTABILITY
5. MEMORY REQUIRED
6. INTERACTION WITH VARIANCE REDUCTION TECHNIQUES

3. STATE OF THE ART SURVEY: OUTLINE

SCHMEISER / 18

3.1 CONTINUOUS UNIVARIATE
- -- CLOSED FORM INVERSES
- -- NORMAL
- -- GAMMA
- -- BETA

3.2 UNIVARIATE DISCRETE
- -- POISSON
- -- BINOMIAL
- -- NEGATIVE BINOMIAL

3.3 MULTIVARIATE
- -- CONTINUOUS
- -- DISCRETE
- -- POINT PROCESSES
- -- TIME SERIES
- -- ORDER STATISTICS
- -- GEOMETRIC PROBLEMS

CLOSED FORM INVERSE TRANSFORMATIONS

SCHMEISER / 19

UNIFORM(A,B)	X = A + (B-A) * U
TRIANGULAR(A,B)	X = A + (B-A) * $U^{1/2}$
EXPONENTIAL(A)	X = -A * ALOG(1-U)
WEIBULL(A,B)	X = $\{-A * \text{ALOG}(1-U)\}^B$
CAUCHY(A,B)	X = A + B * TAN(2πU)

NORMAL DISTRIBUTION SCHMEISER / 20

-- NEED ONLY $Y \sim N(0,1)$, SINCE $X = \mu + \sigma Y$ IS $N(\mu,\sigma^2)$.

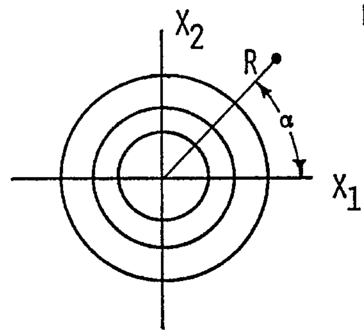
-- CLASSICAL METHOD: DIRECT TRANSFORMATION, BOX AND MULLER (1958)

$R = \{-2 * \text{ALOG}(U_1)\}^{1/2}$

$\alpha = 2\pi U_2$

$X_1 = R * \text{SIN}(\alpha)$

$X_2 = R * \text{COS}(\alpha)$



THEN X_1 AND X_2 ARE NID(0,1).

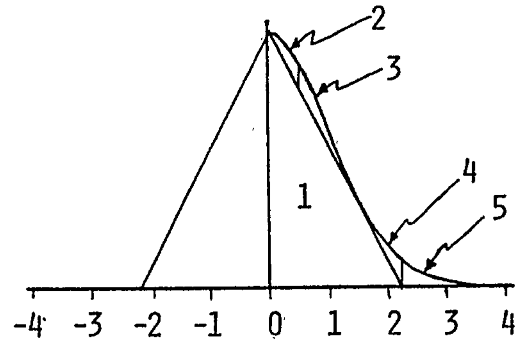
NORMAL DISTRIBUTION SCHMEISER / 21

FASTEST METHOD: MARSAGLIA, MACLAREN AND BRAY (1964)

37 REGION COMPOSITION, REQUIRES BIT MANIPULATION

FASTEST METHOD IN FORTRAN: KINDERMAN AND RAMAGE (1976)

5 REGION COMPOSITION



INVERSE TRANSFORMATION

MACHINE ACCURACY: ODEH AND EVANS (1974)

QUICK AND DIRTY: RAMBERG AND SCHMEISER (1972)

$$X = \{U^{.135} - (1-U)^{.135}\}/.1975$$

GAMMA DISTRIBUTION SCHMEISER / 22

-- DENSITY FUNCTION $f(x) = \exp(-x)\, x^{A-1} / \Gamma(A) \quad I(x)_{(0,\infty)}$

-- CAN LATER RESCALE BY MULTIPLYING BY B.

-- MARGINAL EXECUTION TIMES (MILLISECONDS)

A	CHENG GB	KINDERMAN & MONAHAN GRUB	MARSAGLIA [1] RGAMA	TADIKAMALLA GAMMA	SCHMEISER & LAL G4PE
1.01	.53	.29	.39	.43	.24
1.2	.49	.29	.39	.45	.21
2.	.46	.30	.38	.52	.21
5.	.41	.30	.35	.56	.18
10.	.41	.31	.34	.57	.18
100.	.41	.33	.34	.54	.18
1000.	.40	.34	.34	.56	.18

1 USING KR NORMAL

BETA DISTRIBUTION SCHMEISER / 23

-- $f(x) = x^{P-1}(1-x)^{Q-1} / \beta(P,Q) \quad I(x)_{(0,1)}$

-- CAN RESCALE TO ANY INTERVAL (A,B)

-- MARGINAL EXECUTION TIMES (MILLISECONDS)

P=Q	JÖHNK	RATIO [1] GAMMAS	A & D [2] BN	CHENG BB	S & B B4PE
1.01	1.2	~.50	1.8	.57	.36
1.2	1.4	~.48	.85	.58	.38
2.	3.4	~.45	.77	.61	.44
5.	*	~.41	.72	.62	.23
10.	*	~.37	.69	.65	.23
100.	*	~.37	.67	.65	.24
1000.	*	~.37	.67	.65	.23

1 USING G4PE 2 USING KR FOR NORMAL

POISSON DISTRIBUTION SCHMEISER / 24

-- $F(X) = EXP(-M)\ M^X / X!$ $X = 0, 1, 2, \ldots$

-- $E\{X\} = M$

SMALL MEAN

A. INVERSE TRANSFORMATION VIA $F(X) = F(X-1)\ M / X$

B. SPECIAL PROPERTY: RELATIONSHIP TO EXPONENTIAL

C. COMPOSITION: DISCRETE MIXING

$Y \sim$ POISSON(K) $K \geq M$

$X \sim$ BINOMIAL(N=Y, P=M/K)

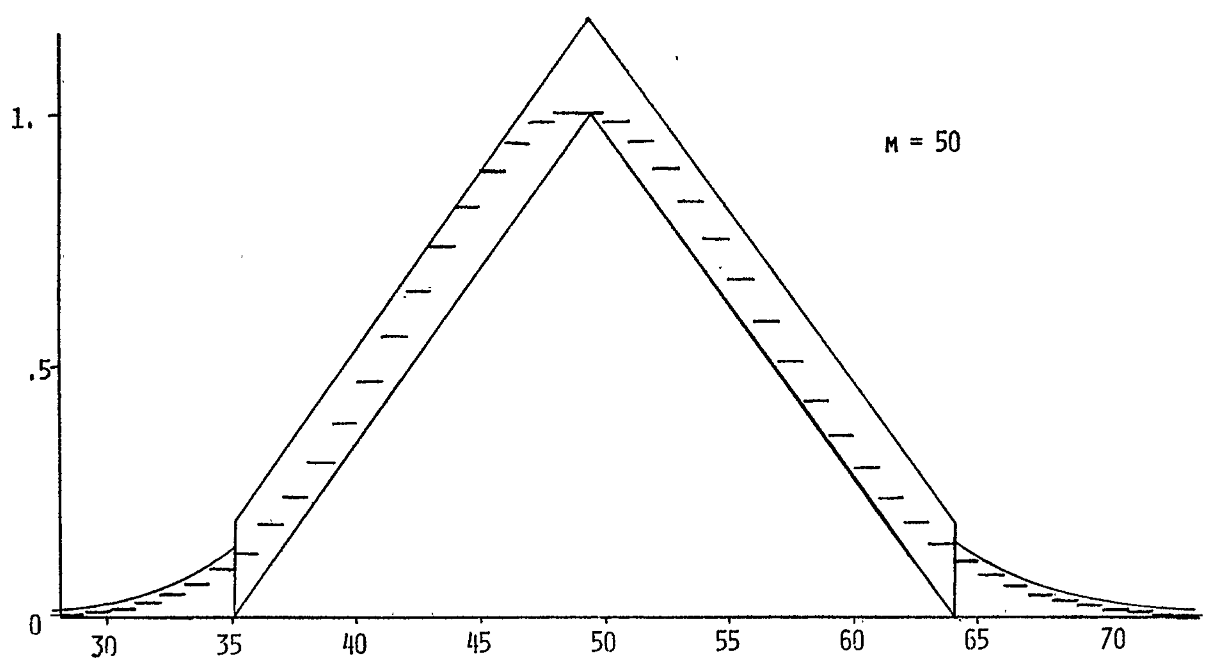
THEN X IS POISSON(M).

ALGORITHM

1. USE METHOD B FOR POISSON(K)
2. ACCEPT EACH EVENT WITH PROBABILITY M/K

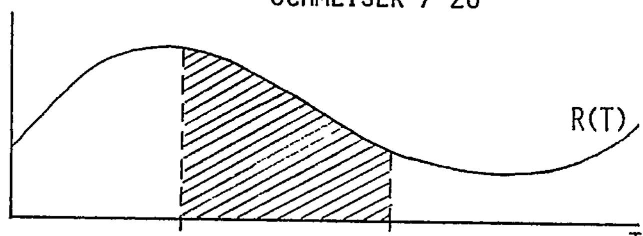
POISSON -- LARGE MEAN SCHMEISER / 25

COMPOSITION, SCHMEISER AND KACHITVICHYANUKUL (1981)



NONHOMOGENEOUS POISSON PROCESS

SCHMEISER / 26



-- INDEPENDENT EVENTS

-- TIME VARYING RATE R(T)

INVERSE TRANSFORMATION

-- $F_{T_I|T_{I-1}}(T_I) = 1 - \text{EXP}\left\{-\int_{T_{I-1}}^{T_I} R(T)\, dT\right\}$

SET TO U AND SOLVE FOR T_I AS A FUNCTION OF T_{I-1}

-- EXAMPLE: R(T) = 2*C*T

THEN $T_I = \{T_{I-1}^2 - \text{ALOG}(1-U)/C\}^{1/2}$

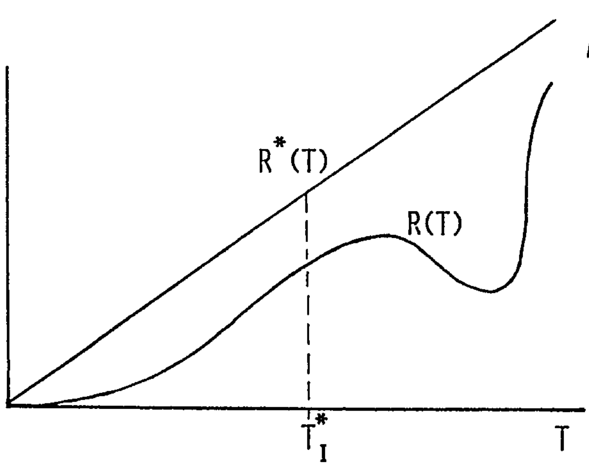
NONHOMOGENEOUS POISSON PROCESS

SCHMEISER / 27

THINNING: LEWIS AND SHEDLER (1979)

-- ANALOGOUS TO ACCEPTANCE/REJECTION

-- GIVEN ANOTHER RATE FUNCTION $R^*(T) \geq R(T)$ FOR ALL T



ALGORITHM

1. GENERATE POISSON PROCESS

   $T_1^*, T_2^*, T_3^*, \ldots$ USING $R^*(T)$

2. REJECT THE EVENT AT TIME T_I^*

   WITH PROBABILITY $R(T_I^*)/R^*(T_I^*)$

COMMON ERRORS SCHMEISER / 28

1. NORMAL VIA $\sum^{12} U_I - 6$

2. BOX - MULLER WITH U(0,1) VIA CONGRUENTIAL

3. ALGORITHMS WITH ASYMPTOTICALLY INFINITE EXECUTION TIMES
   A. BETA VIA JÖHNK WHEN P & / OR Q IS LARGE
   B. ERLANG VIA SUM OF EXPONENTIALS

4. TRUNCATION DUE TO MACHINE ACCURACY, ESPECIALLY WITH ORDER STATISTICS
   EXAMPLE: X = -ALOG(1 - U)

5. NONHOMOGENEOUS POISSON PROCESS USING CURRENT RATE

6. BLINDLY MODELLING BIVARIATE RELATIONSHIPS USING ONLY CORRELATION TO MEASURE DEPENDENCE.

SUMMARY SCHMEISER / 29

1. FEW BASIC CONCEPTS

2. CRITERIA DEPEND ON SITUATION

3. MAJORITY OF LITERATURE ON UNIVARIATE CONTINUOUS

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