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SOCIAL RELATIONS AS ALGEBRAIC RELATIONS

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ABSTRACT: It is convenient to analyze social systems by applying techniques of abstract algebra to the social relations within the system. Two complementary techniques of analysis are given: one heuristic; the other computational. The former is a dynamic use of graphs and their algebraic properties. This can result in the uncovering of new social relations. Examples given include the study of couple relationships, and management structures.

1. INTRODUCTION:

The analysis of social systems is replete with concepts such as structure and stratification. There is good reason to wonder whether these constructs have more than a superficial depth. Although much of mathematics is qualitative in nature (graph theory, abstract algebra, topology, etc.) too little effort has been spent on applying these theories where they may be useful, such as the analysis of social systems. It will be shown here that the relations in a social system can be analyzed by precise methods from algebra and graph theory to illuminate the structure imposed by these relations. It will further be shown that this can be done in a practical and even convenient manner.

There are a multitude of theories that deal effectively with qualitative relationships. We will draw upon algebra for its wealth. Graphs are used as a heuristic device to manipulate algebras. In both cases the emphasis on binary relations is practical and not theoretical (the extension of graphs to greater-than-binary relations is "hypergraphs"): we usually process (understand) binary relations with more clarity than greater-than-binary relations. Since graph theory is essentially the theory of binary relations it is no surprise that we can use graphs and algebra in a synergistic manner. From this there is also an apparent approach from the theory of automata and we will touch upon that lightly.

The class of social relations we will address is exhaustive. Emphasis will be placed on algebraic analysis of binary social relations. The algebra will be groupoids, that is, any algebra with one operation and closure. Even the closure condition is not strictly binding, as a new dummy product can be added to the algebra which represents "none of the above". Thus maximum generality is achieved. Section 6, "Social Systems" is here for theoretical completeness. Since the substance is better learned from examples, I recommend that the reader skim that section.

In particular we will study but not be limited to semigroups. For this reason matrix techniques are avoided since matrix multiplication is associative. In the analysis of social relations it will not be necessary (although it is interesting) to observe whether the relation is reciprocal, asymetric or transitive. This is not important to the actual analysis.

The partition classes generated by a hormormorphism are precisely "congruence" classes. These are subpopulations that function and are *acted on* as a unit. Congruence classes are extended to right semicongruence classes which only act as a unit. An example of the advantage of the latter is to consider the political notion of the "fifth column". A fifth column is a group of people who act as a unit but whose very purpose is not to be recognized (until too late). Again, through right semicongruence classes, maximum generality is achieved.

A very simple and efficient algorithm "the method of congruences" is given for identifying all right semicongruence and congruence classes. (It is appropriate here to identify the genesis of this method. In 1936, Todd and Coxeter developed a method for testing the abstract definition of a finite group. It has many other uses and is best discussed in Coxeter and Moser (1972, pp. 12-18). The method of Todd and Coxeter is based upon multiplication tables of cosets and elements of the group so that the defining relations are exploited. Neumann (1967) extended the same technique to semigroups. The difference between Todd and Coxeter, and Neumann is just that, in semigroups, inverses are not considered (e.g., in a group with U, V, W as cosets, and T an element, UT = W and VT = W imply U = V;

this is not necessarily true in semigroups). The method of congruences which is much simpler grew out of an attempt to extend Neumann's method to arbitrary groupoids. The right semicongruence relations that will be explored yield a very large class of partitions. I believe that this is a deeper and richer theory than the theory of blockmodels so popular in mathematical-sociology.

In an earlier work (Cargal, 1978) much use was made of graphs, many identical to those of other researchers. However, in that work use was made of the fact that graphs can represent algebraic groups as well as social structures. Graphs of groups are extended here to graphs of arbitrary groupoids and even to graphs of right semicongruences. This yields an important technique for uncovering *new* social relations. It is one thing to know how to anlayze social relations, but how does one know what relations to analyze? There will never be a satisfactory answer to this question, but the best device for uncovering such relations is to use graphs of groupoids. One tries to find the relations whose graphs yield the most algebraic structure. Another method is to relabel existing graphs in order to yield more algebraic structure. Both techniques are illustrated in the earlier paper. In this way graphs become many times more important. Rather than just representing structures they become a dynamic part of the analysis. The genesis of this is the work of A. Cayley (1878a, 1878b, 1889) who first studied graphs of groups.

2. SOCIAL RELATIONS

The relations we are interested in are said to be "social" because they are on people. The extension to other species is obvious and unnecessary to pursue. Otherwise, social relations are similar to mathematical relations. "A general n-ary social relation" is a set of ordered n-tuples of people within the system of our interest.

Good examples of general binary social relations might be "A likes B," or "A dislikes C," or "A communicates with D" or perhaps "A sleeps with E." We will define a "social relation" as a general social relation that is also a commutative function. The property of commutativity only enters if the relation is more than binary. For example, in the military example below, the top ranking soldier is independent of the order of the soldiers. The function requirement is not the restriction it appears. "A likes B" is replaced, for n people, by n social relations, "A likes X the most," ..., "A likes X the fourth," ..., "A likes X the least." An example of a social relation is the military system of rank. It is designed (in theory) so that given any n personnel there is a highest ranking member of that set. Rank is a stratifying (partitioning) relation on the military. For many purposes, this is the only relation on the military of any consequence.

This paper will concentrate on binary social relations. The primary tool of analysis will be the extension of these binary relations via operators to ternary relations. As a result, the method used applies very well to ternary relations which, by the above, will be commutative groupoids. However, the binary relations will usually yield groupoids that are not commutative, but will be what we will call "right semiquasigroups."

3. GROUPOIDS

A groupoid is a set with a well-defined binary operation. We will generally refer to the operation of the groupoid as "multiplication." It is convenient to view a groupoid on n elements by its corresponding multiplication table. This table is a matrix with n rows and n columns corresponding to the groupoid elements. The i-jth entry is defined to be $i \cdot j$. Since any of the n elements can be defined to be $i \cdot j$, there are $n^{(n^2)}$ groupoids of n elements.

In an organization of ten people, there are 10^{100} (the well known "gogool") possible groupoids on those people. Since for n = 10, the number of groupoids, 10^{100} is something like the number of subatomic particles in the universe it is a reasonable assertion that if none of these groupoids can yield insight into the social structure of the n people, then perhaps the concept of social structure is essentially valueless. Algebraically, many of these groupoids on n people are isomorphic (that is, just different labelings of the same algebraic structure). Socially, they are often not isomorphic. "Isomorphism" is an equivalence of size and structure and is sensitive to the particular theory. As a rule we do not always see individuals as equivalent (although sometimes we can). Consider the following groupoids that are algebraically isomorphic. We have a business organization of ten people, A, B, ..., J. Suppose that multiplication is defined by $X \cdot Y = Z$ where Z is that individual that X and Y choose in the organization (closure) that they like least. By definition, the groupoid is commutative. We will consider two cases:

1. $X \cdot Y = A$ for every X, Y in the organization, and
2. $X \cdot Y = J$ for every X, Y in the organization.

Both groupoids are said to be "zero semigroups," and they are algebraically isomorphic. If A is the manager and J is a secretary, we would view these cases quite differently. If both A and J were secretaries we might (or might not) view the groupoids as equivalent. Incidentally, the relation defined above is a commutative ternary function. For n people, if each individual is considered to have unique attributes (of interest) the $n^{(n^2)}$ corresponding groupoids are all nonisomorphic.

The best known types of groupoids are groups, and the literature on groups is extensive. Associative groupoids are said to be "semigroups." If a groupoid G has the property that for any a,b εG, ax = b and ya = b have *unique* solutions, G is said to be a "quasigroup." Of particular interest to us will be groupoids with the property that a,b εG, implies, ax = b has a unique solution. We will call these right semiquasigroups, with left semiquasigroups being the dual case. There should be no confusion between semiquasigroups and semigroup/quasigroups since the latter are precisely groups.

4. GRAPHS OF GROUPOIDS

A model of the use of the methods that are the subject of this paper is the precursor paper (Cargal, 1978). In it, the primary groupoids are groups. Also, great use is made of graphs of groups. It is remarked (p. 158) that "any groupoid...may be graphically represented..." This will be expanded upon here because graphs of groupoids have enormous utility. The reason this has received so little attention is that most algebraic studies are theoretical. Our interest is quite different. We desire the techniques to manipulate particular (social) groupoids so as to uncover truths peculiar to each (social) groupoid.

Given a groupoid, G, with n elements, A, B, ..., N, the corresponding (directed) graph has n vertices labeled by the elements of G. Each vertex has outgoing degree n and each arc is also labeled by an element of G. Specifically, one arc corresponding to each element of G leaves each vertex. Given the vertex, x, the arc labeled y leaving x, has the endpoint xy. That is, each type of edge corresponds to multiplication on the right by its corresponding element of G. This is the "complete" graph of the groupoid. Since types of arcs are differentiated, the graph is a "multigraph." Examples of graphs of groups are given in the prior paper.

Unlike the case of groups, tracing an arc against its direction does not necessarily correspond to multiplying on the right by an inverse. This is because in an arbitrary groupoid inverses do not necessarily exist and when they do, they are not necessarily unique. It is convenient (more often necessary) to have graphs of groupoids where not all of the multipliers (arcs) are represented. That is most graphs of groupoids (order n) will have outgoing degree m, with m labels, and m < n. We say such a graph is a graph of a particular type of groupoid (e.g., quasigroup) if it can be extended to a complete graph of that type of groupoid. Such extensions are not always unique. The extension of the graph of a semigroup with an identity and a defining set of generators (to the complete graph of a semigroup) is unique. In general, the graph of a groupoid of order n with m labels (on arcs) can be extended $n^{(n-m)n}$ ways.

A graph can be labeled as a graph of a groupoid if all vertices are of equal outgoing degree. In general, if a graph can be labeled *or relabeled* as a groupoid with significant structure, greater insight can be gained into the structure of the subject of study. This is a very important strategy to keep in mind, but it carries the caveat that the new labels should be meaningful to the analyst. An example of this sort is given in the earlier paper (Cargal, 1978, pp. 161-162) where relations are shown on the subsections of the Kariera tribe of Australia. The two relations are "subsection of son" and "subsection of daughter." The edges are relabeled according to "subsection of child of the same sex" and "subsection of child of the opposite sex." The change in structure is from a non-associative groupoid without well defined inverses, to a group. This new perspective was *not* based on insight: I relabeled for group structure and found afterward that the relabeling was meaningful. Furthermore looking at subsections rather than sections is based on sound anthropological principles. I believe that the previously undiscovered algebraic group of subsections is more significant that the group of sections (clans).

Graphs of groups were discussed in detail in the previous paper. The graph of a quasigroup is distinguished by no parallel arcs, and by every generator entering every vertex once. Graphs of semigroups will be discussed later. For convenience, undirected arcs represent two arcs, one in each direction.

5. CONGRUENCES

Structure in social systems implies the existence of stratification, both parallel and perpendicular to the parameters of the analyst's interest (such as income). Stratification is the partitioning of a system. We can always look at a partition or collection of partitions at the most particular level: as a mathematical partition. (Note: the intersection of mathematical partitions is another mathematical partition.) Such a partition is the separation of elementsinto classes with no element in two classes. The binary relation of belonging to the same class is thus an equivalence relation. Similarly, an equivalence relation on a set partitions the set.

Within a groupoid, H, given an equivalence relation τ ($a\tau b$ if only if a and b belong to the same τ-class) τ is said to be a "right semicongruence" if $a,b,x \varepsilon H$ and $a\tau b$ implies $ax\tau bx$. It is a "left semicongruence" if a, b, $x\varepsilon H$ and $a\tau b$ implies $xa\tau xb$ (Neumann, 1967, p. 1024). τ is a "congruence" if it is both a left and right semicongruence. If ψ is a homomorphism from groupoid M onto groupoid N ($\psi(ab) = \psi(a)\psi(b)$) the inverse classes of N ($\psi^{-1}(x)$, $x\varepsilon N$) in M form a congruence on M. If τ is a congruence on K, there is a homomorphism from K to the natural groupoid formed from its congruence classes. For any groupoid there are two trivial congruences; the congruence consisting of one class, and the congruence where each element constitutes a congruence class. Right semicongruence relations can be graphed by letting arcs correspond to multiplication on the right. If it is a congruence, this merely yields the graph of a groupoid.

6. SOCIAL SYSTEMS

Let G be a collection of n people, A, B, ..., N. Let ϕ be a collection of K($K \leq n$) binary *social* relations $\phi_1, \ldots, \phi_k$, on G. (Note that in general ϕ_i is *not* an equivalence relation.) Assign elements of G to ϕ arbitrarily on a one-to-one basis. If $x\varepsilon G$ and x is assigned to ϕ_j, define for all $y\varepsilon G$, $y \cdot x = \phi_j(x)$ (which is well defined since ϕ_j is a function). For $x\varepsilon G$, not assigned to ϕ, define for all $z\varepsilon G$, $z \cdot x = z$. G is thus a groupoid under this multiplication. If τ is a partition (equivalence relation) on G such that $ax\tau bx$ for every x corresponding to an element of ϕ (let us say $x\varepsilon\phi$) then for any $y\not\varepsilon\phi$, $ay\tau by$ also, and τ is a right semicongruence on G. We will consider G, ϕ and the defined multiplication a "social system" usually referred to only by "G".

The preceding blithely assumes that there are no more than n social relations on a system of n individuals. There is no reason the techniques discussed here cannot be extended to more than n relations. However, in most cases the existence of more than n generators will imply that some are redundant.

If τ is a nontrivial right semicongruence relation on a social system $G(\phi, \cdot)$, then τ partitions G into classes that act collectively with regard to the relations ϕ and the other τ classes. This indeed constitutes social structure by most reasonable criteria. The rest of this paper is devoted to showing by example that the converse is generally true: that non-trivial social structure implies right semicongruences. It is interesting to note that given a set of people the only social structures we can assume are the two trivial cases corresponding to the only two right semicongruences we can assume: the identity relation and the "unit" relation (where every element belongs to the same class). Given an arbitrary set of people, we can always view them collectively or individually and if there are not complex relationships amongst them (for instance, if they are strangers) these may be the only sensible viewpoints. Also, the following technique will be offered that along with graphs of groupoids facilitate analysis of social systems. Examples to be presented were *not* specially selected. They are representative of the methods of this paper.

7. THE METHOD OF CONGRUENCES

We desire to partition social systems meaningfully via right semicongruences and congruences (homomorphisms). Consider a multiplication table of a groupoid G. Let $x\varepsilon G$. We are interested in minimal right semicongruence classes containing x. In general, the identity relation leaves x as an entire right semicongruence class. The right semicongruence (or congruence) containing x and y ($y\varepsilon G$, $x \neq y$) in a minimal class is a different matter.

Postulate x and y ($x \neq y$) as belonging to the same class. This can always be satisfied by the unit relation. We want the right semicongruence containing x and y in the smallest class. The method itself will show that the minimal class is unique. Check the x and y rows of the multiplication table. Any elements in those rows in the same column are also in the same class. That is, if a is in the x-z position (row x, column z) and b is in the y-z position, a and b are in the same right semicongruence class. The extension to three or more elements is in the obvious way. The only rule is that if any element falls in two classes, the two classes are the same. That is, if x is in the same class as y, and x is in the same class as z, then x, y and z are all in the same class. We start again with our new class; any elements found in the same column of the x, y, z rows are in the same class. Every time a class is enlarged we begin again. (Note we begin with certain elements defined to be in the same class; the other elements are assumed to be separate classes.) Once the process has ended, i.e., the rows have been checked for each class without enlarging any class, the classes themselves define the relation ($r\tau s$ if and only if r and s belong to the same class).

If a, b, ..., n belong to the same left semicongruence class, any elements in their columns and in the same row, are in the same left semicongruence class. By applying this and the above, the congruence with the smallest class containing a particular set of elements can be found. For twenty elements or less, I have found this technique surprisingly easy. For groupoids with many elements the technique is easily computerized.

My technique for finding congruence classes is to construct a table with columns numbered as high as necessary. I put elements of the same class in the same column. When an element occurs in two columns, those columns are merged.

8. TWO IMPORTANT GENERALIZATIONS

The principal algebraic approach we have is the analysis of right semicongruences. In general, we will extend m (m < n) relations to n relations in order to attain a groupoid. This makes it possible to take advantage of a well developed theory. However, it is clear that the method of congruences applied to right semicongruences works whether m < n, m = n or m > n. If m relations on n objects are graphed, we have the structure of a finite automaton (Hopcroft and Ullman, 1979, Chapter 2) with the exception that start and final states are not indicated. This implies that the theory of automata and languages can be utilized as well (it has been (Skvoretz and Fararo, 1980).

A useful and well written introduction to finite automata is by Hartmanis (1967). The example below is borrowed from his paper (with a slight change of notation).

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} d & c \\ c & b \\ c & a \\ b & c \end{bmatrix} \end{array}$$

It is easy to see that these are no nontrivial right semicongruences in this system. Hartmanis discusses a type of extension we will not persue here, but is important to be aware of. It is "set system decompositions". He produces the system below:

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ c' \\ d \end{array} & \begin{bmatrix} d & c \\ c' & b \\ c' & a \\ c & a \\ b & c \end{bmatrix} \end{array}$$

This is identical to the prior example, except that c has been separated into two states. These two states, c and c', can be considered as two roles of c. What is accomplished is that now (through the role partition c to c and c') we have the following right semicongruence classes, abc|c'd. This second example from Hartmanis is extremely important for an entirely new reason. It is not hard to show that the relations can not be extended to five relations (thus yielding groupoids) (depending on how the relations are relabelled)) and such that there are still nontrivial right semicongruences. This shows that considering m relations (m < n or m > n) is a genuine extension of the theory.

9. MARRIAGE SYSTEMS

In the previous paper it was pointed out that marriage rules are significantly more general than the Australian systems studied in that paper would yield quasigroups: "Axioms 1 through 7 give a quasigroup in that where one gives and receives wives is uniquely determined (Cargal, 1978, p. 159)." If a society is divided into non-overlapping blocks such that any block receives wives from a particular class, this structure is a right semiquasigroup. If any of the tribes discussed in the previous paper were viewed on the individual level (rather than sections or subsections) appropriate marriage and child relations should have yielded the section perspective found by the ethnographers. However, it should have also found subsections where they existed that were not known to the ethnographers. Ethnographers using only pencil, paper, and insight are limited to the obvious structural partitionings and those explicitly stated (such as clans) by the subjects.

10. COUPLES

A very simple and yet instructive application of the method of congruences is the analysis of friendship on four people as partitioned by the relation of marriage. We will assume married couples functioning as couples (i.e., not separated or otherwise at war) and we will assume heterosexual couples. It is interesting that this last understanding does affect the analysis. It is implicit, and important for this particular analysis that as functioning couples, people are more fond of their spouses than they are of others.

Initially we will assume two married couples with c married to d and a married to b. The first relation will be i (identity) XiX. The second will be m (married to) amb, bma, cmd, and dmc. The third and fourth operators will be 1 and 2 which will rank the remaining two people by order of preference.

(We will not pursue a rigorous definition here, but the ranking should answer the question "who do you enjoy talking with the most?") Note we have defined a right semiquasigroup structure. Note also that the multiplication table of a (finite) right semiquasigroup is characterized by no element occuring twice in the same row.

Consider the following matrix. We will label the columns twice, according to operator and secondly to yield a groupoid (and left semicongruences and thereby congruences). Note that in labeling the columns by groupoid elements we have (4!=) 24 arbitrary choices. However, to achieve worthwhile congruences, the trick is to find a labeling such that one element is a left identity. This can always be done with a right semiquasigroup, and in n ways (where n is the order of the quasigroup).

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \\ i & m & 1 & 2 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} a & b & c & d \\ b & a & c & d \\ c & d & a & b \\ d & c & a & b \end{bmatrix} \end{array}$$

Often in couple relations, one couple will associate with the other couple for the company of one of the other individuals. In this example, both couples are doing that. a and b like c. c and d like a. There are four right semicongruences: 1) ab|c|d; 2) cd|a|b; 3) ac|db; and 4) ab|cd. The first right semicongruence shows that a and b as a unit regard c and d separately, and that c and d, since they are individuals, are consistent with themselves in how they regard each other and a and b. As an illustration of this relationship, observe that if d leaves town, a and b will have c over for supper. If c leaves town, a and b are likely to go on vacation themselves.

The second right semicongruence is symmetric to the first. The third divides the set into "popular and unpopular" classes. Since it is not a congruence, the new classes *do not* constitute a new groupoid under the relations i, m, 1, and 2. However the graph of the right semicongruence is given in Figure 1. If we had a congruence and hence a new groupoid, the new relations shown in Figure 1 would be consistent.

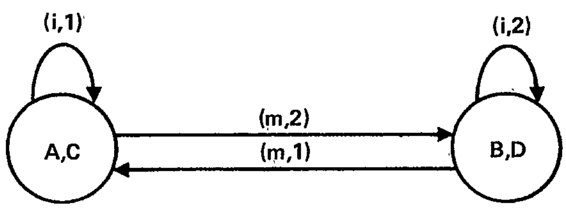

Figure 1. Two Couples Split into Popular and Unpopular Classes (Not a Groupoid)

The fourth right semicongruence is also a congruence. Note that in all of the possible relations as defined here on two couples, the couples themselves form a congruence and therefore a groupoid structure which is, in fact, a group. This is shown in Figure 2. So, as is to be expected, a system built on two couples yields structures where the couples function as classes.

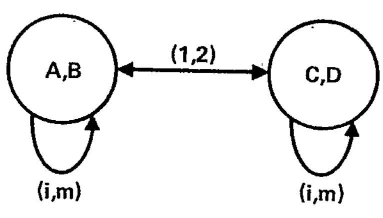

Figure 2. The Group of Two (Happy) Couples

If we assume that one couple associates with the other more for one individual than the other, but that the second couple is split with regard to the first couple, we get the following multiplication table (or one like it).

$$\begin{array}{cc} & \begin{matrix} a & b & c & d \\ i & m & 1 & 2 \end{matrix} \\ \begin{matrix} a\\ b\\ c\\ d \end{matrix} & \begin{bmatrix} a & b & c & d \\ b & a & c & d \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \end{array}$$

In this case, the right semicongruences are: 1) ab|c|d; and 2) ab|cd. We have already stated that 2) is a congruence and thus a right semicongruence. The interesting thing about 1) is that the dual case cd|a|b no longer holds. c and d as a class do not consistently regard a and b as individuals. Figure 3 gives the graph for ab|c|d which is not a groupoid since it is not a congruence.

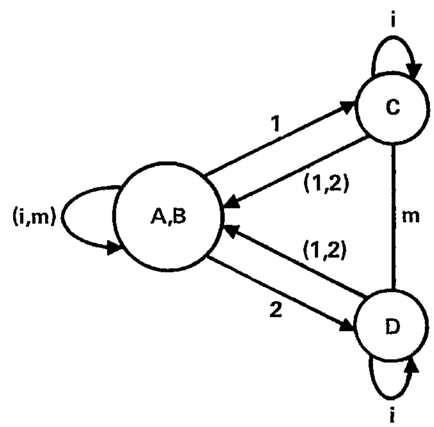

Figure 3. Two Couples With D Unpopular (Not a Groupoid)

We can, of course, have that both couples are split with regard to how they feel toward each other. There are two such cases. The first is:

$$\begin{array}{cc} & \begin{matrix} a & b & c & d \\ i & m & 1 & 2 \end{matrix} \\ \begin{matrix} a\\ b\\ c\\ d \end{matrix} & \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & b & a \\ d & c & a & b \end{bmatrix} \end{array}$$

It is clear from its graph shown in Figure 4 that this structure is a group. In this graph, the solid arcs represent m and the dotted arcs are 1. 2 is not shown because it is consistently 1's inverse. (The consistency has to be checked if we don't know yet that we have a group.) The generator m is also superfluous. The group is then the cyclic group Z_4. If the columns are labeled in the other three ways such that there is a left identity (badc, cdba, and dcab) the resultant groupoid is still the group Z_4 but with identities b, c and d, respectively. Note that there are no other right semicongruences other than ab|cd.

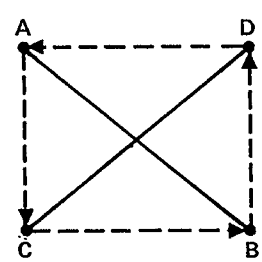

Figure 4. The Group of Two Couples Who Will Stop Seeing Each Other

The other "splitting" case is:

$$\begin{array}{c|cccc} & a & b & c & d \\ & i & m & 1 & 2 \\ \hline a & a & b & c & d \\ b & b & a & d & c \\ c & c & d & a & b \\ d & d & c & b & a \end{array}$$

Again we have a group (verify by graph if you don't recognize it) and it is $Z_2 \times Z_2$. This time ab|cd, ac|bd, ad|bc are all congruences. The groupoids given by ac|db and ad|bc are again groups and are shown in Figure 5. The last congruence reflects a consistent way of looking at the structure, but does not reflect a physical partitioning likely to occur.

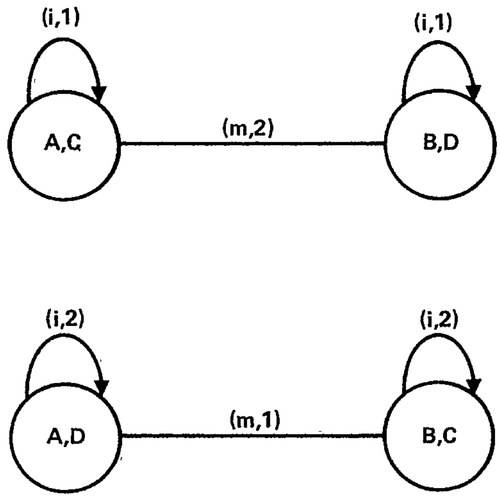

Figure 5. Two Groups on Two Very Compatible Couples

11. A FUNDAMENTAL PRINCIPLE

An empirical rule borne out by the examples of this paper and the previous paper can now be given. *In general the more congruences (homomorphisms) or right semicongruences a social system yields, the more cohesive it is for its given size.*

In the couples example just given, there were four cases with one congruence in common. Beyond that, the first case had three right semicongruences, the second has one, the third had none, and the fourth had two congruences. Similarly, the fourth is clearly the most cohesive, the third is the least cohesive, and the first and second fall in between.

The congruence in common reflects that all four cases have two couples who, by assumption, are drawn to each other (i.e., spouse to spouse). In the first and second case, we might reasonably expect a and c also to be drawn into intercouple conversations. I would argue that case 1 is slightly more stable (it has more right semicongruences) since b and d being cool to each other can just read while a and c fascinate one another. In case 2, d is in danger of driving b up the wall. If you don't accept this, view cases 1 and 2 as being close. Case 3 with no extra right semicongruences or congruences is cohesive as water. a likes c likes b likes d likes a. Furthermore, the person x likes the most, likes x the least. An interesting additional insight is that one couple is consistently drawn to the opposite sex (in the other couple) and the other couple is consistently drawn to the same sex (in the first couple).

We have all known (at least I have) couples that go out a lot together where they split nicely. This is like case 4 (with the additional two congruences). When they relate intercouple, a concentrates on c and b concentrates on d. Note that the division is either man-woman or man-man and woman-woman. These are two separate structures (when sex operators are considered (sex relations actually)) but both are highly stable.

12. MORE COUPLES

It is instructive to view the relations of a couple and any two other people. For the other two people (c and d) it is natural to redefine m as 1, 1 as 2, and 2 as being the least preferred person.

If c and d both prefer each other, we have the previous cases with the automatic ab|cd congruence. If a and b both prefer c over d (which is symmetric to d over c), then we have the right semicongruence ab|c|d. Assuming c didn't choose d first (above) then c chose d second or third, and we might reasonably expect d to be dropped from the association. In the case below, we have a right semicongruence which means little or nothing.

$$\begin{array}{cc} & \begin{array}{cccc} i & m & 1 & 2 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & a & d & b \\ d & b & c & a \end{bmatrix} \end{array}$$

There are other cases like this (with bc|a|d and bc|ad) which bear examination.

$$\begin{array}{cc} & \begin{array}{cccc} i & m & 1 & 2 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & a & d & b \\ d & c & b & a \end{bmatrix} \end{array}$$

However, in all cases if d (or c) is simply dropped from the picture, we have the following (or its dual equivalent) left.

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \\ i & m & 1 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} a & b & c \\ b & a & c \\ c & a & b \end{bmatrix} \end{array}$$

It has two right semicongruences, ab|c and a|bc (and no congruence). The second right semicongruence is the grouping of b and c according to their high regard for a. (Perhaps they both listen well together to a.) This certainly corresponds to my experience that a couple in dealing with two other people, not a couple, will usually drop one; and, as a rule, such groups are not cohesive. Consider the popular wisdom "it's a couple's world."

13. POPULARITY

A telling experiment is to consider n people who are absolutely consistent in their likes. We will assume six people, a, b, c, d, e and f, whose order of popularity is fixed (alphabetically). The multiplication table for the six people is as follows:

	i	1	2	3	4	5
a	a	b	c	d	e	f
b	b	a	c	d	e	f
c	c	a	b	d	e	f
d	d	a	b	c	e	f
e	e	a	b	c	d	f
f	f	a	b	c	d	e

The right semicongruences include ab|cd|ef, abc|def, abcd|ef, abcde|f, a|bc|def, a|bcd|ef, a|bcde|f, a|bcdef, ab|cde|f, ab|cdef. There are, in fact, thirty-two right semicongruences (and no possible left ones) on the above multiplication table. They all have one thing in common; each semicongruence is partitioned by popularity. The minimum right semicongruence containing m and n contains m, n and those elements intermediate (in popularity) to m and n. Conversely, every set containing a popularity sequence (e.g., c-d-e) is a right semicongruence class. It can be shown that this property is unrelated to the number of people in the system.

Consider the right semicongruence, abc|def. This roughly partitions the six people given above into popular and unpopular classes. However, there is no reason to expect the lower half to be chummy. They don't like each other either. However, abc|def has the sub-right semicongruence abc|d|e|f. This (again) reflects the reality. In high school, where popularity as a social relation perhaps reaches its zenith, the unpopular are thrown together by necessity. Their togetherness is possibly less cohesive than that of the more popular. (Lastly, it is easy to show by induction, that for a popularity system of n people, there are 2^{n-1} right semicongruence relations.)

14. MANAGEMENT AND SEMIGROUPS

Management structures we will consider will be characterized by most employees having one immediate supervisor. For generality, we will consider secretaries as having more than one supervisor. We will look at (small company) structures with four generators: S_1, S_2, S_3 and C. XS_i is the i^{th} subordinate of X. XC is the secretary. Figure 6 gives a good example of such a management structure. (This example is to challenge the algebra; more realistic examples are less challenging. In general, I have had consistent success representing different types of management structures as semigroups.) Labeling of vertices is almost never unique. In Figure 6, secretaries are diamonds and others are circles. The boss is defined to be the multiplicative identity 1. The generators S_2, S_2, S_3 are placed fron left to right under each vertex. One of the realistic features of such a graph is that whereas the subordinate relation is theoretically transitive, it is actually only weakly so. In general, it is bad form to fire one's subordinate's subordinate. The result can be a midlevel manager enraged over usurped power.

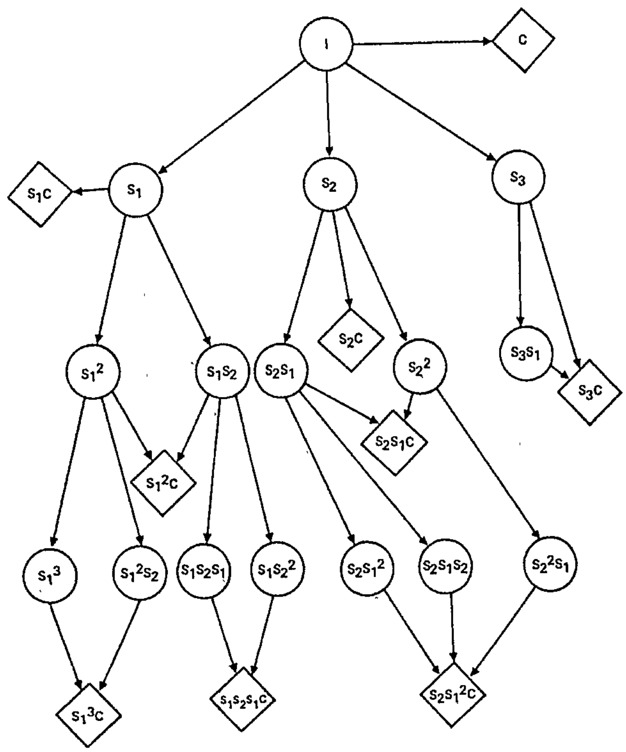

Figure 6. A Management Structure with Secretaries Assigned to Hierarchies (Not a Semigroup)

The key to graphing semigroups is adjoining a 1 if there isn't one already. Such an element can always be added (even if there is a 1 already) by the rule $1 \cdot x = x \cdot 1 = x$ for each x in the semigroup. A graph with a 1 and a set of generators is then the graph of a semigroup, if every word relation on 1 holds at every vertex.

A word is just a product of generators (no inverses!). A word relation is of the form $W_1 = W_2$. Thus a graph, with a vertex labeled 1, is a semigroup if every equality $W_1 = W_2$ (where W_1 and W_2 are products) that holds at 1, holds at all other vertices. The act of graphing management structures is to satisfy the semigroup condition since semigroups have perhaps the most important property one can ask in a groupoid. A real-life justification is that in a semigroup the boss's perspective is true (often trivially) for subordinates as well. The boss is in a better position to make appropriate decisions. This rather compensates for the frequent human failing of lack of empathy.

In Figure 6 the operators S_2 and S_3 are often not explicit due to lack of appropriate subordinates. Secretaries have only one possibility, for XC a secretary, $XC = XC^2 = XCS_1 = XCS_2 = XCS_3$. For an employee, U, with only a secretary subordinate, $U = US_1 = US_2 = US_3$ (instead of $US_1 = UC$). Consistent with this, I prefer for higher level vertices U, to define $US_3 = U$ rather than $US_3 = US_2$. Often to achieve a semigroup, I have to do both ($US_3 = U$; $VS_3 = VS_2$) in the same graph. The result seems to cause no problems except the need for extra care in writing the multiplication table. Both strategies produce meaningful (intuitive) results.

Again in Figure 6, there is the relation $S_3C = S_3S_1C$. Whether we define $S_2S_3 = S_2$ or $S_2S_3 = S_2^2$, this relation is not true at S_2. Therefore, the graph in Figure 6 is not a semigroup. The graph in

Figure 7 treats S_3C as two individuals and achieves a semigroup by the rule that, for mid-level vertices U, if S_3 isn't defined, $US_3 = U$. Note that there is nothing intrinsically wrong with a relation like $S_3C = S_3S_1C$. The problem in Figure 6 is that it's inconsistent with the rest of the structure. With respect to that organization, S_3C seems to have two jobs. To a degree, associativity implies consistency. (Relations on each vertex are consistent with relations on the 1.)

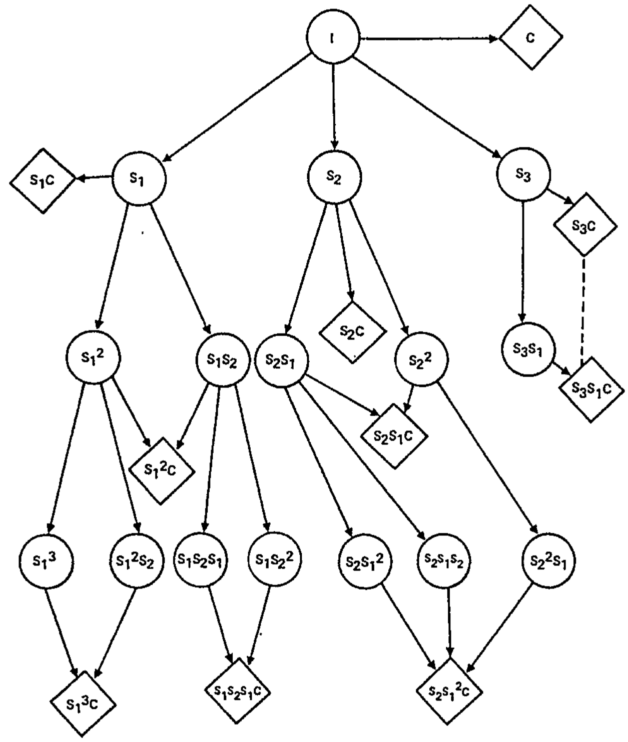

Figure 7. The Management Structure of Figure 6, as a Semigroup

The graph we have just examined is characterized by secretaries being assigned to employees of a particular hierarchy. It is more common in industry for secretaries to be assigned to a division as in Figure 8. With the same convention as before ($S_1S_3 = S_1$; $S_2S_3 = S_2$; $S_3S_2 = S_3^2 = S_3$) the graph in Figure 8 represents a semigroup. A multiplication table for this graph is given below. Note that the secretaries are all left zeros ($ZU = Z$ for all U). In examining the multiplication table and disregarding order, no two rows will have exactly the same elements occuring. Similarly, no two columns will have exactly the same elements. (This follows in a management type structure from elementary results of Green (1951, 163-172).)

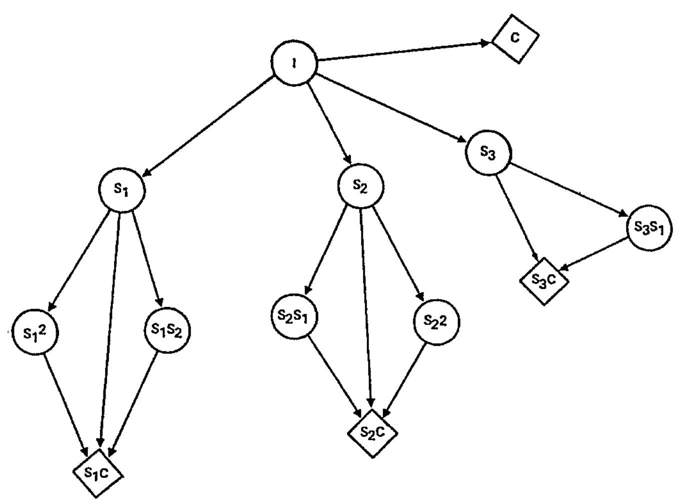

Figure 8. A Management Structure Where Secretaries are Assigned to Divisions (A Semigroup)

	1	s_1	s_2	s_3	s_1^2	s_1s_2	s_2s_1	s_2^2	s_3s_1	c	s_1c	s_2c	s_3c
1	1	s_1	s_2	s_3	s_1^2	s_1s_2	s_2s_1	s_2^2	s_3s_1	c	s_1c	s_2c	s_3c
s_1	s_1	s_1^2	s_1s_2	s_1	s_1^2	s_1^2	s_1s_2	s_1s_2	s_1^2	s_1c	s_1c	s_1c	s_1c
s_2	s_2	s_2s_1	s_2^2	s_2	s_2s_1	s_2s_1	s_2^2	s_2^2	s_2s_1	s_2c	s_2c	s_2c	s_2c
s_3	s_3	s_3s_1	s_3	s_3	s_3s_1	s_3s_1	s_3s_1	s_3	s_3s_1	s_3c	s_3c	s_3c	s_3c
s_1^2	s_1^2	s_1^2	s_1^2	s_1^2	s_1^2	s_1^2	s_1^2	s_1^2	s_1^2	s_1c	s_1c	s_1c	s_1c
s_1s_2	s_1s_2	s_1s_2	s_1s_2	s_1s_2	s_1s_2	s_1s_2	s_1s_2	s_1s_2	s_1s_2	s_1c	s_1c	s_1c	s_1c
s_2s_1	s_2s_1	s_2s_1	s_2s_1	s_2s_1	s_2s_1	s_2s_1	s_2s_1	s_2s_1	s_2s_1	s_2c	s_2c	s_2c	s_2c
s_2^2	s_2^2	s_2^2	s_2^2	s_2^2	s_2^2	s_2^2	s_2^2	s_2^2	s_2^2	s_2c	s_2c	s_2c	s_2c
s_3s_1	s_3s_1	s_3s_1	s_3s_1	s_3s_1	s_3s_1	s_3s_1	s_3s_1	s_3s_1	s_3s_1	s_3c	s_3c	s_3c	s_3c
c	c	c	c	c	c	c	c	c	c	c	c	c	c
s_1c	s_1c	s_1c	s_1c	s_1c	s_1c	s_1c	s_1c	s_1c	s_1c	s_1c	s_1c	s_1c	s_1c
s_2c	s_2c	s_2c	s_2c	s_2c	s_2c	s_2c	s_2c	s_2c	s_2c	s_2c	s_2c	s_2c	s_2c
s_3c	s_3c	s_3c	s_3c	s_3c	s_3c	s_3c	s_3c	s_3c	s_3c	s_3c	s_3c	s_3c	s_3c

I have performed several studies on Figure 8 that I cannot report here for reasons of space. None of the results were of a theoretical nature. However, I feel that examining right semicongruences and congruences is an important tool for restructuring an organization. If, for example, we decide to move Y into X's division and Z is in every congruence class containing X and Y, perhaps Z should also be reassigned to X. By working with congruences, we disrupt as few relations as we have to. Also, it is usually instructive to graph the resultant homorphic structure (of the congruence classes).

An example of one study on Figure 8 is the listing of all congruence classes containing S_2^2. Using the method given in this paper and many photocopies of the multiplication table, I computed the smallest congruence class containing S_2^2 and each other element. For example, if we assume S_2^2 and S_1^2 belong to the same congruence class, so do S_1S_2 and S_2S_1. That is, every congruence class that contains S_2^2 and S_1^2 contains S_1S_2 and S_2S_1. Since congruences are associated in a one-to-one manner with homorphisms, another way of stating this is that if we view S_1^2 and S_2^2 as equivalent, so are S_1S_2 and S_2S_1 (and every pair of the four).

In every congruence where S_2^2 is not isolated, S_2S_1 is in the same class as S_2^2. S_2S_1 is the only element which has this property. In other words, anything that affects S_2^2 and anyone else affects the other employee of that rank in that division, S_2S_1. If I were boss, I would think twice before giving S_2^2 a raise and not giving S_2S_1 a raise. This is consistent with my experience. The company I worked for (at this writing), SAI, is organized into divisions of often around fifteen employees (range 1-100) and maybe two secretaries. Secretaries are answerable to only one division (SAI has about 3,000 employees). In general, employees are oblivious to the happenstance of employees of nearly equal rank in other divisions. Most, however, are acutely sensitive to the rise and fall of co-workers in their own division. There is a custom against discussing salaries.

15. FURTHER SEMIGROUPS ON MANAGEMENT STRUCTURE

There is no reason not to look at management structures bottom up. However, this will rarely result in a semigroup. Also, one can go for inverses of sorts by defining a new operator P where XP is the highest ranking immediate superior of X. This will usually give a structure that is not a semigroup or a quasigroup. I feel that the semigroup structures shown so far are much more useful.

A new "dummy" member can be added to any management structure. This is an individual, who is an individual who is subordinate to everyone. In the management semigroups we looked at earlier, this individual would be a zero. If we define "less than" to mean "subordinate to" and "greater than" in the obvious manner, we have a complete lattice. For every subset there is also a least subordinate. We can define a multiplication *, where a*b is that individual who is the lowest ranking superior to both a and b. If a is b's boss, a*b = b. Similarly, we could define a*b to be the highest ranking subordinate to both a and b. Both of these multiplications give rise to commutative semigroups (Petrich, 1973, p. 12). Let * be as in the former case, and let S be an arbitrary subset of a management structure M. Then S under * generates in M a subsemigroup $\overline{S}$. $\overline{S}$ contains all the managers that are directly responsible for S. The product of all the elements in S (which is then in $\overline{S}$) is the lowest ranking superior to all of S.

16. A REAL SOCIOGRAM

One day I went to the eleventh floor of my building (different company) and with the help of the receptionist, found six individuals who were acquainted with one another. I gave them each a slip of paper and asked each to write his or her name on the top, and the names of the other five in the order they liked them. The subjects themselves reformulated the problem as listing the others in the order of who they *wanted* to spend the most time with. To the question of how to handle ties, I replied, "I don't care." The whole process took five minutes. I would not recognize them again. I was even unaware of the male-to-female ratio. Fortunately, they appeared to take my request seriously, partly because they had little time to entertain uncooperative ideas. I have, for reasons of ethics, resisted all impulses to return (to the receptionist) and test my hypotheses. The data is given in the table below with $X \cdot i = X$ and (for $j = 1, 2, 3, 4, 5$) $X \cdot j$ = the individual X likes j^{th}. There is vastly more qualitative data that could be added. The main drawback is that we're never certain whether X likes Y less than Z or Z more than Y. We will use only the multiplication table and the fact that A, B C are the women and D, E, F the men (I couldn't ignore the names).

	i	1	2	3	4	5
A	A	B	D	E	C	F
B	B	F	D	C	E	A
C	C	B	D	E	A	F
D	D	C	B	F	E	A
E	E	F	B	A	C	D
F	F	E	B	C	A	D

The simplest metric is more than adequate to show B is most popular and A the least. In the above matrix C, D, E, F are all roughly intermediate in popularity. No matter how we label the columns, we don't seem to get non-trivial left semicongruences. Therefore, we will look for right semicongruences instead of congruences. There are two non-trivial right semicongruences: AC|B|D|E|F and AC|EF|B|D. The contrast between the pairs E and F, and A and C is illuminating. If any individuals should be lumped together they are E and F. E and F are clearly buddies. They list each other first followed by B and they seem to share a dislike for D (who ranks second with all the women). E and F are the only pair who list each other first. A and C are together because they are alike. They certainly don't appear to be close. They agree on everything including low regard for one another. One should speculate about their regard for themselves and keep in mind the old saw that what people like least in others is what they like least in themselves. It is the fact that A and C function alike that makes them a right semi-congruence class. B on the other hand is singular in her popularity and her view of the others. D is the lone male who, as we remarked earlier, is well liked by the women and not by the men. A graph of this right semicongruence structure is provided in Figure 9.

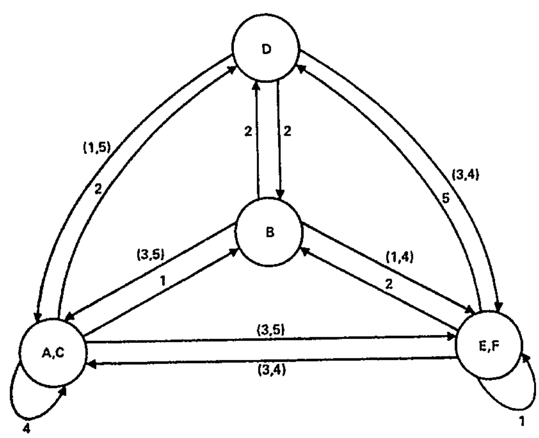

Figure 9. The Partitioning By Friendships of Six (Real) People (Not a Groupoid)

17. ADJUSTMENTS

There is, in large structures, a tendency to consider "90%" or "95%" congruence classes. That is, one wants to overlook these elements that spoil an "almost" congruence or right semicongruence. This is a common technique in using blockmodels. Another way to do this is simply to try the congruence without certain elements. Remember that a slight change in a multiplication table can change the congruence classes considerably. If in the previous example we remove F, we get the following table:

	i	1	2	3	4
A	A	B	D	E	C
B	B	D	C	E	A
C	C	B	D	E	A
D	D	C	B	E	A
E	E	B	A	C	D

In this case, BCD|E|A is a right semicongruence. The individuals B, C, D give rise to a permutation matrix. They cannot be decomposed at all: B likes D who likes C who likes B. However, they are all in total agreement about E and A.

That the removal of one individual so affects the structure is largely because of the small size of the example. It follows that a good way to use the method of congruences is to use it on the $2^n - 1$ nonempty subsets of the system. It may then become obvious that one or two individuals are inconsistent with the system. By the Fundamental Principle given earlier, if one individual is all the difference between a commutative group and a groupoid devoid of nontrivial right semicongruences, perhaps that individual should be transferred. For large systems, the various subsets can of course be only analyzed selectively. All these approaches are statistical and thus transform the methodology from qualitative to quantitative. For large social systems, statistical methods become attractive. However, this is outside of the scope of this paper.

18. FINAL REMARKS

Little has been said about n-ary relations for the obvious reason that they are a lot of trouble. However, they should not be forgotten. Often binary relations imply higher order relations. For example, in the commutative semigroup of the management lattice, the relation on n individuals that gives their least ranking superior is their product, i.e., the repeated application of the defined binary operation. Most n-ary relations cannot be simplified to binary relations.

One of the hindrances to mathematical-sociology is that the practitioners are theory-and-academia oriented (as this paper is) and not problem oriented. An exception is the work by Lipton and Budd (1978) in data security. In that paper they are concerned with protecting computer memory from unauthorized access. Some of the "objects" of this analysis are people, since people are what the system is protected against. Although this work (Lipton and Budd's) is not considered social analysis, it is, as surely as the study of marriage systems is, and it has a more clearly defined product. They use in an integrated fashion, graph theory and the theory of languages (similar to our use of algebra). (Significantly, they did not draw upon the mathematical-sociology literature.) Although it is similar to this work they produced primarily computational results via dynamic programming. Budd (1980) corrected and extended their earlier work, and I have been able to generalize some of this considerably, to the point of calculating a very large class of social transformations on a finite population in cubic time. The algorithm and many results are Budd's, and my work (for which there is a manuscript) has been to show precisely how large is the class of problems solved (it is very large). As a final result OAO Corporation now has an operational data security analyser (DSA).

Two types of techniques have been touched upon in this paper. Graphing is a paper-and-pencil treatment of groupoids that leads to a fast visual understanding. Congruences, and semicongruences can be enumerated quite efficiently with computers. Graphs give the strongest method for analyzing relations. They are a tool to be used to recognize the important relations and to discover new significant relations. That this is something of an art is not surprising. Although it means that these techniques can be dismissed as "ad hoc," by that token, group theory is ad hoc, given the unsolvability of the word problem.

There are many questions that need to be explored. We know something about finding new social relations, given an initial set of relations, but how does one choose the initial set? What can be done to raise the labelling of graphs of groupoids beyond an art? How does one recognize for example, when a graph of social relations can be decomposed into two graphs with significant structure? At present, the application of these techniques is not difficult but usually requires some ingenuity. Also, a knowledge of people is as important as the knowledge of mathematical techniques.

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However, I did 90 percent of this research and all of the previous paper (Cargal, 1978) sitting at a pizza stand in 1976. It was done only because of the material support (in 1976) of my father Dr. Buchanan Cargal. His confidence and emotional support were singular at times when Professors were contemptuous of this work. Anyone who ever worked with Buck knows that he performed applied mathematics rather than the theory of applied mathematics that is taught. His work was realized in military applications for over twenty-five years. His integrity manifested itself in all his work. He died suddenly August 11, 1980, the day that this paper went onto camera ready copy. Individuals who knew him mourn the many losses that his death brought. All (who knew him) mourn our loss of his humor. Most mourn our loss of his integrity. Quite a few mourn their loss of his knowledge of laser-testing and laser-safety. I mourn the loss of twenty-five years of private research in number theory.

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