A PRAGMATIC COMPARISON OF EXPLICIT INTEGRATION ALGORITHM TECHNIQUES

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ABSTRACT

Seven Runge-Kutta and seven predictor-corrector integration algorithms are reviewed as candidates for solution of differential equations in a real-time environment. The algorithms are evaluated pragmatically in terms of maximum error, error versus time, and computation time required. The time error history of Runge Kutta algorithms is shown to be proportional to the errors produced by an exactly computed power series of the same order as the algorithm. The algorithms found to have the lowest errors are Hamming, Adams-Moulton, Adams-Bashforth, Runge-Kutta-Ralston and Runge-Kutta-Merson. The Runge-Kutta-Ralston algorithm is the most stable and is favored for reasonable speed and accuracy. When time is critical, rectangular integration is the fastest method, but, however, the Corrected Euler algorithm is shown to give significantly more than twice the accuracy for approximately twice the computation time and should also be considered for fast integration. Error tables are presented for solution of the harmonic oscillator problem as functions of oscillator frequency, time-step, time interval, and samples per cycle. These tables may be used to predict the error expected for a given integration algorithm and time-step when the solution frequency is known apriori or postpriori.

INTRODUCTION

The choice of a suitable integration algorithm for the solution of differential equations in continuous systems is a double optimization problem whose purpose is to minimize the computation error and minimize the computation time subject to the constraints of an acceptable solution. These constraints usually place an error limit on the maximum absolute permissable error or on RMS (root mean square) error and usually impose a time limit determined by the maximum permissable frame time (for a solution run in real time) or a dollar ceiling for off-line or batch programs. The optimization problem would seem to possess no global solution because computation speed and accuracy are inversely related. However, for a given algorithm, error may be expected to decrease with decreasing time-step up to a certain point beyond which error will increase due to accumulation of error within the computation loop. This error may be due to round-off of the least significant bit in the machine word or may be due to error generated by the integration algorithm. Typical variation of error with time-step is shown in Figure 1.

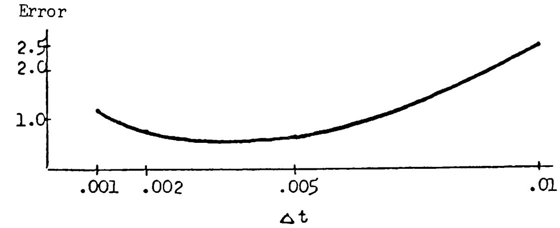

FIGURE 1.

Error is defined in this paper as the difference, computed value minus true value:

$$E_i = X_{c_i} - X_i \tag{1}$$

where X_i is the true value of the ith solution and X_{c_i} is is the value computed by the algorithm. If equation 1 is taken to define the components of an error vector $\bar{E}$, the maximum absolute error is the maximum absolute component of the vector,

$$E_{max} = \text{MAX} \left| E_i \right|_{i=1}^{N} \tag{2}$$

where N is the number of solutions generated. The RMS error is,

$$E_{RMS} = \frac{\sqrt{\bar{E} \cdot \bar{E}}}{N} \tag{3}$$

The computation time, which we are interested in minimizing, may be expressed as,

$$T_C = T_K + N\, t_f \tag{4}$$

where T_C is the total computation time for a program which computes N sets of solutions with a frametime of t_f and T_K is a constant time value which includes initialization, bookkeeping and other tasks accomplished outside the integration computation loop. The computation time factor (Real Time Factor) is then,

$$RTF = \frac{T_c}{N\Delta t} \tag{5}$$

where Δt is the time-step.
Combining equations 4 and 5,

$$RTF = \frac{T_k}{N\Delta t} + \frac{t_f}{\Delta t} \tag{6}$$

The first term in equation 6 is a constant because $N\Delta t$ is a constant equal to the solution end-time. The frame time t_f for a given problem is usually also a constant and the Real Time Factor is therefore inversely proportional to the time step used. This is shown in Figure 2.

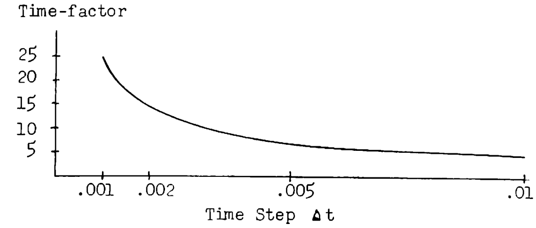

FIGURE 2.

Data in the literature on how much maximum error we can expect candidate integration algorithms to generate as functions of solution frequency, time, and time-step is generally lacking or not detailed enough for continuous system simulation purposes. The bibliography provides a list of sources from which the algorithms in this paper were obtained. In this list, the most extensive comparison of integration algorithms is done by Martens (8).

In this paper Taylor series, Runge-Kutta, and predictor-corrector algorithms are reviewed. Solutions of the harmonic oscillator problem are used as a test to determine how well these algorithms perform as functions of oscillator frequency and time-step, and how error may be expected to behave with time for damped and undamped solutions. This information may be used to predict error behavior for an algorithm when natural frequency and time step are known. The first algorithms to be considered are Taylor Series based -- this set includes the rectangular, or Euler, and trapezoidal integration algorithms. Runge-Kutta algorithms will then be considered and finally predictor-corrector algorithms will be examined.

TAYLOR SERIES

The Taylor series expansion for an arbitrary function is,

$$f(t) \simeq f(t_o) + f'(t_o)\,(t-t_o) + f''(t_o)\,(t-t_o)^2/2! + f'''(t_o)\,(t-t_o)^3/3! + \ldots \tag{7}$$

How good an approximation this expansion is to f(t) depends upon how fast the series converges. If $t-t_o$ is less than unity the series will converge provided the derivatives are finite and do not diverge faster than the convergence of the time step powers. Another condition on the validity of equation 7 is that the derivatives must exist (be single-valued) in the closed interval $[t_o, t]$. This means that when discontinuities occur in the function or in its derivatives, computation of the solution must stop and be restarted with the new initial conditions. Let the integration time-step Δt be,

$$\Delta t = t_1 - t_o \tag{8}$$

where t_o is the current time and t_1 is the time at the end of the time-step. Then the following integration algorithm may be derived from equation 7:

$$f(t_1) = \int_{t_o}^{t_1} f'(t)\,dt \simeq f(t_o) + f'(t_o)\,\Delta t + f''(t_o)\,(\Delta t)^2/2! + f'''(t_o)\,(\Delta t)^3/3! + \ldots \tag{9}$$

For small values of Δt, and well-behaved derivative terms, this series will converge quickly. To use this algorithm the current value of the function and the values of its derivatives are needed at time t_o. For a set of Nth order differential equations only the Nth order derivatives will in general be defined and the higher order derivatives will have to be approximated computationally. In the Runge-Kutta algorithm this occurs indirectly by matching terms with a Taylor expansion. Rough estimates of derivative values may be obtained from the definition of a derivative,

$$f'(t) = \frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t) - f(t - \Delta t)}{\Delta t} \tag{10}$$

The accuracy of equation 10 will depend upon the smallness of the time-step Δt, or upon how linear f(t) is within the interval $(t - \Delta t, t)$. For a fixed time-step we can expect the accuracy of equation 10 to decrease with increasing frequency (or increasing severity of inflections). The truncation error after summing n terms in equation 9 is,

$$E_{n+1} \geq f^{[n+1]}(t_o)\,\frac{(\Delta t)^{n+1}}{(n+1)} \tag{11}$$

This could be used as an estimate of the lower error bound, but errors due to higher derivatives and errors induced through equation 10 will make actual errors greater than this. Repeated application of equation 10 to higher order derivatives may result in no significant improvement in the results of equation 9: Taylor series corrections carried out to ninth order yield errors which are no lower than those of a third order Taylor series for a 5 cps solution, and a fourth order Taylor series algorithm is not significantly better than a third order Taylor series. This is due to: (1) the error induced by using a finite value for Δt in equation 10, and (2) the convergence of higher order terms in equation 9 to corrections which are less than the least significant bit in the machine word.

In practical applications of integrating algorithms we are interested in how much maximum error the algorithm will generate over the period our solution is run and in how short a time it takes to generate the solution. Error and computation time data are given in Tables 1-4 for first through fourth order Taylor series integration algorithm solutions to the differential equation for an undamped harmonic oscillator,

$$\ddot{X} + \omega^2 X = 0 \tag{12}$$

The first order Taylor series is also called rectangular or Euler integration. The errors oscillate at the natural frequency of the solution and accumulate to diverge with time. The maximum errors which were observed to occur during one second time intervals are given in the tables. The real time factor was obtained by dividing the amount of time required by the computer to solve the problem by the amount of time the oscillator was run (ten seconds for these runs). The time behavior of the errors for selected frequencies and time steps are plotted in Figures 3-17. The effect of using terms higher than third order in the Taylor series integration algorithm may be seen by comparing Tables 3 and 4. Fourth order terms result in no significant improvement over third order. This is due to the rate at which the series converges. The algorithm was run single precision on a UNIVAC 1240 with 30-bit word length. For single precision 30-bit machine words, the fourth term is smaller than the least significant bit in the word (except for frequencies $\geq 10\pi$ radians/second, where the fourth derivative is large enough to delay convergence). For some algorithms it therefore may be desirable to decrease the order of the algorithm when decreasing step-size, in order to avoid cycling through insignificant additions in the integration algorithm. An example of this may be seen in Tables 2 and 3 for a frequency of one cps and time step of .001, where the errors are essentially the same for the two algorithms. Errors for Taylor series algorithms are lowest when explicit expressions for the highest derivatives are available. Table 5 gives errors which result from a fourth order Taylor series when exact derivatives are available. Figures 18-21 show the time behavior of these errors at selected frequencies and time-steps. Table 5 represents the least errors which can be expected from a Runge-Kutta computation, because the Runge-Kutta algorithm is based on a Taylor series fit. This is discussed in the next section.

The largest errors are to be expected for undamped oscillatory solutions such as those given in Tables 1-4. When a system is damped or has controlled feedback, errors will decrease with time as the steady-state solution is driven to its final value. Error and computation time data are given in Tables 6-9 for Taylor series solutions to the damped harmonic oscillator equation,

$$\ddot{X} + \dot{X} + (\omega^2 + .25)\, X = 0 \qquad (13)$$

The time behavior of these errors is shown in Figures 22-36 for selected frequencies and time-steps.

Plots of maximum errors as surface functions of frequency and time-step, obtained from the data in Tables 1-9 are shown in Figures 37-46.

RUNGE-KUTTA INTEGRATION

Taylor series methods are most accurate when solution frequencies are low or higher derivatives can be computed exactly without resorting to use of equation 10. When derivatives cannot be computed, specialized fits to Taylor series, such as Runge-Kutta, are used.

The Runge-Kutta method is derived by assuming a calculable form for the solution to a first order differential equation and then matching terms with a Taylor series to determine relations and values of constants. Once the first order expression is obtained it may be extended by successive substitution to apply to simultaneous and higher order differential equations. Derivation of the fourth order Runge-Kutta for a first order differential equation will be outlined first.

Let

$$\dot{X}(X,t) = f(X,t) \qquad (14)$$

The expression for a fourth order Runge-Kutta solution is,

$$X(t_1) = X(t_o) + a\, A_1 + b\, A_2 + c\, A_3 + d\, A_4 \qquad (15)$$

where,

$$t_1 = t_o + \Delta t$$

$$A_1 = \Delta t\, f(x_o, t_o)$$

$$A_2 = \Delta t\, f(x_o + m\, A_1, t_o + m\, \Delta t)$$

$$A_3 = \Delta t\, f(x_o + r\, A_2 + (n-r)A_1, t_o + n\, \Delta t)$$

$$A_4 = \Delta t\, f(x_o + s\, A_2 + t\, A_3 + (p-s-t)A_1, t_o + p\, \Delta t)$$

a, b, c, and d are determined by matching terms with a fourth order Taylor series expansion:

$$\begin{aligned} X(t_1) &= X(t_o + \Delta t) \\ &= X(t_o) + \Delta t\, X'(t_o) + \frac{(\Delta t)^2 X''(t_o)}{2!} + \frac{(\Delta t)^3 X'''(t_o)}{3!} + \frac{(\Delta t)^4 X^{iv}(t_o)}{4!} \\ &= X(t_o) + \Delta t\, f(x_o,t_o) + \frac{(\Delta t)^2 f'(x_o,t_o)}{2!} + \frac{(\Delta t)^3 f''(x_o,t_o)}{3!} + \frac{(\Delta t)^4 f'''(x_o,t_o)}{4!} \end{aligned} \qquad (16)$$

Expanding the expressions for A_2, A_3, and A_4 in equation 15 in two-dimensional Taylor series about the point (x_o, t_o) results in partial derivative terms which may be collected and matched term by term with partial derivatives collected from chain-rule expansions of f', f", and f"' in equation 16. The resulting relations are,

$$\begin{aligned} a + b + c + d &= 1 \\ bm + cn + dp &= \tfrac{1}{2} \\ bm^2 + cn^2 + dp^2 &= 1/3 \\ bm^3 + cn^3 + dp^3 &= \tfrac{1}{4} \\ cmr + dnt + dms &= 1/6 \\ cmnr + dnpt + dmps &= 1/8 \\ cm^2r + dn^2t + dm^2s &= 1/12 \\ dmrt &= 1/24 \end{aligned} \qquad (17)$$

RUNGE-KUTTA-SIMPSON

Equations 17 are eight equations for ten unknowns. To solve them, two of the constants are chosen arbitrarily. Various choices result in the various fourth order Runge-Kutta algorithms. The most popular choice is b = c and m = n, which results in the Runge-Kutta-Simpson formula:

$$\begin{aligned} A_1 &= \Delta t\, f(x_o, t_o) \\ A_2 &= \Delta t\, f(x_o + \tfrac{1}{2} A_1\, ,\ t_o + \tfrac{1}{2}\Delta t) \\ A_3 &= \Delta t\, f(x_o + \tfrac{1}{2} A_2\, ,\ t_o + \tfrac{1}{2}\Delta t) \\ A_4 &= \Delta t\, f(x_o + A_3\, ,\ t_o + \Delta t) \end{aligned} \qquad (18)$$

$$X(t_1) = X(t_o) + \frac{1}{6}(A_1 + 2\, A_2 + 2\, A_3 + A_4)$$

Equations 18 will generate numeric solutions to first order differential equations of the form

$$X' = f(X,t) \qquad (19)$$

To generate solutions to second order differential equations of the form

$$X'' = f(x,x',t) \qquad (20)$$

equations 18 may be extended by the following technique:

Let

$$X' = Y$$
$$Y' = X'' = f(X,Y,t) \qquad (21)$$

The equation for Y' is now a function of three variables. For equations of many variables,

$$x' = f(x,y,z, \cdots, t)$$
$$y' = g(x,y,z, \ldots, t) \qquad (22)$$
$$z' = h(x,y,z, \cdots, t)$$
$$\vdots$$

Equations 22 may be expressed in vector form as,

$$\overline{X}' = \overline{F}(\overline{X}, t)$$
where
$$X_i' = F_i(X_1, X_2, X_3, \ldots, t) \qquad (23)$$

Then equations 18 assume the form,

$$\overline{A}_1 = \Delta t\ \overline{F}(\overline{X}_o, t_o)$$
$$\overline{A}_2 = \Delta t\ \overline{F}(\overline{X}_o + \tfrac{1}{2}\overline{A}_1, t_o + \tfrac{1}{2}\Delta t)$$
$$\overline{A}_3 = \Delta t\ \overline{F}(\overline{X}_o + \tfrac{1}{2}\overline{A}_2, t_o + \tfrac{1}{2}\Delta t) \qquad (24)$$
$$\overline{A}_4 = \Delta t\ \overline{F}(\overline{X}_o + \overline{A}_3, t_o + \Delta t)$$
$$\overline{X}(t_o + \Delta t) = \overline{X}(t_o) + \frac{1}{6}(\overline{A}_1 + 2\overline{A}_2 + 2\overline{A}_3 + \overline{A}_4)$$

Equations 21 are a special case of equations 22 and, using equations 23, solutions to equation 20 therefore may be generated by,

$$A_{11} = \Delta t\ \dot{x}_o$$
$$A_{12} = \Delta t\ f(x_o, \dot{x}_o, t_o)$$
$$A_{21} = \Delta t\ (\dot{x}_o + \tfrac{1}{2}A_{12})$$
$$A_{22} = \Delta t\ f(x_o + \tfrac{1}{2}A_{11}, \dot{x}_o + \tfrac{1}{2}A_{12}, t_o + \tfrac{1}{2}\Delta t)$$
$$A_{31} = \Delta t\ (\dot{x}_o + \tfrac{1}{2}A_{22})$$
$$A_{32} = \Delta t\ f(x_o + \tfrac{1}{2}A_{21}, \dot{x}_o + \tfrac{1}{2}A_{22}, t_o + \tfrac{1}{2}\Delta t) \qquad (25)$$
$$A_{41} = \Delta t\ (\dot{x}_o + A_{32})$$
$$A_{42} = \Delta t\ f(x_o + A_{31}, \dot{x}_o + A_{32}, t_o + \Delta t)$$
$$x(t_1) = x(t_o) + \frac{1}{6}(A_{11} + 2A_{21} + 2A_{31} + A_{41})$$
$$\dot{x}(t_1) = \dot{x}(t_o) + \frac{1}{6}(A_{12} + 2A_{22} + 2A_{32} + A_{42})$$

where,

$$t_1 = t_o + \Delta t$$

Equations 25 are a set of equations which may be used to generate solutions to any general non-linear second order differential equation. The methods used to develop equations 23 and 24 may be extended to other algorithms and to any simultaneous set of differential equations reduced to a first order set.

The errors produced when the Runge-Kutta-Simpson algorithm was used to solve the harmonic oscillator problem are shown in Table 10. The time behavior of these errors is identical to that shown in Table 5 and Figures 18-21 for the fourth order Taylor series. Other variants of the Runge-Kutta algorithm are given below.

RUNGE-KUTTA-GILL

The Runge-Kutta-Gill algorithm, introduced by Gill (5), is obtained when b and c in equations 17 are chosen to be,

$$b = \frac{1}{3}\left[1 - \frac{1}{\sqrt{2}}\right]$$
$$c = \frac{1}{3}\left[1 + \frac{1}{\sqrt{2}}\right] \qquad (26)$$

These values were originally chosen because they are well-suited to a technique in which temporary storage space is minimized when large sets of first order differential equations are solved. The first order form of the algorithm is:

$$A_1 = \Delta t\ f(x_o, t_o)$$
$$A_2 = \Delta t\ f(x + \tfrac{1}{2}A_1, t + \tfrac{1}{2}\Delta t)$$
$$A_3 = \Delta t\ f\left[x + (-\tfrac{1}{2} + \frac{1}{\sqrt{2}})A_1 + (1 - \frac{1}{\sqrt{2}})A_2, t + \tfrac{1}{2}\Delta t\right]$$
$$A_4 = \Delta t\ f\left[x - \frac{1}{\sqrt{2}}A_2 + (1 + \frac{1}{\sqrt{2}})A_3, t + \Delta t\right] \qquad (27)$$
$$X(t_1) = X(t_o) + \frac{1}{6}\left[A_1 + 2(1 - \frac{1}{\sqrt{2}})A_2 + 2(1 + \frac{1}{\sqrt{2}})A_3 + A_4\right]$$

Errors resulting from application of the Runge-Kutta-Gill algorithm to the undamped harmonic oscillator equation are summarized in Table 11.

RUNGE-KUTTA-RALSTON

This version was derived by Ralston (10) to minimize the truncation error. It does not really minimize truncation error but lowers the bound for truncation error. The formulas are:

$$A_1 = \Delta t\ f(x_o, t_o)$$
$$A_2 = \Delta t\ f(x + .4A_1, t + .4\Delta t)$$
$$A_3 = \Delta t\ f(x + .296978A_1 + .158760A_2, t + .455737\Delta t) \qquad (28)$$
$$A_4 = \Delta t\ f(x + .2181A_1 - 3.05096A_2 + 3.83286A_3, t + \Delta t)$$
$$X(t_1) = X(t_o) + .17476A_1 - .551481A_2 + 1.20553A_3 + .171185A_4$$

Errors resulting when the Runge-Kutta-Ralston algorithm is used to solve the harmonic oscillator equation are given in Table 12.

RUNGE-KUTTA-MERSON

The Runge-Kutta-Merson algorithm was designed by Merson (9) to provide an equation to estimate the truncation error of the Runge-Kutta formula. The Merson formulas are:

$$\begin{aligned}
A_1 &= \frac{1}{3}\Delta t\, f(x_o,t_o) \\
A_2 &= \frac{1}{3}\Delta t\, f(x + A_1,\ t + \frac{1}{3}\Delta t) \\
A_3 &= \frac{1}{3}\Delta t\, f(x + \tfrac{1}{2} A_1 + \tfrac{1}{2} A_2,\ t + \frac{1}{3}\Delta t) \\
A_4 &= \frac{1}{3}\Delta t\, f(x + \frac{3}{8} A_1 + \frac{9}{8} A_3,\ t + \tfrac{1}{2}\Delta t) \\
A_5 &= \frac{1}{3}\Delta t\, f(x + \frac{3}{2} A_1 - \frac{9}{2} A_3 + 6 A_4,\ t + \Delta t) \\
X(t_1) &= X(t_o) + 2 A_4 + \tfrac{1}{2}(A_1 + A_5)
\end{aligned} \tag{29}$$

An estimate of the truncation error is given by,

$$E_t \simeq \frac{1}{30}(2 A_1 - 9 A_3 + 8 A_4 - A_5) \tag{30}$$

This formula is only valid when $f(x,t)$ is linear in x and in t.

Errors resulting in application of the Runge-Kutta-Merson algorithm to the harmonic oscillator are given in Table 13.

RUNGE-KUTTA-BLUM

The formulas for the Runge-Kutta-Blum algorithm are:

$$\begin{aligned}
A_1 &= \Delta t\, f(x_o,t_o) \\
A_2 &= \Delta t\, f(x + \tfrac{1}{2} A_1,\ t + \Delta t) \\
A_3 &= \Delta t\, f(x + \tfrac{1}{2} A_2,\ t + \tfrac{1}{2}\Delta t) - \tfrac{1}{2} A_2 \\
A_4 &= \Delta t\, f(x + \tfrac{1}{2} A_2 + A_3,\ t + \Delta t) + 2 A_3 \\
X(t_1) &= X(t_o) + \tfrac{1}{2} A_2 + \frac{1}{6}(A_1 + A_4)
\end{aligned} \tag{31}$$

Errors resulting when the Runge-Kutta-Blum algorithm is used to solve the harmonic oscillator are summarized in Table 14.

RUNGE-KUTTA-KUTTA

Formulas for the Kutta form of the Runge-Kutta algorithm are:

$$\begin{aligned}
A_1 &= \Delta t\, f(x_o,t_o) \\
A_2 &= \Delta t\, f(x + \frac{1}{3} A_1,\ t + \frac{1}{3}\Delta t) \\
A_3 &= \Delta t\, f(x - \frac{1}{3} A_1 + A_2,\ t + \frac{2}{3}\Delta t) \\
A_4 &= \Delta t\, f(x + A_1 - A_2 + A_3,\ t + \Delta t) \\
X(t_1) &= X(t_o) + \frac{1}{8}(A_1 + 3 A_2 + 3 A_3 + A_4)
\end{aligned} \tag{32}$$

Errors obtained when the Runge-Kutta-Kutta algorithm is used to solve the harmonic oscillator are shown in Table 15.

RUNGE-KUTTA-ENGLAND

The Runge-Kutta-England algorithm was designed to provide a built-in estimate of the error accumulated during a pass through the algorithm. Time is stepped twice within the algorithm and the error is estimated by summing the coefficients. If the error is acceptable then the solution may be completed; otherwise the time-step may be readjusted. Equations for the Runge-Kutta-England algorithm are:

$$\begin{aligned}
A_o &= \Delta t\, f(X_o,t_o) \\
A_1 &= \Delta t\, f(X_o + \tfrac{1}{2} A_o,\ t_o + \tfrac{1}{2}\Delta t) \\
A_2 &= \Delta t\, f(X_o + \tfrac{1}{4}(A_o + A_1),\ t_o + \tfrac{1}{2}\Delta t) \\
A_3 &= \Delta t\, f(X_o - A_1 + 2 A_2,\ t_o + \Delta t) \\
t_1 &= t_o + \Delta t \\
X_1 &= X(t_1) = X_o + \frac{1}{6}(A_o + 4 A_2 + A_3) \\
A_4 &= \Delta t\, f(X_1,\ t_o + \Delta t) \\
A_5 &= \Delta t\, f(X_1 + \tfrac{1}{2} A_4,\ t_1 + \tfrac{1}{2}\Delta t) \\
A_6 &= \Delta t\, f(X_1 + \tfrac{1}{4}(A_4 + A_5),\ t_1 + \tfrac{1}{2}\Delta t) \\
A_7 &= \Delta t\, f(X_o + \frac{1}{6}(- A_o - 96A_1 + 92 A_2 - 121 A_3 \\
&\qquad + 144 A_4 + 6 A_5 - 12 A_6),\ t_1 + \Delta t)
\end{aligned} \tag{33}$$

Local error accumulation estimate is,

$$R = \frac{1}{90}(- A_o + 4 A_2 + 17 A_3 - 23 A_4 + 4 A_6 - A_7)$$

$\leq$ Error Criteria else adjust Δt and repeat above

$$A_8 = \Delta t\, f(X_1 - A_5 + 2 A_6,\ t_1 + \Delta t)$$

$$X(t_2) = X(t_1 + \Delta t) = X_1 + \frac{1}{6}(A_4 + 4 A_6 + A_8)$$

Errors and solution times obtained using the Runge-Kutta-England algorithm to solve the harmonic oscillator with fixed time-steps and without examining the error estimate are summarized in Table 16.

SUMMARY OF THE RUNGE-KUTTA METHODS

The Runge-Kutta-Merson and Runge-Kutta-Ralston algorithms give the lowest errors. Ralston's version gives slightly lower errors for low frequency solutions ($\leq$ 2 cps) while the Merson version gives slightly lower errors for high frequency ($\geq$ 5 cps) solutions and samples of ten or more per cycle. In view of the amount of computation time used, the faster Ralston version is preferable to the Merson version.

The fastest of the Runge-Kutta algorithms is the Kutta version, which also generates the greatest amount of error. Ordering the Runge-Kutta versions from the fastest to the slowest - Kutta, Blum, Gill, England, Simpson, Ralston, Merson - note that this also roughly corresponds to the order of increasing accuracy.

PREDICTOR-CORRECTOR ALGORITHMS

Predictor-corrector algorithms offer an advantage of improved accuracy and, if not too many iterations are required, improved speed for variable step algorithms. Step-size determination for variable step algorithms will not be examined nor discussed in detail in this paper. A method commonly used for step-size determination is to carry out the computation in duplicate using two integration algorithms (say a fourth order and a third order algorithm). When the two solutions agree, the step-size is permitted to be increased (by 50% to 100%) for the next integration interval. When the two solutions differ by more than a given number, step-size is decreased and the integration computations are repeated until they agree. This method is useful for controlling step-size in Runge-Kutta algorithms. In predictor-corrector algorithms, step-size control may be built into the algorithm.

A predictor-corrector algorithm uses an equation to compute a predicted (approximate) solution value at the end of a time step; a second equation uses the predicted solution to compute a corrected solution which hopefully is closer to the true solution. The corrected value may be used iteratively to generate better corrections until successive corrections converge to differ by no more than a given amount. Step-size control may be built into an iterative predictor-corrector algorithm by decreasing step-size whenever two or more iterations are necessary to converge the solution. The Corrected Euler algorithm is used below to illustrate the predictor-corrector process.

CORRECTED EULER

In Figure 47, it is desired to integrate f(x,t) over the time interval (t_o,t_1) to obtain a solution near the true solution x_1. Rectangular (Euler) integration results in multiplication of the slope at (x_o,t_o) by the time-step Δt to achieve the approximate (predicted) solution x_p. A better solution (the corrected solution x_c) can be obtained if the average of the slope at (x_o,t_o) and the slope at (x_1,t_1) is used. Because x_1 is not known, x_p is used to approximate the slope at (x_1,t_1):

$$f(x_1,t_1) \simeq f(x_p,t_1) \tag{34}$$

The algorithm for the Corrected Euler is:

$$\begin{aligned} x_p(t_1) &= x(t_o) + \Delta t\, f(x_o,t_o) \\ x_c(t_1) &= x(t_o) + \tfrac{1}{2}\Delta t \left[f(x_o,t_o) + f(x_p,t_1) \right] \end{aligned} \tag{35}$$

These are also the equations for second order Runge-Kutta integration. By iteration, the corrected value may be substituted back into the second of equations 35, in place of x_p, to yield a better correction, and iteration continued until successive solutions differ by a prescribed amount; Beckett and Hurt (1) have shown that this does not always yield a better solution. The error tables presented in this paper for predictor-corrector algorithms are based on the use of one predictor term and one corrector term with no iterations in the corrector equations. Table 17 summarizes the errors produced when the corrected Euler algorithm is used to solve the undamped harmonic oscillator equation. Note that the Corrected Euler algorithm is faster than the fastest Runge-Kutta routine (Kutta's version) and gives smaller errors. It is not, however, more accurate than the Ralston or Merson versions of the Runge-Kutta method except for an extremely low frequency solution (.1 cps).

EULER PREDICTOR-CORRECTOR

The Euler Predictor-Corrector algorithm was derived to give truncation errors of the order of $(\Delta t)^3$. Its equations are,

$$\begin{aligned} x_p(t_2) &= x(t_o) + 2\,\Delta t\, f(x_1,t_1) \\ x_c(t_2) &= x(t_1) + \tfrac{1}{2}\Delta t \left[f(x_1,t_1) + f(x_p,t_2) \right] \end{aligned} \tag{36}$$

where,

$$t_1 = t_o + \Delta t$$

$$t_2 = t_1 + \Delta t$$

The truncation errors for the corrector equation are,

$$E_c(t_2) = \frac{1}{12} (\Delta t)^3\, X'''(t') \qquad t_2 \le t' \le t_1 \tag{37}$$

and the truncation errors for the predictor equation are,

$$E_p(t_2) = -\frac{1}{3} (\Delta t)^3\, X'''(t') \tag{38}$$

The Euler Predictor-corrector algorithm requires that two start-up values, x_o and x_1, be computed and stored for use at each integration step. A Runge-Kutta routine is usually used to begin initial computation, because the Runge-Kutta method requires no start-up values. If, during computation, a discontinuity is encountered, where initial conditions for the next integration interval are abruptly changed, the start-up routine must be called again. The errors produced when the algorithm is used to solve the harmonic oscillator equation, shown in Table 18, are greater than those of the Corrected Euler algorithm. These errors may be reduced significantly, however, by using the truncation error estimates of equations 37 and 38 to modify the final result. This produces the Modified Euler Predictor-corrector algorithm described below.

MODIFIED EULER PREDICTOR-CORRECTOR

A better approximation for $x(t_2)$ may be obtained by subtracting the truncation errors of equations 37 and 38 from the predictor and corrector expressions given in equations 36:

$$x(t_2) \simeq x_p + \frac{1}{3} (\Delta t)^3\, X'''(t') \tag{39}$$

$$x(t_2) \simeq x_c - \frac{1}{12} (\Delta t)^3\, X'''(t') \tag{40}$$

Solving equations 39 and 40 simultaneously for $x(t_2)$,

$$x(t_2) = \frac{x_p + 4\, x_c}{5} \tag{41}$$

Equation 41, coupled with equations 36, is the Modified Euler Predictor-Corrector Algorithm. Errors generated by this algorithm, shown in Table 19, for solutions of the harmonic oscillator equation are significantly lower than those of the Corrected Euler or Euler Predictor-corrector for large time steps and high frequencies.

ADAMS-BASHFORTH PREDICTOR-CORRECTOR

The Adams-Bashforth Predictor-corrector algorithm is formed from fourth order backward-difference approximations for the predictor and corrector equations:

$$x_p(t_4) = x_3 + \frac{\Delta t}{24}\Big[55\ f(x_3,t_3) - 59\ f(x_2,t_2)$$

$$+ 37\ f(x_1,t_1) - 9\ f(x_o,t_o) \Big]$$

$$+ \frac{251}{720}(\Delta t)^5\ f^V(t')$$

$$x_c(t_4) = x_3 + \frac{\Delta t}{24}\Big[f(x_1,t_1) - 5\ f(x_2,t_2) \qquad (42)$$

$$+ 19\ f(x_3,t_3) + 9\ f(x_p,t_4) \Big]$$

$$- \frac{19}{720}(\Delta t)^5\ f^V(t')$$

where $t_4 \geq t' \geq t_3$

The last terms in equations 42 are the truncation errors. Note that four pairs of start-up values - (x_o,t_o), (x_1,t_1), (x_2,t_2), and (x_3,t_3) - are needed to use the algorithm. An integration routine constructed using equations 42, ignoring the truncation terms, yields the errors tabulated in Table 20 when applied to the harmonic oscillator equation. These errors are slightly less than those of the Modified Euler Predictor-corrector. The truncation terms are used in the Adams-Moulton algorithm, described below.

ADAMS-MOULTON

The truncation errors in equations 42 may be used to attempt a lowering of the final error in the same manner as described for the Modified Euler Predictor-corrector. This results in the Modified Adams-Bashforth, or Adams-Moulton, predictor-corrector. The final form is,

$$x(t_4) = \frac{251\ x_c + 19\ x_p}{270} \qquad (43)$$

where x_c and x_p are computed from equations 42. Errors tabulated in Table 21 for the Adams-Moulton Algorithm solution to the harmonic oscillator are not significantly better than those of the Adams-Bashforth algorithm.

MILNE PREDICTOR-CORRECTOR

A fourth order algorithm developed by Milne uses the equations,

$$x_p(t_4) = x_o + \frac{4\Delta t}{3}\Big[2\ f(x_3,t_3) - f(x_2,t_2)$$

$$+ 2\ f(x_1,t_1) \Big]$$

$$x_c(t_4) = x_2 + \frac{\Delta t}{3}\Big[f(x_p,t_4) + 4\ f(x_3,t_3) \qquad (44)$$

$$+ f(x_2,t_2) \Big]$$

$$x(t_4) = \frac{1}{29}(28\ x_c + x_p)$$

Like the Adams-Bashforth Algorithm, these equations require four pairs of start-up values. Errors resulting when the Milne Predictor-Corrector is used to solve the undamped harmonic oscillator are shown in Table 22. For low frequencies, $\leq$ 1 cps, the errors are the same order of magnitude as those of the Adams-Moulton and Adams-Bashforth algorithms, but for solutions above 5 cps the algorithm is unstable for samples less than 100 per cycle.

HAMMING'S MODIFIED PREDICTOR-CORRECTOR

Hamming (6) developed an algorithm similar to the Milne Predictor-corrector but which has greater stability. It uses the same predictor equation as the Milne algorithm; the corrector equation is exact for fourth order polynomials. The magnitude of the truncation error in the corrector equation is greater than in Milne's algorithm, but the corrector equation has greater stability. Equations for the Hamming algorithm are:

$$x_p(t_4) = x_o + \frac{4}{3}\Delta t\Big[2\ f(x_3,t_3) - f(x_2,t_2)$$

$$+ 2\ f(x_1,t_1) \Big]$$

$$x_m(t_4) = x_p(t_4) - \frac{112}{121}\Big[x_p(t_3) - x_c(t_3) \Big]$$

$$x_c(t_4) = \frac{1}{8}\Big\{ 9\ x_3 - x_1 + 3\ \Delta t\Big[f(x_m(t_4),t_4) \qquad (45)$$

$$+ 2\ f(x_3,t_3) - f(x_2,t_2) \Big]\Big\}$$

$$x_4 = x_c(t_4) + \frac{9}{121}\Big[x_p(t_4) - x_c(t_4) \Big]$$

x_m is a modifier which is computed for use in the corrector equation. Initial values at four previous time steps plus the predicted and corrected values at the last time-step are required for start-up; these are: x_o, x_1, t_1, x_2, t_2, x_3, t_3, $x_p(t_3)$, $x_c(t_3)$.

Comparing Table 23, the table of errors produced when the Hamming's Modified Predictor Corrector is used to solve the harmonic oscillator, with the errors produced by the other predictor-corrector algorithms, we see that the error is slightly, but not significantly, larger than the error of the Milne algorithm for low frequency solutions. For high frequency (5-20 cps) solutions the Hamming algorithm is stable and has significantly lower errors than the Milne algorithm. The Hamming and Adams-Moulton algorithms are nearly equivalent in their error and computation time profiles.

SUMMARY OF THE PREDICTOR-CORRECTOR INTEGRATION METHODS

Of all the predictor-corrector algorithms considered here, the Adams-Moulton and Hamming algorithms have the lowest errors. Ranking the predictor-corrector algorithms in order of least to greatest error, the order is: Hamming, Adams-Moulton, Adams-Bashforth, Modified-Euler, Corrected-Euler, Euler Predictor-Corrector, Milne. Ranking the algorithms in order of computation speed, from fastest to slowest, the order is: Corrected Euler, Euler Predictor-Corrector, Modified Euler, Milne, Adams-Bashforth, Adams-Moulton, Hamming. If computation time and accuracy are both important criteria in choosing an algorithm, the Modified Euler is probably the best choice. The Modified Euler algorithm is more stable than Hamming's; it has slightly lower errors at .1 cps and errors are comparable to those of Hamming's algorithm at frequencies $\geq$ 5 cps, provided sample rates are 100 or more per solution cycle.

SUMMARY

Of all the algorithms considered - Runge-Kutta and predictor-corrector - the Runge Kutta-Ralston is the most stable and the most accurate. The Modified Euler Predictor-Corrector offers the best compromise between speed and accuracy. When the most important criteria is speed, the Euler method (rectangular integration, or first order Taylor series) is fastest. However, for approximately twice the computing time as the Euler method, significantly more than twice the accuracy can be obtained by use of the Corrected Euler Algorithm, and both the Corrected Euler and the Euler algorithms should be considered for fast integration.

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EULER INTEGRATION ALGORITHM FOR UNDAMPED SOLUTIONS

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL) 1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACT
0.1	0.2 PI	.010	0.161	0.178	0.178	0.637	0.994	1.037	0.992	0.531	1.468	1.996	1000	0.86
0.1	0.2 PI	.005	0.081	0.090	0.100	0.332	0.496	0.520	0.473	0.210	0.721	0.993	2000	1.59
0.1	0.2 PI	.002	0.031	0.032	0.018	0.101	0.196	0.205	0.167	0.186	0.362	0.404	5000	3.79
0.1	0.2 PI	.001	0.014	0.034	0.030	0.091	0.105	0.095	0.100	0.224	0.273	0.271	10000	7.43
0.5	1.0 PI	.010	5.056	10.368	15.948	21.810	28.037	34.510	41.311	48.456	55.965	63.854	200	0.86
0.5	1.0 PI	.005	2.498	5.058	7.696	10.386	13.144	15.956	18.853	21.822	24.864	27.999	400	1.59
0.5	1.0 PI	.002	0.993	1.995	3.005	4.028	5.059	6.103	7.154	8.217	9.288	10.371	1000	3.79
0.5	1.0 PI	.001	0.496	0.992	1.493	1.993	2.497	3.004	3.527	4.068	4.612	5.159	2000	7.43
1.0	2.0 PI	.010	21.771	48.272	80.528	119.755	168.450	226.904	298.068	384.706	490.187	618.573	100	0.86
1.0	2.0 PI	.005	10.368	21.810	34.510	48.456	63.854	80.727	99.460	120.133	142.946	168.337	200	1.59
1.0	2.0 PI	.002	4.034	8.223	12.572	17.105	21.819	26.733	31.834	37.141	42.654	48.395	500	3.79
1.0	2.0 PI	.001	1.995	4.028	6.103	8.217	10.371	12.569	14.806	17.095	19.416	21.780	1000	7.43
2.0	4.0 PI	.010	118.421	375.022	928.574	999.999	999.999	999.999	999.999	999.999	999.999	999.999	50	0.86
2.0	4.0 PI	.005	48.272	119.785	226.904	384.706	618.573	958.205	999.999	999.999	999.999	999.999	100	1.59
2.0	4.0 PI	.002	17.139	37.171	60.572	88.028	120.172	157.938	202.040	253.687	314.187	355.032	250	3.79
2.0	4.0 PI	.001	8.223	17.105	26.733	37.141	48.395	60.582	73.735	88.009	103.379	119.977	500	7.43

TABLE 1.

SECOND ORDER TAYLOR SERIES, UNDAMPED HARMONIC OSCILLATOR

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.161	0.178	0.178	0.637	0.994	1.037	0.992	0.531	1.468	1.996	1000	0.86
0.1	0.2 PI	.005	0.081	0.090	0.100	0.332	0.496	0.520	0.473	0.210	0.721	0.993	2000	1.59
0.1	0.2 PI	.002	0.031	0.032	0.018	0.101	0.196	0.205	0.167	0.186	0.362	0.404	5000	3.79
0.1	0.2 PI	.001	0.014	0.034	0.030	0.091	0.105	0.095	0.100	0.224	0.273	0.271	10000	7.43
0.5	1.0 PI	.010	5.056	10.368	15.948	21.810	28.037	34.510	41.311	48.456	55.965	63.854	200	0.86
0.5	1.0 PI	.005	2.498	5.058	7.696	10.386	13.144	15.956	18.853	21.822	24.864	27.999	400	1.59
0.5	1.0 PI	.002	0.993	1.995	3.005	4.028	5.059	6.103	7.154	8.217	9.288	10.371	1000	3.79
0.5	1.0 PI	.001	0.496	0.992	1.493	1.993	2.497	3.004	3.527	4.068	4.612	5.159	2000	7.43
1.0	2.0 PI	.010	21.771	48.272	80.528	119.755	168.450	226.904	298.068	384.706	490.187	618.573	100	0.86
1.0	2.0 PI	.005	10.368	21.810	34.510	48.456	63.854	80.727	99.460	120.133	142.946	168.337	200	1.59
1.0	2.0 PI	.002	4.034	8.223	12.572	17.105	21.819	26.733	31.834	37.141	42.654	48.395	500	3.79
1.0	2.0 PI	.001	1.995	4.028	6.103	8.217	10.371	12.569	14.806	17.095	19.416	21.780	1000	7.43
2.0	4.0 PI	.010	118.421	375.022	928.574	999.999	999.999	999.999	999.999	999.999	999.999	999.999	50	0.86
2.0	4.0 PI	.005	48.272	119.785	226.904	384.706	618.573	958.205	999.999	999.999	999.999	999.999	100	1.59
2.0	4.0 PI	.002	17.139	37.171	60.572	88.028	120.172	157.938	202.040	253.687	314.187	355.032	250	3.79
2.0	4.0 PI	.001	8.223	17.105	26.733	37.141	48.395	60.582	73.735	88.009	103.379	119.977	500	7.43

TABLE 2.

THIRD ORDER TAYLOR SERIES, UNDAMPED HARMONIC OSCILLATOR

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.184	0.297	0.312	0.290	0.167	0.197	0.327	0.349	0.327	0.180	1000	2.05
0.1	0.2 PI	.005	0.001	0.002	0.008	0.012	0.010	0.005	0.020	0.033	0.029	0.006	2000	3.82
0.1	0.2 PI	.002	0.001	0.005	0.019	0.027	0.022	0.014	0.055	0.087	0.099	0.069	5000	9.05
0.1	0.2 PI	.001	0.002	0.022	0.019	0.027	0.026	0.067	0.141	0.174	0.173	0.118	10000	17.75
0.5	1.0 PI	.010	1.528	1.560	1.592	1.623	1.686	1.777	1.869	1.960	1.990	1.963	200	2.05
0.5	1.0 PI	.005	0.003	0.006	0.043	0.102	0.104	0.049	0.019	0.071	0.037	0.184	400	3.82
0.5	1.0 PI	.002	0.003	0.024	0.042	0.158	0.140	0.065	0.225	0.394	0.570	0.744	1000	9.05
0.5	1.0 PI	.001	0.007	0.060	0.049	0.156	0.320	0.492	0.665	0.837	1.008	1.181	2000	17.75
1.0	2.0 PI	.010	2.491	2.770	3.050	3.330	3.729	4.137	4.543	4.950	5.191	5.368	100	2.05
1.0	2.0 PI	.005	0.053	0.114	0.263	0.435	0.468	0.414	0.355	0.302	0.579	0.974	200	3.82
1.0	2.0 PI	.002	0.017	0.073	0.099	0.330	0.307	0.235	0.576	0.916	1.255	1.589	500	9.05
1.0	2.0 PI	.001	0.024	0.157	0.139	0.392	0.722	1.053	1.383	1.712	2.037	2.369	1000	17.75
2.0	4.0 PI	.010	6.678	8.978	11.257	13.504	16.232	18.962	21.691	24.419	26.857	29.110	50	2.05
2.0	4.0 PI	.005	0.653	1.217	1.941	2.713	3.128	3.475	3.806	4.130	5.276	6.362	100	3.82
2.0	4.0 PI	.002	0.112	0.291	0.273	0.427	0.393	0.954	1.703	2.448	3.199	3.917	250	9.05
2.0	4.0 PI	.001	0.073	0.330	0.307	0.916	1.588	2.267	2.936	3.617	4.275	4.951	500	17.75
5.0	10.0 PI	.010	13.821	41.485	81.790	125.997	184.542	201.923	203.778	199.643	178.151	141.421	20	2.05
5.0	10.0 PI	.005	1.690	3.479	13.133	18.927	25.328	27.390	29.350	36.845	44.986	67.398	40	3.82
5.0	10.0 PI	.002	0.212	0.385	0.405	0.234	0.393	1.317	1.671	2.018	3.737	4.276	100	9.05
5.0	10.0 PI	.001	0.018	0.016	0.061	0.106	0.315	0.372	0.722	0.788	1.279	1.874	200	17.75
10.0	20.0 PI	.010	283.297	399.229	770.681	999.999	999.999	999.999	999.999	999.999	999.999	999.999	10	2.05
10.0	20.0 PI	.005	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	20	3.82
10.0	20.0 PI	.002	67.693	109.458	117.742	151.600	192.465	268.544	303.420	334.973	385.456	404.836	50	9.05
10.0	20.0 PI	.001	1.152	1.204	2.735	3.759	6.985	8.357	12.684	14.383	19.743	25.730	100	17.75

TABLE 3.

FOURTH ORDER TAYLOR SERIES FOR UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.184	0.297	0.312	0,290	0.167	0.197	0.327	0.349	0.327	0.180	1000	2.56
0.1	0.2 PI	.005	0.001	0.002	0.008	0,012	0.010	0.005	0.020	0.033	0.029	0.006	2000	4.83
0.1	0.2 PI	.002	0.001	0.005	0.019	0,027	0.022	0.014	0.055	0.087	0.089	0.069	5000	11.57
0.1	0.2 PI	.001	0.002	0.022	0.019	0,027	0.026	0.067	0.141	0.174	0.173	0.118	10000	22.79
0.5	1.0 PI	.010	1.528	1.559	1.591	1,622	1.685	1.776	1.867	1.959	1.989	1.961	200	2.56
0.5	1.0 PI	.005	0.003	0.006	0.043	0,102	0.104	0.048	0.019	0.071	0.037	0.184	400	4.83
0.5	1.0 PI	.002	0.003	0.024	0.042	0,158	0.140	0.065	0.225	0.394	0.570	0.744	1000	11.57
0.5	1.0 PI	.001	0.007	0.060	0.049	0,156	0.320	0.492	0.665	0.837	1.008	1.181	2000	22.79
1.0	2.0 PI	.010	2.489	2.771	3.052	3,334	3.717	4.123	4.527	4.932	5.172	5.347	100	2.56
1.0	2.0 PI	.005	0.052	0.113	0.263	0,435	0.466	0.411	0.353	0.300	0.581	0.966	200	4.83
1.0	2.0 PI	.002	0.017	0.072	0.100	0,330	0.307	0.235	0.576	0.916	1.254	1.588	500	11.57
1.0	2.0 PI	.001	0.024	0.157	0.139	0,392	0.722	1.053	1.383	1.712	2.037	2.369	1000	22.79
2.0	4.0 PI	.010	6.670	8.989	11.291	13,568	16.080	18.795	21.515	24.239	26.677	28.937	50	2.56
2.0	4.0 PI	.005	0.623	1.189	1.919	2,695	3.075	3.460	3.794	4.123	5.272	6.370	100	4.83
2.0	4.0 PI	.002	0.109	0.289	0.271	0,428	0.395	0.953	1.702	2.447	3.199	3.917	250	11.57
2.0	4.0 PI	.001	0.073	0.329	0.307	0,916	1.588	2.267	2.936	3.617	4.274	4.950	500	22.79
5.0	10.0 PI	.010	11.655	32.797	71.072	117,442	190.915	220.719	232.641	232.703	221.012	187.483	20	2.56
5.0	10.0 PI	.005	2.049	2.978	13.116	18,812	25.153	27.205	28.418	35.961	44.218	67.796	40	4.83
5.0	10.0 PI	.002	0.181	0.322	0.335	0,106	0.232	1.122	1.442	1.757	3.444	3.960	100	11.57
5.0	10.0 PI	.001	0.014	0.013	0.048	0,091	0.295	0.346	0.692	0.755	1.242	1.833	200	22.79
10.0	20.0 PI	.010	326.132	375.379	999.999	999,999	999.999	999.999	999.999	999.999	999.999	999.999	10	2.56
10.0	20.0 PI	.005	999.999	999.999	999.999	999,999	999.999	999.999	999.999	999.999	999.999	999.999	20	4.83
10.0	20.0 PI	.002	999.999	999.999	999.999	999,999	999.999	999.999	999.999	999.999	999.999	999.999	50	11.57
10.0	20.0 PI	.001	22.889	23.863	42.895	53,097	83.837	94.180	125.134	135.458	166.143	196.202	100	22.79

TABLE 4.

FOURTH ORDER TAYLOR SERIES, EXACT HIGHER DERIVATIVES, UNDAMPED OSCILLATOR

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0	0.002	0.003	0,003	0.002	0.010	0.026	0.036	0.033	0.012	1000	2.40
0.1	0.2 PI	.005	0.001	0.002	0.008	0,012	0.010	0.005	0.021	0.033	0.029	0.006	2000	4.67
0.1	0.2 PI	.002	0.001	0.005	0.019	0,027	0.022	0.014	0.054	0.087	0.089	0.069	5000	11.48
0.1	0.2 PI	.001	0.002	0.022	0.019	0,028	0.026	0.067	0.141	0.174	0.173	0.118	10000	22.85
0.5	1.0 PI	.010	0.001	0.007	0.014	0,021	0.012	0.058	0.108	0.160	0.153	0.088	200	2.40
0.5	1.0 PI	.005	0.033	0.010	0.020	0,068	0.062	0.018	0.080	0.144	0.102	0.093	400	4.67
0.5	1.0 PI	.002	0.003	0.021	0.046	0,163	0.146	0.059	0.217	0.384	0.557	0.730	1000	11.48
0.5	1.0 PI	.001	0.007	0.060	0.049	0,155	0.319	0.491	0.663	0.835	1.006	1.179	2000	22.85
1.0	2.0 PI	.010	0.004	0.018	0.031	0,044	0.034	0.130	0.231	0.331	0.333	0.201	100	2.40
1.0	2.0 PI	.005	0.007	0.020	0.058	0,157	0.153	0.050	0.179	0.306	0.270	0.259	200	4.67
1.0	2.0 PI	.002	0.008	0.052	0.132	0,374	0.359	0.169	0.497	0.826	1.153	1.475	500	11.48
1.0	2.0 PI	.001	0.021	0.163	0.146	0,382	0.709	1.039	1.366	1.692	2.013	2.342	1000	22.85
2.0	4.0 PI	.010	0.013	0.043	0.070	0,102	0.082	0.262	0.450	0.639	0.683	0.420	50	2.40
2.0	4.0 PI	.005	0.018	0.044	0.130	0,328	0.333	0.135	0.388	0.646	0.604	0.567	100	4.67
2.0	4.0 PI	.002	0.024	0.110	0.317	0,798	0.783	0.399	1.055	1.706	2.364	2.997	250	11.48
2.0	4.0 PI	.001	0.052	0.374	0.359	0,826	1.475	2.130	2.776	3.434	4.069	4.722	500	22.85
5.0	10.0 PI	.010	0.066	0.132	0.199	0,267	0.294	0.178	0.327	0.469	0.450	0.241	20	2.40
5.0	10.0 PI	.005	0.001	0.003	0.059	0,136	0.127	0.023	0.006	0.005	0.012	0.242	40	4.67
5.0	10.0 PI	.002	0.005	0.018	0.010	0,027	0.024	0.066	0.138	0.180	0.596	0.687	100	11.48
5.0	10.0 PI	.001	0.004	0.008	0.015	0,042	0.169	0.191	0.421	0.439	0.771	1.181	200	22.85
10.0	20.0 PI	.010	4.219	8.716	13.440	18,344	20.348	5.148	7.908	10.949	12.924	15.111	10	2.40
10.0	20.0 PI	.005	0.132	0.267	0.294	0,469	0.431	0.822	0.947	1.056	1.267	0.852	20	4.67
10.0	20.0 PI	.002	0.020	0.073	0.042	0,100	0.086	0.268	0.557	0.727	2.364	2.727	50	11.48
10.0	20.0 PI	.001	0.018	0.027	0.066	0,180	0.687	0.780	1.695	1.768	3.088	4.709	100	22.85
20.0	40.0 PI	.010	97.289	101.390	100.830	100,054	99.995	33.960	35.939	37.858	37.628	35.093	5	2.40
20.0	40.0 PI	.005	8.716	18.344	20.348	10,949	15.111	59.211	68.627	77.308	85.609	53.191	10	4.67
20.0	40.0 PI	.002	0.066	0.263	0.207	0,173	0.419	1.016	2.163	2.841	8.349	9.695	25	11.48
20.0	40.0 PI	.001	0.073	0.100	0.268	0,727	2.727	3.094	6.683	6.967	12.085	18.257	50	22.85

TABLE 5.

EULER INTEGRATION ALGORITHM FOR DAMPED SOLUTIONS

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL) 1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.092	0.105	0.230	0.242	0.217	0.121	0.033	0.052	0.052	0.040	1000	1.25
0.1	0.2 PI	.005	0.047	0.051	0.118	0.124	0.110	0.061	0.014	0.024	0.025	0.020	2000	1.99
0.1	0.2 PI	.002	0.018	0.025	0.039	0.041	0.039	0.024	0.009	0.013	0.013	0.009	5000	4.32
0.1	0.2 PI	.001	0.009	0.005	0.028	0.032	0.028	0.012	0.010	0.011	0.010	0.006	10000	8.23
0.5	1.0 PI	.010	3.161	3.925	3.707	3.247	2.660	2.006	1.485	1.069	0.755	0.526	200	1.25
0.5	1.0 PI	.005	1.557	1.911	1.791	1.543	1.249	0.954	0.707	0.505	0.353	0.240	400	1.99
0.5	1.0 PI	.002	0.617	0.755	0.693	0.617	0.500	0.373	0.268	0.188	0.130	0.088	1000	4.32
0.5	1.0 PI	.001	0.307	0.374	0.350	0.296	0.232	0.172	0.124	0.087	0.060	0.041	2000	8.23
1.0	2.0 PI	.010	13.377	17.988	18.457	17.449	15.046	12.472	10.024	7.887	6.111	4.681	100	1.25
1.0	2.0 PI	.005	6.346	8.093	8.054	7.191	5.902	4.632	3.502	2.586	1.877	1.336	200	1.99
1.0	2.0 PI	.002	2.461	3.040	2.992	2.592	2.062	1.556	1.138	0.813	0.570	0.393	500	4.32
1.0	2.0 PI	.001	1.217	1.491	1.458	1.250	0.983	0.736	0.532	0.376	0.260	0.178	1000	8.23
2.0	4.0 PI	.010	72.968	141.756	216.255	305.233	418.677	565.592	759.628	999.999	999.999	999.999	50	1.25
2.0	4.0 PI	.005	29.455	44.389	50.638	52.082	51.688	49.842	46.733	43.187	39.539	35.652	100	1.99
2.0	4.0 PI	.002	10.403	13.680	13.900	12.850	10.930	8.837	6.934	5.323	4.022	2.990	250	4.32
2.0	4.0 PI	.001	4.993	6.306	6.264	5.536	4.492	3.472	2.592	1.893	1.356	0.960	500	8.23
5.0	10.0 PI	.010	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	20	1.25
5.0	10.0 PI	.005	628.413	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	40	1.99
5.0	10.0 PI	.002	102.231	228.188	407.992	686.228	999.999	999.999	999.999	999.999	999.999	999.999	100	4.32
5.0	10.0 PI	.001	38.728	61.956	75.760	83.890	88.399	90.972	91.977	92.499	92.522	92.121	200	8.23

TABLE 6.

SECOND ORDER TAYLOR SERIES FOR DAMPED HARMONIC OSCILLATOR SOLUTIONS

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL) 1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.183	0.196	0.174	0.100	0.035	0.016	0.017	0.016	0.009	0.003	1000	1.72
0.1	0.2 PI	.005	0.091	0.097	0.086	0.053	0.019	0.007	0.007	0.006	0.003	0.001	2000	3.03
0.1	0.2 PI	.002	0.037	0.041	0.036	0.013	0.002	0.004	0.006	0.006	0.004	0.002	5000	6.87
0.1	0.2 PI	.001	0.019	0.018	0.015	0.015	0.009	0.007	0.009	0.008	0.006	0.002	10000	13.24
0.5	1.0 PI	.010	1.290	0.824	0.524	0.334	0.215	0.141	0.091	0.059	0.037	0.023	200	1.72
0.5	1.0 PI	.005	0.632	0.390	0.249	0.165	0.103	0.060	0.034	0.020	0.013	0.010	400	3.03
0.5	1.0 PI	.002	0.249	0.163	0.088	0.032	0.021	0.024	0.022	0.017	0.013	0.009	1000	6.87
0.5	1.0 PI	.001	0.128	0.059	0.040	0.055	0.052	0.043	0.033	0.024	0.017	0.012	2000	13.24
1.0	2.0 PI	.010	2.857	2.094	1.508	1.059	0.732	0.510	0.347	0.233	0.154	0.099	100	1.72
1.0	2.0 PI	.005	1.386	0.926	0.625	0.440	0.296	0.183	0.113	0.070	0.045	0.032	200	3.03
1.0	2.0 PI	.002	0.547	0.361	0.224	0.092	0.051	0.053	0.049	0.039	0.030	0.022	500	6.87
1.0	2.0 PI	.001	0.270	0.158	0.091	0.106	0.103	0.088	0.069	0.051	0.036	0.026	1000	13.24
2.0	4.0 PI	.010	7.074	6.924	6.219	5.006	3.790	2.801	1.979	1.368	0.929	0.618	50	1.72
2.0	4.0 PI	.005	2.975	2.570	2.054	1.614	1.184	0.810	0.541	0.362	0.242	0.170	100	3.03
2.0	4.0 PI	.002	1.154	0.855	0.608	0.312	0.177	0.173	0.148	0.115	0.085	0.061	250	6.87
2.0	4.0 PI	.001	0.572	0.385	0.215	0.236	0.230	0.196	0.153	0.114	0.081	0.057	500	13.24
5.0	10.0 PI	.010	20.041	59.130	67.991	64.552	37.644	13.415	13.727	8.514	7.111	7.719	20	1.72
5.0	10.0 PI	.005	1.039	3.152	7.324	7.381	7.080	6.112	4.114	3.246	2.429	1.887	40	3.03
5.0	10.0 PI	.002	0.217	0.314	0.316	0.144	0.146	0.237	0.201	0.157	0.118	0.114	100	6.87
5.0	10.0 PI	.001	0.040	0.036	0.047	0.047	0.068	0.058	0.058	0.043	0.038	0.027	200	13.24
10.0	20.0 PI	.010	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	10	1.72
10.0	20.0 PI	.005	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	20	3.03
10.0	20.0 PI	.002	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	50	6.87
10.0	20.0 PI	.001	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	100	13.24

TABLE 7.

THIRD ORDER TAYLOR SERIES INTEGRATION ALGORITHM, DAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.183	0.196	0.173	0.100	0.035	0.016	0.017	0.015	0.009	0.003	1000	2.23
0.1	0.2 PI	.005	0.091	0.097	0.086	0.053	0.019	0.007	0.007	0.006	0.003	0.001	2000	4.05
0.1	0.2 PI	.002	0.037	0.041	0.036	0.013	0.002	0.004	0.006	0.006	0.004	0.002	5000	9.42
0.1	0.2 PI	.001	0.019	0.018	0.015	0.015	0.009	0.007	0.009	0.008	0.006	0.002	10000	18.34
0.5	1.0 PI	.010	1.273	0.785	0.487	0.301	0.189	0.121	0.077	0.049	0.030	0.018	200	2.23
0.5	1.0 PI	.005	0.626	0.381	0.240	0.157	0.097	0.055	0.031	0.017	0.011	0.008	400	4.05
0.5	1.0 PI	.002	0.249	0.162	0.087	0.031	0.020	0.023	0.021	0.017	0.013	0.009	1000	9.42
0.5	1.0 PI	.001	0.128	0.059	0.040	0.054	0.052	0.043	0.033	0.024	0.017	0.012	2000	18.34
1.0	2.0 PI	.010	2.810	1.845	1.220	0.801	0.525	0.353	0.233	0.152	0.098	0.062	100	2.23
1.0	2.0 PI	.005	1.374	0.863	0.553	0.374	0.243	0.144	0.084	0.050	0.031	0.023	200	4.05
1.0	2.0 PI	.002	0.545	0.351	0.212	0.082	0.044	0.047	0.044	0.036	0.028	0.020	500	9.42
1.0	2.0 PI	.001	0.270	0.156	0.088	0.104	0.101	0.086	0.067	0.050	0.036	0.025	1000	18.34
2.0	4.0 PI	.010	6.109	5.013	3.907	2.894	2.076	1.483	1.025	0.697	0.467	0.306	50	2.23
2.0	4.0 PI	.005	2.929	2.088	1.480	1.086	0.758	0.488	0.310	0.198	0.128	0.092	100	4.05
2.0	4.0 PI	.002	1.147	0.776	0.515	0.229	0.119	0.122	0.110	0.089	0.067	0.049	250	9.42
2.0	4.0 PI	.001	0.570	0.364	0.193	0.217	0.213	0.183	0.143	0.107	0.077	0.054	500	18.34
5.0	10.0 PI	.010	12.217	16.655	18.790	18.730	17.036	13.685	8.803	5.186	2.801	1.376	20	2.23
5.0	10.0 PI	.005	1.269	1.455	3.032	2.913	2.520	2.028	1.126	0.876	0.658	0.607	40	4.05
5.0	10.0 PI	.002	0.223	0.237	0.230	0.090	0.079	0.115	0.094	0.070	0.052	0.050	100	9.42
5.0	10.0 PI	.001	0.046	0.044	0.037	0.037	0.049	0.041	0.040	0.029	0.026	0.018	200	18.34
10.0	20.0 PI	.010	190.830	133.283	158.916	146.519	145.443	136.633	148.209	152.700	155.826	157.982	10	2.23
10.0	20.0 PI	.005	24.896	44.525	52.177	40.848	21.565	7.408	2.474	0.389	0.940	1.257	20	4.05
10.0	20.0 PI	.002	2.406	3.277	3.326	2.223	2.154	2.414	2.007	1.530	1.127	0.927	50	9.42
10.0	20.0 PI	.001	0.342	0.343	0.386	0.384	0.463	0.398	0.370	0.283	0.238	0.169	100	18.34

TABLE 8.

FOURTH ORDER TAYLOR SERIES INTEGRATION ALGORITHM, DAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.183	0.196	0.173	0.100	0.035	0.016	0.017	0.015	0.009	0.003	1000	2.74
0.1	0.2 PI	.005	0.091	0.097	0.086	0.053	0.019	0.007	0.007	0.006	0.003	0.001	2000	5.06
0.1	0.2 PI	.002	0.037	0.041	0.036	0.013	0.002	0.004	0.006	0.006	0.004	0.002	5000	11.95
0.1	0.2 PI	.001	0.019	0.018	0.015	0.015	0.009	0.007	0.009	0.008	0.006	0.002	10000	23.42
0.5	1.0 PI	.010	1.273	0.785	0.487	0.301	0.189	0.121	0.077	0.049	0.030	0.018	200	2.74
0.5	1.0 PI	.005	0.626	0.381	0.240	0.157	0.097	0.055	0.031	0.017	0.011	0.008	400	5.06
0.5	1.0 PI	.002	0.249	0.162	0.087	0.031	0.020	0.023	0.021	0.017	0.013	0.009	1000	11.95
0.5	1.0 PI	.001	0.123	0.059	0.040	0.054	0.052	0.043	0.033	0.024	0.017	0.012	2000	23.42
1.0	2.0 PI	.010	2.806	1.844	1.218	0.799	0.524	0.352	0.232	0.152	0.098	0.061	100	2.74
1.0	2.0 PI	.005	1.374	0.863	0.553	0.374	0.243	0.144	0.084	0.050	0.031	0.023	200	5.06
1.0	2.0 PI	.002	0.545	0.351	0.212	0.082	0.044	0.047	0.044	0.036	0.028	0.020	500	11.95
1.0	2.0 PI	.001	0.270	0.156	0.098	0.104	0.101	0.086	0.067	0.050	0.036	0.025	1000	23.42
2.0	4.0 PI	.010	6.084	4.976	3.874	2.869	2.058	1.475	1.020	0.695	0.466	0.305	50	2.74
2.0	4.0 PI	.005	2.926	2.084	1.477	1.083	0.757	0.487	0.309	0.197	0.128	0.092	100	5.06
2.0	4.0 PI	.002	1.147	0.776	0.515	0.228	0.119	0.122	0.110	0.089	0.067	0.049	250	11.95
2.0	4.0 PI	.001	0.570	0.365	0.193	0.217	0.213	0.183	0.143	0.107	0.077	0.054	500	23.42
5.0	10.0 PI	.010	10.524	13.651	16.687	16.790	17.315	14.330	9.694	6.007	3.407	1.747	20	2.74
5.0	10.0 PI	.005	1.180	1.079	2.669	2.588	2.269	1.847	0.992	0.786	0.600	0.576	40	5.06
5.0	10.0 PI	.002	0.204	0.214	0.206	0.068	0.062	0.103	0.085	0.063	0.049	0.047	100	11.95
5.0	10.0 PI	.001	0.043	0.041	0.034	0.034	0.047	0.039	0.039	0.029	0.025	0.017	200	23.42
10.0	20.0 PI	.010	199.792	138.700	214.702	207.937	275.298	359.993	439.780	529.104	627.893	750.125	10	2.74
10.0	20.0 PI	.005	68.463	91.232	91.562	58.814	20.316	4.561	5.514	4.449	1.577	2.788	20	5.06
10.0	20.0 PI	.002	2.098	2.909	2.962	1.922	1.899	2.258	1.897	1.462	1.087	0.911	50	11.95
10.0	20.0 PI	.001	0.303	0.302	0.342	0.341	0.433	0.373	0.353	0.270	0.230	0.163	100	23.42

TABLE 9.

FOURTH ORDER RUNGE-KUTTA-SIMPSON FOR UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACT
0.1	0.2 PI	.010	0.000	0.002	0.003	0.003	0.002	0.010	0.026	0.036	0.033	0.012	1000	5.22
0.1	0.2 PI	.005	0.001	0.002	0.008	0.012	0.010	0.005	0.021	0.033	0.029	0.006	2000	8.72
0.1	0.2 PI	.002	0.001	0.005	0.019	0.027	0.023	0.014	0.054	0.087	0.089	0.069	5000	18.74
0.1	0.2 PI	.001	0.002	0.022	0.019	0.028	0.026	0.067	0.141	0.174	0.173	0.118	10000	35.50
0.5	1.0 PI	.010	0.001	0.007	0.013	0.021	0.011	0.058	0.108	0.161	0.153	0.088	200	5.22
0.5	1.0 PI	.005	0.003	0.009	0.020	0.068	0.063	0.017	0.079	0.142	0.101	0.095	400	8.72
0.5	1.0 PI	.002	0.003	0.021	0.046	0.163	0.146	0.059	0.216	0.382	0.556	0.729	1000	18.74
0.5	1.0 PI	.001	0.007	0.060	0.049	0.155	0.319	0.491	0.664	0.835	1.006	1.179	2000	35.50
1.0	2.0 PI	.010	0.004	0.018	0.031	0.044	0.035	0.131	0.232	0.332	0.334	0.203	100	5.22
1.0	2.0 PI	.005	0.007	0.020	0.058	0.157	0.153	0.050	0.179	0.305	0.268	0.260	200	8.72
1.0	2.0 PI	.002	0.008	0.052	0.132	0.373	0.358	0.169	0.498	0.826	1.152	1.475	500	18.74
1.0	2.0 PI	.001	0.021	0.163	0.146	0.381	0.707	1.037	1.365	1.692	2.015	2.344	1000	35.50
2.0	4.0 PI	.010	0.012	0.042	0.069	0.100	0.080	0.264	0.452	0.641	0.686	0.423	50	5.22
2.0	4.0 PI	.005	0.018	0.044	0.131	0.331	0.334	0.132	0.384	0.642	0.601	0.573	100	8.72
2.0	4.0 PI	.002	0.024	0.111	0.316	0.796	0.782	0.401	1.057	1.708	2.367	2.999	250	18.74
2.0	4.0 PI	.001	0.052	0.373	0.358	0.826	1.475	2.131	2.779	3.438	4.073	4.725	500	35.50
5.0	10.0 PI	.010	0.066	0.131	0.198	0.265	0.293	0.177	0.325	0.468	0.449	0.240	20	5.22
5.0	10.0 PI	.005	0.001	0.003	0.060	0.137	0.128	0.024	0.008	0.004	0.013	0.245	40	8.72
5.0	10.0 PI	.002	0.005	0.018	0.010	0.027	0.024	0.066	0.138	0.180	0.596	0.687	100	18.74
5.0	10.0 PI	.001	0.004	0.008	0.015	0.044	0.169	0.192	0.422	0.441	0.773	1.182	200	35.50
10.0	20.0 PI	.010	4.219	8.715	13.440	18.344	20.347	5.150	7.909	10.950	12.924	15.111	10	5.22
10.0	20.0 PI	.005	0.131	0.265	0.293	0.468	0.430	0.819	0.944	1.052	1.263	0.851	20	8.72
10.0	20.0 PI	.002	0.020	0.073	0.043	0.099	0.084	0.270	0.560	0.731	2.370	2.732	50	18.74
10.0	20.0 PI	.001	0.018	0.027	0.066	0.180	0.687	0.780	1.696	1.769	3.090	4.713	100	35.50
20.0	40.0 PI	.010	97.289	101.390	100.830	100.054	99.995	33.960	35.939	37.858	37.628	35.093	5	5.22
20.0	40.0 PI	.005	8.715	18.344	20.347	10.950	15.111	59.204	68.618	77.298	85.598	53.183	10	8.72
20.0	40.0 PI	.002	0.066	0.265	0.207	0.173	0.419	1.018	2.166	2.844	8.357	9.704	25	18.74
20.0	40.0 PI	.001	0.073	0.099	0.270	0.731	2.732	3.103	6.693	6.979	12.099	18.275	50	35.50

Table 10.

FOURTH ORDER RUNGE-KUTTA GILL, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0	0.001	0.003	0.003	0.002	0.010	0.027	0.036	0.034	0.013	1000	4.51
0.1	0.2 PI	.005	0.001	0.002	0.008	0.012	0.010	0.005	0.020	0.033	0.029	0.006	2000	8.87
0.1	0.2 PI	.002	0.001	0.005	0.018	0.027	0.022	0.014	0.055	0.088	0.090	0.069	5000	21.97
0.1	0.2 PI	.001	0.002	0.022	0.019	0.028	0.027	0.067	0.142	0.175	0.174	0.118	10000	43.82
0.5	1.0 PI	.010	0.001	0.007	0.013	0.020	0.012	0.059	0.110	0.163	0.156	0.092	200	4.51
0.5	1.0 PI	.005	0.002	0.009	0.021	0.069	0.064	0.015	0.076	0.139	0.097	0.100	400	8.87
0.5	1.0 PI	.002	0.003	0.022	0.045	0.161	0.144	0.062	0.220	0.387	0.561	0.734	1000	21.97
0.5	1.0 PI	.001	0.007	0.059	0.049	0.156	0.320	0.492	0.665	0.836	1.008	1.181	2000	43.82
1.0	2.0 PI	.010	0.003	0.016	0.029	0.041	0.038	0.135	0.237	0.337	0.340	0.209	100	4.51
1.0	2.0 PI	.005	0.007	0.019	0.059	0.159	0.156	0.047	0.175	0.300	0.263	0.265	200	8.87
1.0	2.0 PI	.002	0.009	0.054	0.130	0.371	0.353	0.173	0.502	0.831	1.159	1.483	500	21.97
1.0	2.0 PI	.001	0.022	0.161	0.144	0.385	0.712	1.041	1.368	1.696	2.019	2.350	1000	43.82
2.0	4.0 PI	.010	0.011	0.040	0.065	0.095	0.082	0.271	0.461	0.651	0.696	0.434	50	4.51
2.0	4.0 PI	.005	0.016	0.041	0.135	0.335	0.340	0.124	0.375	0.632	0.591	0.585	100	8.87
2.0	4.0 PI	.002	0.026	0.113	0.312	0.791	0.776	0.409	1.066	1.718	2.378	3.011	250	21.97
2.0	4.0 PI	.001	0.054	0.371	0.353	0.831	1.483	2.139	2.787	3.446	4.083	4.737	500	43.82
5.0	10.0 PI	.010	0.065	0.132	0.198	0.266	0.293	0.183	0.333	0.477	0.458	0.250	20	4.51
5.0	10.0 PI	.005	0.001	0.003	0.062	0.140	0.131	0.028	0.008	0.004	0.014	0.250	40	8.87
5.0	10.0 PI	.002	0.005	0.018	0.011	0.026	0.023	0.068	0.140	0.183	0.601	0.691	100	21.97
5.0	10.0 PI	.001	0.005	0.007	0.016	0.045	0.172	0.194	0.425	0.445	0.776	1.188	200	43.82
10.0	20.0 PI	.010	4.218	8.714	13.436	18.337	20.339	5.156	7.914	10.953	12.924	15.107	10	4.51
10.0	20.0 PI	.005	0.132	0.266	0.293	0.477	0.439	0.819	0.943	1.051	1.261	0.869	20	8.87
10.0	20.0 PI	.002	0.021	0.075	0.045	0.099	0.085	0.275	0.565	0.736	2.384	2.749	50	21.97
10.0	20.0 PI	.001	0.018	0.026	0.068	0.183	0.691	0.785	1.705	1.778	3.105	4.734	100	43.82
20.0	40.0 PI	.010	97.287	101.389	100.830	100.054	99.995	33.960	35.939	37.858	37.628	35.093	5	4.51
20.0	40.0 PI	.005	8.714	18.337	20.339	10.953	15.107	59.166	68.572	77.245	85.540	53.139	10	8.87
20.0	40.0 PI	.002	0.070	0.271	0.206	0.172	0.416	1.033	2.184	2.866	8.409	9.763	25	21.97
20.0	40.0 PI	.001	0.075	0.099	0.275	0.736	2.749	3.122	6.728	7.019	12.160	18.358	50	43.82

Table 11.

FOURTH ORDER RUNGE-KUTTA RALSTON, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0	0.002	0.004	0,005	0.003	0.008	0.023	0.033	0.031	0.011	1000	5.07
0.1	0.2 PI	.005	0.001	0.003	0.007	0,011	0.009	0.006	0.023	0.036	0.032	0.004	2000	9.99
0.1	0.2 PI	.002	0.002	0.004	0.019	0,027	0.021	0.010	0.053	0.088	0.091	0.073	5000	24.77
0.1	0.2 PI	.001	0.005	0.020	0.017	0,024	0.023	0.071	0.144	0.174	0.172	0.115	10000	49.36
0.5	1.0 PI	.010	0.002	0.010	0.018	0,027	0.013	0.049	0.097	0.148	0.139	0.072	200	5.07
0.5	1.0 PI	.005	0.003	0.012	0.017	0,063	0.056	0.024	0.088	0.154	0.114	0.082	400	9.99
0.5	1.0 PI	.002	0.004	0.019	0.049	0,168	0.152	0.052	0.208	0.372	0.544	0.716	1000	24.77
0.5	1.0 PI	.001	0.007	0.060	0.051	0,153	0.316	0.485	0.656	0.825	0.996	1.167	2000	49.36
1.0	2.0 PI	.010	0.006	0.023	0.039	0,056	0.041	0.113	0.211	0.306	0.306	0.172	100	5.07
1.0	2.0 PI	.005	0.010	0.026	0.049	0,145	0.139	0.069	0.202	0.331	0.296	0.227	200	9.99
1.0	2.0 PI	.002	0.005	0.045	0.143	0,388	0.373	0.148	0.473	0.798	1.122	1.442	500	24.77
1.0	2.0 PI	.001	0.019	0.168	0.152	0,372	0.695	1.022	1.345	1.667	1.986	2.312	1000	49.36
2.0	4.0 PI	.010	0.019	0.055	0.089	0,127	0.108	0.225	0.407	0.590	0.630	0.360	50	5.07
2.0	4.0 PI	.005	0.023	0.056	0.113	0,306	0.306	0.171	0.430	0.695	0.655	0.508	100	9.99
2.0	4.0 PI	.002	0.019	0.098	0.335	0,822	0.809	0.362	1.011	1.655	2.308	2.934	250	24.77
2.0	4.0 PI	.001	0.045	0.388	0.373	0,798	1.442	2.089	2.730	3.382	4.012	4.658	500	49.36
5.0	10.0 PI	.010	0.070	0.139	0.210	0,281	0.310	0.168	0.316	0.457	0.438	0.228	20	5.07
5.0	10.0 PI	.005	0.003	0.006	0.056	0,132	0.122	0.017	0.016	0.014	0.026	0.231	40	9.99
5.0	10.0 PI	.002	0.003	0.015	0.007	0,028	0.024	0.058	0.129	0.169	0.574	0.663	100	24.77
5.0	10.0 PI	.001	0.003	0.008	0.012	0,039	0.163	0.183	0.407	0.424	0.749	1.151	200	49.36
10.0	20.0 PI	.010	4.237	8.753	13.499	18,427	20.439	5.192	7.969	11.029	13.025	15.233	10	5.07
10.0	20.0 PI	.005	0.139	0.281	0.310	0,457	0.419	0.869	1.003	1.122	1.344	0.836	20	9.99
10.0	20.0 PI	.002	0.018	0.069	0.039	0,089	0.075	0.258	0.545	0.712	2.309	2.665	50	24.77
10.0	20.0 PI	.001	0.015	0.028	0.058	0,169	0.663	0.752	1.648	1.714	3.009	4.601	100	49.36
20.0	40.0 PI	.010	97.295	101.390	100.830	100,054	99.995	33.960	35.939	37.858	37.628	35.093	5	5.07
20.0	40.0 PI	.005	8.753	18.427	20.439	11,029	15.233	59.461	68.902	77.600	85.907	53.486	10	9.99
20.0	40.0 PI	.002	0.060	0.252	0.237	0,210	0.481	0.990	2.136	2.814	8.169	9.500	25	24.77
20.0	40.0 PI	.001	0.069	0.089	0.258	0,712	2.665	3.020	6.539	6.802	11.828	17.899	50	49.36

Table 12.

FOURTH ORDER RUNGE-KUTTA MERSON, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0	0.002	0.003	0,003	0.002	0.010	0.026	0.036	0.033	0.012	1000	6.06
0.1	0.2 PI	.005	0.001	0.002	0.008	0,012	0.010	0.005	0.021	0.033	0.029	0.006	2000	11.96
0.1	0.2 PI	.002	0.001	0.005	0.019	0,027	0.023	0.014	0.054	0.087	0.089	0.069	5000	29.67
0.1	0.2 PI	.001	0.002	0.022	0.019	0,027	0.026	0.067	0.141	0.174	0.173	0.118	10000	59.16
0.5	1.0 PI	.010	0.001	0.007	0.014	0,021	0.012	0.058	0.108	0.161	0.154	0.088	200	6.06
0.5	1.0 PI	.005	0.003	0.009	0.020	0,068	0.063	0.017	0.079	0.143	0.101	0.095	400	11.96
0.5	1.0 PI	.002	0.003	0.021	0.046	0,163	0.146	0.059	0.217	0.384	0.557	0.731	1000	29.67
0.5	1.0 PI	.001	0.007	0.060	0.049	0,155	0.319	0.491	0.664	0.835	1.006	1.179	2000	59.16
1.0	2.0 PI	.010	0.004	0.017	0.030	0,043	0.036	0.132	0.233	0.332	0.334	0.203	100	6.06
1.0	2.0 PI	.005	0.007	0.020	0.058	0,158	0.154	0.050	0.179	0.305	0.269	0.259	200	11.96
1.0	2.0 PI	.002	0.008	0.052	0.131	0,373	0.358	0.168	0.497	0.826	1.152	1.474	500	29.67
1.0	2.0 PI	.001	0.021	0.163	0.146	0,382	0.709	1.040	1.367	1.694	2.016	2.346	1000	59.16
2.0	4.0 PI	.010	0.010	0.038	0.063	0,092	0.085	0.276	0.466	0.657	0.701	0.440	50	6.06
2.0	4.0 PI	.005	0.017	0.043	0.132	0,331	0.334	0.131	0.383	0.641	0.600	0.574	100	11.96
2.0	4.0 PI	.002	0.024	0.111	0.316	0,796	0.782	0.401	1.056	1.707	2.365	2.998	250	29.67
2.0	4.0 PI	.001	0.052	0.373	0.358	0,826	1.474	2.130	2.778	3.437	4.072	4.724	500	59.16
5.0	10.0 PI	.010	0.007	0	0	0,001	0.050	0.168	0.313	0.452	0.432	0.218	20	6.05
5.0	10.0 PI	.005	0.001	0.002	0.060	0,137	0.127	0.023	0.011	0.021	0.019	0.243	40	11.96
5.0	10.0 PI	.002	0.005	0.017	0.010	0,028	0.024	0.065	0.138	0.180	0.596	0.687	100	29.67
5.0	10.0 PI	.001	0.004	0.037	0.015	0,043	0.170	0.192	0.422	0.440	0.772	1.182	200	59.16
10.0	20.0 PI	.010	0.430	0.161	0.307	0,455	3.644	3.780	3.890	4.001	5.423	6.838	10	6.06
10.0	20.0 PI	.005	0.005	0.001	0.168	0,452	0.413	0.109	0.023	0.061	0.050	0.736	20	11.96
10.0	20.0 PI	.002	0.020	0.073	0.042	0,105	0.091	0.269	0.559	0.729	2.380	2.744	50	29.67
10.0	20.0 PI	.001	0.017	0.028	0.065	0,180	0.687	0.780	1.696	1.768	3.089	4.713	100	59.16
20.0	40.0 PI	.010	29.980	75.058	116.286	141,037	145.200	63.222	69.165	68.989	61.433	44.426	5	6.06
20.0	40.0 PI	.005	0.346	0.455	3.780	4,001	6.838	7.706	3.118	3.788	4.945	11.576	10	11.96
20.0	40.0 PI	.002	0.066	0.262	0.140	0,407	0.349	0.990	2.129	2.796	9.073	10.493	25	29.67
20.0	40.0 PI	.001	0.073	0.105	0.269	0,729	2.744	3.115	6.727	7.018	12.167	18.375	50	59.16

Table 13.

FOURTH ORDER RUNGE-KUTTA BLUM, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0	0.002	0.003	0.003	0.002	0.010	0.026	0.036	0.034	0.012	1000	4.25
0.1	0.2 PI	.005	0.001	0.002	0.008	0.012	0.010	0.005	0.021	0.033	0.029	0.006	2000	8.35
0.1	0.2 PI	.002	0.001	0.005	0.019	0.027	0.022	0.014	0.055	0.087	0.089	0.069	5000	20.65
0.1	0.2 PI	.001	0.002	0.022	0.019	0.028	0.026	0.067	0.141	0.174	0.173	0.118	10000	41.17
0.5	1.0 PI	.010	0.001	0.007	0.013	0.021	0.012	0.058	0.108	0.161	0.154	0.089	200	4.25
0.5	1.0 PI	.005	0.003	0.009	0.020	0.068	0.063	0.017	0.079	0.143	0.101	0.095	400	8.35
0.5	1.0 PI	.002	0.003	0.021	0.046	0.162	0.145	0.060	0.218	0.384	0.558	0.732	1000	20.65
0.5	1.0 PI	.001	0.007	0.060	0.049	0.155	0.319	0.491	0.663	0.835	1.006	1.179	2000	41.17
1.0	2.0 PI	.010	0.004	0.018	0.030	0.043	0.035	0.132	0.233	0.332	0.335	0.203	100	4.25
1.0	2.0 PI	.005	0.007	0.020	0.058	0.158	0.154	0.050	0.179	0.305	0.268	0.260	200	8.35
1.0	2.0 PI	.002	0.008	0.052	0.131	0.373	0.358	0.169	0.497	0.826	1.153	1.476	500	20.65
1.0	2.0 PI	.001	0.021	0.162	0.145	0.383	0.710	1.041	1.367	1.694	2.016	2.346	1000	41.17
2.0	4.0 PI	.010	0.012	0.042	0.069	0.100	0.081	0.265	0.453	0.642	0.685	0.423	50	4.25
2.0	4.0 PI	.005	0.018	0.043	0.132	0.331	0.335	0.131	0.384	0.642	0.602	0.571	100	8.35
2.0	4.0 PI	.002	0.025	0.111	0.316	0.796	0.781	0.401	1.057	1.708	2.367	3.000	250	20.65
2.0	4.0 PI	.001	0.052	0.373	0.358	0.826	1.476	2.132	2.779	3.438	4.073	4.725	500	41.17
5.0	10.0 PI	.010	0.066	0.132	0.198	0.265	0.292	0.177	0.325	0.468	0.449	0.240	20	4.25
5.0	10.0 PI	.005	0.001	0.003	0.060	0.137	0.128	0.025	0.008	0.004	0.014	0.245	40	8.35
5.0	10.0 PI	.002	0.005	0.018	0.010	0.028	0.024	0.065	0.137	0.180	0.596	0.686	100	20.65
5.0	10.0 PI	.001	0.004	0.008	0.015	0.043	0.170	0.192	0.422	0.440	0.772	1.182	200	41.17
10.0	20.0 PI	.010	4.218	8.715	13.439	18.343	20.346	5.150	7.909	10.950	12.923	15.109	10	4.25
10.0	20.0 PI	.005	0.132	0.265	0.292	0.468	0.430	0.818	0.943	1.052	1.263	0.852	20	8.35
10.0	20.0 PI	.002	0.020	0.073	0.042	0.100	0.086	0.268	0.557	0.728	2.367	2.730	50	20.65
10.0	20.0 PI	.001	0.018	0.028	0.065	0.180	0.686	0.779	1.695	1.767	3.088	4.711	100	41.17
20.0	40.0 PI	.010	97.289	101.390	100.830	100.054	99.995	33.960	35.939	37.858	37.628	35.093	5	4.25
20.0	40.0 PI	.005	8.715	18.343	20.346	10.950	15.109	59.201	68.615	77.295	85.595	53.180	10	8.35
20.0	40.0 PI	.002	0.067	0.266	0.207	0.173	0.417	1.018	2.167	2.845	8.359	9.707	25	20.65
20.0	40.0 PI	.001	0.073	0.100	0.268	0.728	2.730	3.099	6.691	6.977	12.099	18.277	50	41.17

Table 14.

FOURTH ORDER RUNGE-KUTTA KUTTA, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.003	0.020	0.056	0.099	0.123	0.123	0.080	0.134	0.216	0.247	1000	4.11
0.1	0.2 PI	.005	0	0.008	0.038	0.063	0.063	0.062	0.052	0.025	0.101	0.126	2000	8.06
0.1	0.2 PI	.002	0.002	0.010	0.008	0.010	0.025	0.025	0.053	0.110	0.122	0.114	5000	19.90
0.1	0.2 PI	.001	0.002	0.020	0.017	0.037	0.037	0.058	0.141	0.185	0.186	0.140	10000	39.62
0.5	1.0 PI	.010	0.618	1.241	1.868	2.498	3.133	3.769	4.410	5.055	5.707	6.363	200	4.11
0.5	1.0 PI	.005	0.309	0.619	0.929	1.240	1.554	1.868	2.183	2.498	2.815	3.130	400	8.06
0.5	1.0 PI	.002	0.124	0.247	0.371	0.496	0.620	0.744	0.912	1.082	1.262	1.466	1000	19.90
0.5	1.0 PI	.001	0.064	0.124	0.190	0.319	0.475	0.651	0.826	1.003	1.178	1.354	2000	39.62
1.0	2.0 PI	.010	2.493	5.048	7.667	10.350	13.127	15.946	18.836	21.798	24.849	27.975	100	4.11
1.0	2.0 PI	.005	1.241	2.498	3.769	5.055	6.363	7.681	9.017	10.371	11.740	13.124	200	8.06
1.0	2.0 PI	.002	0.495	0.990	1.491	1.994	2.498	3.002	3.508	4.019	4.517	5.050	500	19.90
1.0	2.0 PI	.001	0.247	0.496	0.744	1.060	1.466	1.871	2.278	2.686	3.091	3.499	1000	39.62
2.0	4.0 PI	.010	10.279	21.594	34.049	47.756	63.630	80.486	99.059	119.522	142.118	167.000	50	4.11
2.0	4.0 PI	.005	5.048	10.350	15.946	21.798	27.975	34.358	41.133	48.248	55.717	63.724	100	8.06
2.0	4.0 PI	.002	1.994	4.026	6.101	8.220	10.372	12.565	14.799	17.081	19.344	21.710	250	19.90
2.0	4.0 PI	.001	0.990	1.994	3.002	4.019	5.030	6.068	7.270	8.547	9.853	11.152	500	39.62
5.0	10.0 PI	.010	76.282	198.370	382.486	637.543	999.999	999.999	999.999	999.999	999.999	999.999	20	4.11
5.0	10.0 PI	.005	35.447	83.001	150.566	239.014	357.928	494.533	692.948	954.633	999.999	999.999	40	8.06
5.0	10.0 PI	.002	13.126	27.962	44.650	63.588	84.943	109.446	136.822	167.811	202.636	242.315	100	19.90
5.0	10.0 PI	.001	6.359	13.131	20.315	27.942	35.948	44.585	53.500	63.270	73.190	83.570	200	39.62
10.0	20.0 PI	.010	183.709	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	10	4.11
10.0	20.0 PI	.005	198.370	637.543	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	20	8.06
10.0	20.0 PI	.002	63.624	166.950	326.518	586.020	998.375	999.999	999.999	999.999	999.999	999.999	50	19.90
10.0	20.0 PI	.001	27.962	63.588	109.446	167.811	242.315	338.048	459.029	615.828	812.128	999.999	100	39.62
20.0	40.0 PI	.010	323.451	606.496	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	5	4.11
20.0	40.0 PI	.005	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	10	8.06
20.0	40.0 PI	.002	557.222	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	25	19.90
20.0	40.0 PI	.001	166.950	586.020	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	50	39.62

Table 15.

RUNGE-KUTTA ENGLAND INTEGRATION ALGORITHM ERRORS, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
.1	.2 PI	.010	.014	.013	.062	.097	.100	.075	.142	.252	.266	.251	1000	4.28
		.005	.008	.007	.040	.060	.059	.038	.036	.073	.121	.121	2000	8.25
		.002	.003	.008	.006	.009	.016	.016	.078	.131	.137	.118	5000	20.11
		.001	.001	.020	.018	.038	.037	.067	.154	.197	.197	.144	10000	39.80
.5	1.0 PI	.010	.414	.830	1.247	1.666	2.117	2.571	3.027	3.484	3.916	4.347	200	4.28
		.005	.206	.414	.622	.854	1.056	1.251	1.454	1.663	1.912	2.185	400	8.25
		.002	.082	.166	.248	.331	.413	.534	.696	.879	1.064	1.274	1000	20.11
		.001	.041	.082	.134	.265	.440	.615	.805	.994	1.183	1.373	2000	39.80
1.0	2.0 PI	.010	1.665	3.358	5.077	6.826	8.456	10.205	11.983	13.789	15.654	17.548	100	4.28
		.005	.828	1.663	2.488	3.320	4.173	5.068	5.937	6.814	7.696	8.486	200	8.25
		.002	.331	.660	.993	1.327	1.661	1.984	2.383	2.889	3.396	3.907	500	20.11
		.001	.165	.330	.494	.822	1.172	1.524	1.875	2.281	2.687	3.097	1000	39.80
2.0	4.0 PI	.010	6.799	14.061	21.817	30.099	37.887	46.966	56.662	67.021	78.200	90.131	50	4.28
		.005	3.349	6.810	10.293	13.912	17.712	21.850	25.930	30.147	34.501	38.433	100	8.25
		.002	1.328	2.662	4.031	5.414	6.809	8.174	9.572	10.995	12.441	13.911	250	20.11
		.001	.660	1.327	1.988	2.647	3.507	4.498	5.499	6.503	7.512	8.529	500	39.90
5.0	10.0 PI	.010	48.799	120.763	226.564	381.620	618.265	980.145	1524.49	2340.80	3561.94	5381.95	20	4.28
		.005	22.669	50.460	82.758	123.602	174.218	240.181	317.034	411.199	526.531	662.196	40	8.26
		.002	8.563	17.774	27.981	38.969	50.853	63.244	77.000	91.980	108.285	126.030	100	20.13
		.001	4.195	8.581	13.075	17.729	22.611	27.730	32.410	37.881	43.614	49.005	200	39.90
10.0	20.0 PI	.010	263.144	828.322	1005.31	12480.5	41624.6	88519.2	99999.9	99999.9	99999.9	99999.9	10	4.28
		.005	119.976	375.753	985.236	2341.41	5343.59	8749.58	17252.4	32773.8	59288.8	99999.9	20	8.25
		.002	38.393	91.415	166.819	269.945	412.557	609.510	885.077	1268.37	1801.15	2541.38	50	20.11
		.001	17.803	38.945	63.362	92.159	126.288	166.682	208.428	263.377	328.353	396.861	100	39.93

TABLE 16.

CORRECTED EULER METHOD, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	.2 PI	.010	0	0.001	0.003	0.003	0.002	0.010	0.027	0.037	0.034	0.013	1000	1.51
0.1	0.2 PI	.005	0.001	0.002	0.008	0.012	0.010	0.005	0.020	0.033	0.029	0.006	2000	2.90
0.1	0.2 PI	.002	0.001	0.005	0.019	0.027	0.022	0.014	0.055	0.087	0.089	0.069	5000	7.03
0.1	0.2 PI	.001	0.002	0.022	0.019	0.027	0.026	0.067	0.141	0.174	0.173	0.118	10000	13.90
0.5	1.0 PI	.010	0.006	0.012	0.018	0.024	0.064	0.126	0.191	0.257	0.261	0.208	200	1.51
0.5	1.0 PI	.005	0.001	0.004	0.028	0.080	0.077	0.019	0.057	0.117	0.078	0.126	400	2.90
0.5	1.0 PI	.002	0.003	0.022	0.045	0.161	0.144	0.062	0.221	0.388	0.561	0.735	1000	7.03
0.5	1.0 PI	.001	0.007	0.060	0.049	0.155	0.319	0.491	0.664	0.836	1.007	1.180	2000	13.90
1.0	2.0 PI	.010	0.071	0.156	0.241	0.326	0.494	0.697	0.900	1.102	1.154	1.123	100	1.51
1.0	2.0 PI	.005	0.012	0.024	0.125	0.246	0.257	0.153	0.049	0.115	0.146	0.497	200	2.90
1.0	2.0 PI	.002	0.011	0.058	0.122	0.360	0.341	0.190	0.522	0.856	1.186	1.513	500	7.03
1.0	2.0 PI	.001	0.022	0.161	0.144	0.386	0.713	1.042	1.370	1.698	2.022	2.353	1000	13.90
2.0	4.0 PI	.010	0.666	1.423	2.182	2.933	3.764	4.790	5.817	6.845	7.587	8.149	50	1.51
2.0	4.0 PI	.005	0.156	0.326	0.686	1.077	1.154	1.095	1.017	0.962	1.665	2.466	100	2.90
2.0	4.0 PI	.002	0.053	0.170	0.225	0.674	0.653	0.585	1.272	1.954	2.644	3.305	250	7.03
2.0	4.0 PI	.001	0.058	0.360	0.341	0.856	1.513	2.176	2.831	3.496	4.140	4.800	500	13.90
5.0	10.0 PI	.010	2.195	3.381	7.504	13.204	38.223	49.555	61.668	74.353	87.053	99.892	20	1.51
5.0	10.0 PI	.005	0.103	0.212	2.040	2.982	3.846	4.122	2.544	3.312	4.206	10.352	40	2.90
5.0	10.0 PI	.002	0.039	0.088	0.084	0.006	0.028	0.309	0.430	0.526	1.287	1.468	100	7.03
5.0	10.0 PI	.001	0.008	0.007	0.028	0.061	0.213	0.244	0.511	0.542	0.924	1.392	200	13.90
10.0	20.0 PI	.010	50.834	152.078	198.211	196.724	129.995	16.801	113.947	180.022	181.099	132.917	10	1.51
10.0	20.0 PI	.005	3.381	13.204	49.555	74.353	99.892	107.564	125.032	149.127	169.982	193.637	20	2.90
10.0	20.0 PI	.002	0.623	1.449	1.561	1.264	2.126	6.402	8.285	10.208	18.101	20.958	50	7.03
10.0	20.0 PI	.001	0.088	0.080	0.309	0.526	1.468	1.749	3.310	3.656	5.826	8.418	100	13.90

TABLE 17.

EULER PREDICTOR-CORRECTOR, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.001	0.002	0.004	0.004	0.002	0.009	0.024	0.034	0.032	0.011	1000	1.69
0.1	0.2 PI	.005	0.001	0.002	0.008	0.012	0.010	0.005	0.021	0.033	0.029	0.005	2000	3.27
0.1	0.2 PI	.002	0.001	0.005	0.019	0.027	0.022	0.014	0.054	0.087	0.089	0.069	5000	7.94
0.1	0.2 PI	.001	0.002	0.022	0.019	0.027	0.026	0.067	0.141	0.174	0.173	0.118	10000	15.70
0.5	1.0 PI	.010	0.016	0.047	0.079	0.112	0.114	0.068	0.061	0.034	0.076	0.163	200	1.69
0.5	1.0 PI	.005	0.006	0.018	0.009	0.047	0.037	0.047	0.118	0.190	0.151	0.045	400	3.27
0.5	1.0 PI	.002	0.003	0.028	0.008	0.167	0.150	0.053	0.209	0.375	0.547	0.718	1000	7.94
0.5	1.0 PI	.001	0.006	0.060	0.050	0.154	0.318	0.489	0.662	0.833	1.003	1.176	2000	15.70
1.0	2.0 PI	.010	0.167	0.382	0.597	0.812	0.922	1.025	1.124	1.226	1.508	1.830	100	1.69
1.0	2.0 PI	.005	0.047	0.118	0.118	0.058	0.163	0.340	0.519	0.695	0.665	0.387	200	3.27
1.0	2.0 PI	.002	0.004	0.037	0.155	0.405	0.391	0.122	0.442	0.763	1.083	1.398	500	7.94
1.0	2.0 PI	.001	0.020	0.167	0.158	0.373	0.697	1.024	1.349	1.673	1.993	2.320	1000	15.70
2.0	4.0 PI	.010	1.499	3.178	4.836	6.502	7.824	9.243	10.662	12.078	13.887	15.737	50	1.69
2.0	4.0 PI	.005	0.362	0.812	1.025	1.225	1.830	2.518	3.176	3.839	3.799	3.581	100	3.27
2.0	4.0 PI	.002	0.039	0.027	0.500	1.044	1.036	0.441	0.621	1.210	1.805	2.379	250	7.94
2.0	4.0 PI	.001	0.037	0.405	0.391	0.763	1.398	2.038	2.669	3.312	3.932	4.569	500	15.70
5.0	10.0 PI	.010	13.587	29.573	46.671	63.526	69.968	73.435	85.885	96.374	104.324	110.029	20	1.69
5.0	10.0 PI	.005	1.689	3.737	4.188	5.186	7.230	15.115	18.647	22.407	26.404	22.515	40	3.27
5.0	10.0 PI	.002	0.060	0.103	0.331	0.443	0.610	0.641	0.619	0.773	0.894	0.886	100	7.94
5.0	10.0 PI	.001	0.008	0.018	0.032	0.065	0.158	0.188	0.354	0.364	0.616	0.930	200	15.70
10.0	20.0 PI	.010	99.619	105.432	102.878	99.958	99.948	81.844	82.444	83.023	82.954	82.187	10	1.69
10.0	20.0 PI	.005	29.573	63.526	73.435	96.374	110.028	124.084	123.464	118.626	111.504	102.954	20	3.27
10.0	20.0 PI	.002	1.042	2.401	7.351	10.844	15.000	15.882	18.767	23.548	28.057	27.189	50	7.94
10.0	20.0 PI	.001	0.103	0.443	0.641	0.773	0.886	1.146	1.366	1.558	1.884	2.303	100	15.70

TABLE 18.

MODIFIED EULER PREDICTOR-CORRECTOR, UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.001	0.002	0.003	0.003	0.002	0.008	0.025	0.036	0.034	0.015	1000	2.06
0.1	0.2 PI	.005	0.001	0.003	0.008	0.013	0.011	0.008	0.023	0.033	0.028	0.010	2000	3.99
0.1	0.2 PI	.002	0	0.005	0.017	0.022	0.016	0.010	0.050	0.092	0.098	0.084	5000	9.75
0.1	0.2 PI	.001	0.001	0.022	0.019	0.041	0.041	0.053	0.137	0.186	0.187	0.147	10000	19.34
0.5	1.0 PI	.010	0.001	0.007	0.014	0.021	0.013	0.059	0.109	0.161	0.153	0.088	200	2.06
0.5	1.0 PI	.005	0.002	0.009	0.021	0.068	0.061	0.014	0.077	0.142	0.105	0.099	400	3.99
0.5	1.0 PI	.002	0.004	0.022	0.044	0.162	0.148	0.063	0.218	0.383	0.555	0.728	1000	9.75
0.5	1.0 PI	.001	0.008	0.057	0.052	0.159	0.323	0.492	0.666	0.838	1.010	1.183	2000	19.34
1.0	2.0 PI	.010	0.006	0.021	0.035	0.049	0.029	0.133	0.236	0.339	0.341	0.210	100	2.06
1.0	2.0 PI	.005	0.007	0.020	0.059	0.158	0.153	0.048	0.177	0.303	0.270	0.262	200	3.99
1.0	2.0 PI	.002	0.009	0.052	0.132	0.374	0.362	0.170	0.499	0.829	1.155	1.478	500	9.75
1.0	2.0 PI	.001	0.022	0.162	0.148	0.383	0.711	1.040	1.366	1.693	2.016	2.346	1000	19.34
2.0	4.0 PI	.010	0.069	0.138	0.206	0.274	0.355	0.483	0.682	0.876	0.872	0.738	50	2.06
2.0	4.0 PI	.005	0.021	0.049	0.123	0.319	0.341	0.146	0.400	0.659	0.582	0.547	100	3.99
2.0	4.0 PI	.002	0.025	0.112	0.314	0.794	0.784	0.406	1.062	1.714	2.373	3.007	250	9.75
2.0	4.0 PI	.001	0.052	0.374	0.362	0.829	1.478	2.133	2.781	3.440	4.075	4.730	500	19.34
5.0	10.0 PI	.010	2.719	5.344	7.898	10.383	12.474	14.908	17.282	19.587	21.486	23.323	20	2.06
5.0	10.0 PI	.005	0.338	0.674	1.063	1.472	1.701	2.005	2.330	2.650	2.993	3.548	40	3.99
5.0	10.0 PI	.002	0.023	0.055	0.050	0.051	0.091	0.180	0.271	0.333	0.766	0.876	100	9.75
5.0	10.0 PI	.001	0.001	0.015	0.014	0.030	0.152	0.172	0.399	0.413	0.743	1.151	200	19.34
10.0	20.0 PI	.010	36.719	59.961	74.736	84.104	86.804	77.962	80.158	81.689	81.690	81.457	10	2.06
10.0	20.0 PI	.005	5.344	10.383	14.908	19.587	23.323	28.005	31.821	35.417	38.918	41.703	20	3.99
10.0	20.0 PI	.002	0.362	0.755	1.005	1.263	1.694	2.303	2.927	3.432	5.355	6.044	50	9.75
10.0	20.0 PI	.001	0.055	0.051	0.180	0.333	0.876	1.004	1.956	2.065	3.419	5.073	100	19.34

TABLE 19.

ADAMS-BASHFORTH PREDICTOR-CORRECTOR FOR UNDAMPED SOLUTIONS

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL) 1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0	0.002	0.003	0.003	0.002	0.010	0.026	0.036	0.033	0.012	1000	3.65
0.1	0.2 PI	.005	0.001	0.002	0.008	0.012	0.010	0.005	0.020	0.033	0.029	0.006	2000	7.17
0.1	0.2 PI	.002	0.001	0.005	0.019	0.027	0.023	0.014	0.055	0.087	0.089	0.069	5000	17.70
0.1	0.2 PI	.001	0.002	0.022	0.019	0.027	0.026	0.067	0.141	0.174	0.173	0.118	10000	35.25
0.5	1.0 PI	.010	0.001	0.007	0.013	0.021	0.012	0.058	0.109	0.161	0.154	0.089	200	3.65
0.5	1.0 PI	.005	0.003	0.009	0.020	0.068	0.063	0.017	0.079	0.143	0.101	0.094	400	7.17
0.5	1.0 PI	.002	0.003	0.021	0.046	0.163	0.146	0.059	0.217	0.384	0.558	0.731	1000	17.70
0.5	1.0 PI	.001	0.007	0.080	0.089	0.155	0.319	0.491	0.664	0.835	1.007	1.180	2000	35.25
1.0	2.0 PI	.010	0.004	0.017	0.030	0.042	0.037	0.134	0.235	0.334	0.337	0.205	100	3.65
1.0	2.0 PI	.005	0.007	0.020	0.058	0.158	0.154	0.049	0.178	0.304	0.267	0.261	200	7.17
1.0	2.0 PI	.002	0.008	0.052	0.132	0.374	0.359	0.168	0.497	0.827	1.153	1.475	500	17.70
1.0	2.0 PI	.001	0.021	0.163	0.146	0.382	0.710	1.040	1.367	1.694	2.016	2.346	1000	35.25
2.0	4.0 PI	.010	0.006	0.023	0.043	0.064	0.131	0.331	0.531	0.731	0.751	0.496	50	3.65
2.0	4.0 PI	.005	0.017	0.042	0.134	0.333	0.337	0.128	0.379	0.636	0.596	0.580	100	7.17
2.0	4.0 PI	.002	0.024	0.111	0.315	0.796	0.781	0.402	1.057	1.799	2.368	3.000	250	17.70
2.0	4.0 PI	.001	0.052	0.374	0.359	0.827	1.475	2.130	2.777	3.436	4.672	4.725	500	35.25
5.0	10.0 PI	.010	1.289	2.595	3.938	5.298	6.205	7.361	8.533	9.729	11.291	12.874	20	3.65
5.0	10.0 PI	.005	0.046	0.093	0.102	0.053	0.160	0.281	0.333	0.390	0.421	0.341	40	7.17
5.0	10.0 PI	.002	0.005	0.017	0.009	0.029	0.025	0.064	0.136	0.178	0.594	0.684	100	17.70
5.0	10.0 PI	.001	0.004	0.008	0.015	0.043	0.170	0.193	0.422	0.440	0.772	1.182	200	35.25
10.0	20.0 PI	.010	31.728	34.311	76.567	311.414	451.075	913.343	999.999	999.999	999.999	999.999	10	3.65
10.0	20.0 PI	.005	2.595	5.295	7.361	9.729	12.874	16.845	19.942	23.139	26.319	26.880	20	7.17
10.0	20.0 PI	.002	0.019	0.022	0.114	0.228	0.216	0.166	0.379	0.523	2.197	2.540	50	17.70
10.0	20.0 PI	.001	0.017	0.029	0.064	0.178	0.684	0.777	1.693	1.766	3.089	4.714	100	35.25

TABLE 20.

ADAMS-MOULTON PREDICTOR-CORRECTOR FOR UNDAMPED SOLUTIONS

FREQUENCY CPS	FREQUENCY RAD/SEC	TIME STEP	MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL) 1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.001	0.002	0.003	0.003	0.001	0.010	0.026	0.037	0.035	0.014	1000	4.14
0.1	0.2 PI	.005	0.001	0.003	0.008	0.013	0.011	0.008	0.022	0.032	0.028	0.011	2000	8.16
0.1	0.2 PI	.002	0.001	0.004	0.018	0.024	0.018	0.007	0.051	0.089	0.094	0.078	5000	20.17
0.1	0.2 PI	.001	0.001	0.023	0.080	0.035	0.035	0.057	0.137	0.181	0.181	0.137	10000	40.17
0.5	1.0 PI	.010	0.001	0.007	0.014	0.021	0.011	0.057	0.107	0.160	0.153	0.087	200	4.14
0.5	1.0 PI	.005	0.002	0.009	0.022	0.069	0.063	0.015	0.077	0.142	0.102	0.096	400	8.16
0.5	1.0 PI	.002	0.004	0.022	0.045	0.162	0.148	0.063	0.219	0.383	0.556	0.728	1000	20.17
0.5	1.0 PI	.001	0.005	0.057	0.052	0.159	0.323	0.491	0.665	0.837	1.008	1.181	2000	40.17
1.0	2.0 PI	.010	0.004	0.018	0.031	0.044	0.034	0.131	0.231	0.332	0.334	0.202	100	4.14
1.0	2.0 PI	.005	0.007	0.020	0.057	0.156	0.153	0.052	0.181	0.307	0.271	0.257	200	8.16
1.0	2.0 PI	.002	0.008	0.052	0.131	0.372	0.358	0.168	0.497	0.825	1.151	1.474	500	20.17
1.0	2.0 PI	.001	0.022	0.162	0.148	0.383	0.710	1.040	1.368	1.697	2.019	2.348	1000	40.17
2.0	4.0 PI	.010	0.010	0.037	0.061	0.089	0.092	0.284	0.476	0.668	0.692	0.429	50	4.14
2.0	4.0 PI	.005	0.018	0.048	0.131	0.329	0.334	0.134	0.387	0.646	0.603	0.568	100	8.16
2.0	4.0 PI	.002	0.024	0.111	0.317	0.798	0.785	0.397	1.052	1.702	2.360	2.991	250	20.17
2.0	4.0 PI	.001	0.052	0.372	0.358	0.825	1.474	2.128	2.774	3.431	4.066	4.717	500	40.17
5.0	10.0 PI	.010	0.772	1.571	2.372	3.177	4.604	5.407	6.215	7.034	8.207	9.391	20	4.14
5.0	10.0 PI	.005	0.029	0.059	0.065	0.024	0.115	0.182	0.217	0.257	0.272	0.259	40	8.16
5.0	10.0 PI	.002	0.003	0.017	0.009	0.027	0.024	0.067	0.139	0.182	0.597	0.688	100	20.17
5.0	10.0 PI	.001	0.001	0.012	0.011	0.039	0.166	0.189	0.419	0.437	0.769	1.179	200	40.17
10.0	20.0 PI	.010	21.387	29.266	105.956	216.135	264.053	351.560	463.259	487.729	472.485	322.894	10	4.14
10.0	20.0 PI	.005	1.571	3.177	5.407	7.034	9.391	10.112	11.410	13.125	14.751	18.697	20	8.16
10.0	20.0 PI	.002	0.014	0.031	0.085	0.189	0.176	0.144	0.413	0.561	2.193	2.537	50	20.17
10.0	20.0 PI	.001	0.017	0.027	0.067	0.182	0.688	0.781	1.696	1.769	3.088	4.708	100	40.17

TABLE 21.

MILNE PREDICTOR-CORRECTOR FOR UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0.1	0.2 PI	.010	0.001	0.002	0.003	0.003	0.002	0.008	0.025	0.036	0.034	0.014	1000	3.28
0.1	0.2 PI	.005	0.001	0.002	0.008	0.012	0.011	0.006	0.021	0.032	0.028	0.009	2000	6.18
0.1	0.2 PI	.002	0.001	0.006	0.018	0.025	0.019	0.009	0.054	0.092	0.095	0.078	5000	15.00
0.1	0.2 PI	.001	0.002	0.022	0.019	0.036	0.035	0.057	0.138	0.182	0.184	0.140	10000	29.72
0.5	1.0 PI	.010	0.001	0.007	0.014	0.022	0.011	0.057	0.107	0.159	0.152	0.088	200	3.28
0.5	1.0 PI	.005	0.002	0.009	0.022	0.069	0.062	0.015	0.077	0.142	0.102	0.099	400	6.18
0.5	1.0 PI	.002	0.004	0.022	0.044	0.163	0.147	0.060	0.218	0.382	0.554	0.728	1000	15.00
0.5	1.0 PI	.001	0.008	0.057	0.051	0.160	0.325	0.495	0.668	0.840	1.011	1.185	2000	29.72
1.0	2.0 PI	.010	0.004	0.018	0.030	0.044	0.035	0.132	0.233	0.331	0.332	0.203	100	3.28
1.0	2.0 PI	.005	0.007	0.021	0.057	0.156	0.152	0.050	0.179	0.305	0.271	0.260	200	6.18
1.0	2.0 PI	.002	0.008	0.052	0.131	0.374	0.361	0.170	0.497	0.828	1.153	1.476	500	15.00
1.0	2.0 PI	.001	0.022	0.163	0.147	0.382	0.710	1.040	1.369	1.694	2.017	2.349	1000	29.72
2.0	4.0 PI	.010	0.010	0.038	0.065	0.093	0.063	0.260	0.591	1.846	6.831	24.645	50	3.28
2.0	4.0 PI	.005	0.018	0.044	0.132	0.331	0.332	0.126	0.386	0.664	0.601	0.581	100	6.18
2.0	4.0 PI	.002	0.025	0.111	0.316	0.796	0.783	0.401	1.058	1.711	2.373	2.999	250	15.00
2.0	4.0 PI	.001	0.052	0.374	0.361	0.828	1.476	2.133	2.781	3.439	4.077	4.729	500	29.72
5.0	10.0 PI	.010	3.421	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	20	3.28
5.0	10.0 PI	.005	0.013	0.084	6.654	231.347	999.999	999.999	999.999	999.999	999.999	999.999	40	6.18
5.0	10.0 PI	.002	0.003	0.014	0.013	0.052	0.133	0.751	2.879	18.265	93.944	396.450	100	15.00
5.0	10.0 PI	.001	0.001	0.014	0.012	0.037	0.157	0.197	0.425	0.525	1.041	1.262	200	29.72

TABLE 22.

HAMMINGS MODIFIED PREDICTOR-CORRECTOR FOR UNDAMPED SOLUTIONS

MAXIMUM FULL-SCALE PERCENT ERROR IN INDICATED TIME INTERVAL (ONE SECOND PER TIME INTERVAL)

FREQUENCY CPS	RAD/SEC	TIME STEP	1	2	3	4	5	6	7	8	9	10	SAMPLES/ CYCLE	REAL TIME FACTOR
0	0 PI	.010	0.001	0.002	0.003	0.002	0.005	0.005	0.024	0.038	0.037	0.019	0	62.05
0	0 PI	.005	0.001	0.003	0.009	0.016	0.015	0.011	0.023	0.029	0.024	0.021	0	8.29
0	0 PI	.002	0.003	0.003	0.014	0.016	0.022	0.023	0.048	0.101	0.112	0.105	0	20.27
0	0 PI	.001	0.010	0.031	0.027	0.072	0.078	0.064	0.112	0.207	0.233	0.223	0	40.15
0	0 PI	.010	0.001	0.007	0.013	0.021	0.015	0.060	0.110	0.161	0.153	0.087	0	62.05
0	0 PI	.005	0.003	0.009	0.025	0.072	0.060	0.016	0.073	0.141	0.110	0.110	0	8.29
0	0 PI	.002	0.007	0.024	0.039	0.159	0.154	0.075	0.226	0.393	0.557	0.729	0	20.27
0	0 PI	.001	0.014	0.050	0.063	0.174	0.340	0.507	0.674	0.842	1.013	1.185	0	40.15
0	0 PI	.010	0.004	0.017	0.030	0.044	0.038	0.135	0.236	0.335	0.333	0.202	0	62.05
0	0 PI	.005	0.007	0.020	0.060	0.160	0.152	0.044	0.172	0.298	0.273	0.269	0	8.29
0	0 PI	.002	0.011	0.055	0.128	0.369	0.366	0.178	0.508	0.838	1.166	1.492	0	20.27
0	0 PI	.001	0.023	0.159	0.154	0.393	0.722	1.056	1.386	1.716	2.041	2.374	0	40.15
2.0	4.0 PI	.010	0.010	0.039	0.064	0.093	0.084	0.274	0.465	0.655	0.701	0.440	50	62.05
2.0	4.0 PI	.005	0.017	0.042	0.135	0.335	0.333	0.126	0.377	0.634	0.604	0.583	100	8.29
2.0	4.0 PI	.002	0.025	0.114	0.311	0.789	0.788	0.412	1.069	1.723	2.383	3.019	250	20.27
2.0	4.0 PI	.001	0.055	0.369	0.366	0.838	1.492	2.152	2.804	3.466	4.109	4.765	500	40.15
5.0	10.0 PI	.010	0.069	0.131	0.191	0.250	0.360	0.577	0.796	1.010	0.998	0.862	20	62.05
5.0	10.0 PI	.005	0.003	0.004	0.071	0.152	0.143	0.044	0.012	0.008	0.018	0.282	40	8.29
5.0	10.0 PI	.002	0.003	0.008	0.023	0.049	0.046	0.033	0.099	0.135	0.546	0.631	100	20.27
5.0	10.0 PI	.001	0.007	0.028	0.029	0.014	0.117	0.128	0.347	0.353	0.672	1.072	200	40.15
10.0	20.0 PI	.010	45.928	116.347	220.310	373.578	453.539	621.452	918.145	999.999	999.999	999.999	10	62.05
10.0	20.0 PI	.005	0.131	0.250	0.577	1.010	0.986	0.711	0.792	0.918	0.938	2.320	20	8.29
10.0	20.0 PI	.002	0.019	0.071	0.040	0.107	0.092	0.269	0.559	0.730	2.380	2.744	50	20.27
10.0	20.0 PI	.001	0.008	0.049	0.044	0.135	0.631	0.713	1.621	1.679	2.993	4.609	100	40.15
20.0	40.0 PI	.010	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	999.999	5	62.05
20.0	40.0 PI	.005	116.347	373.578	621.452	999.999	999.999	999.999	999.999	999.999	999.999	999.999	10	8.29
20.0	40.0 PI	.002	0.214	0.566	0.460	0.243	0.181	1.930	3.229	4.061	11.442	13.133	25	20.27
20.0	40.0 PI	.001	0.071	0.107	0.269	0.730	2.744	3.117	6.726	7.016	12.159	18.359	50	40.15

TABLE 23.

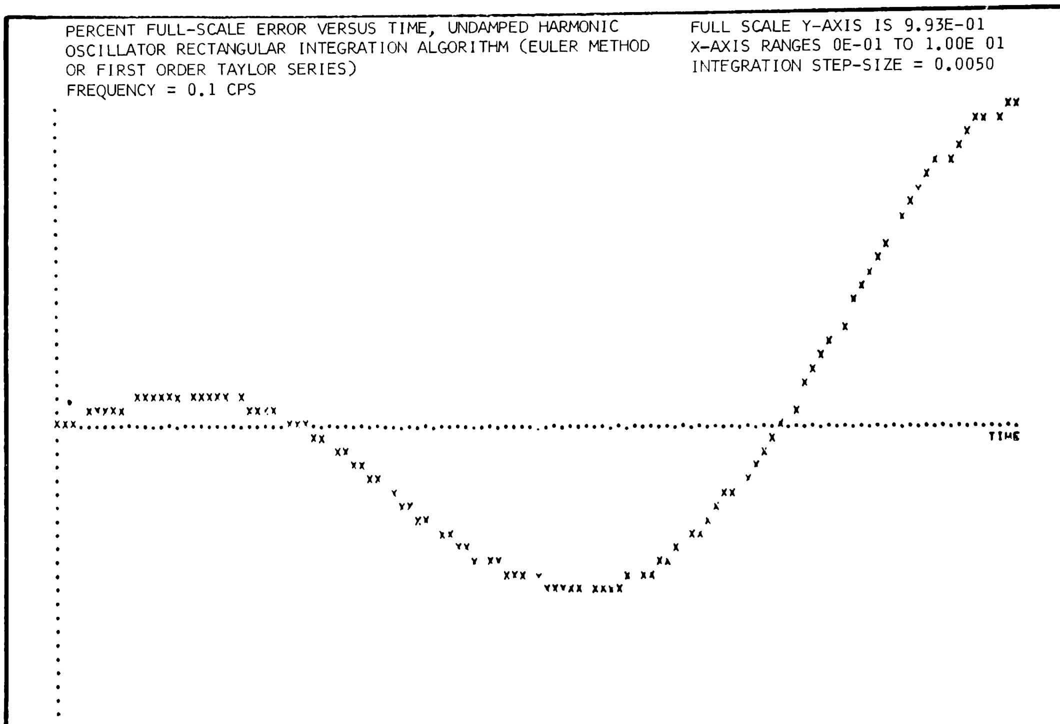


Figure 3.

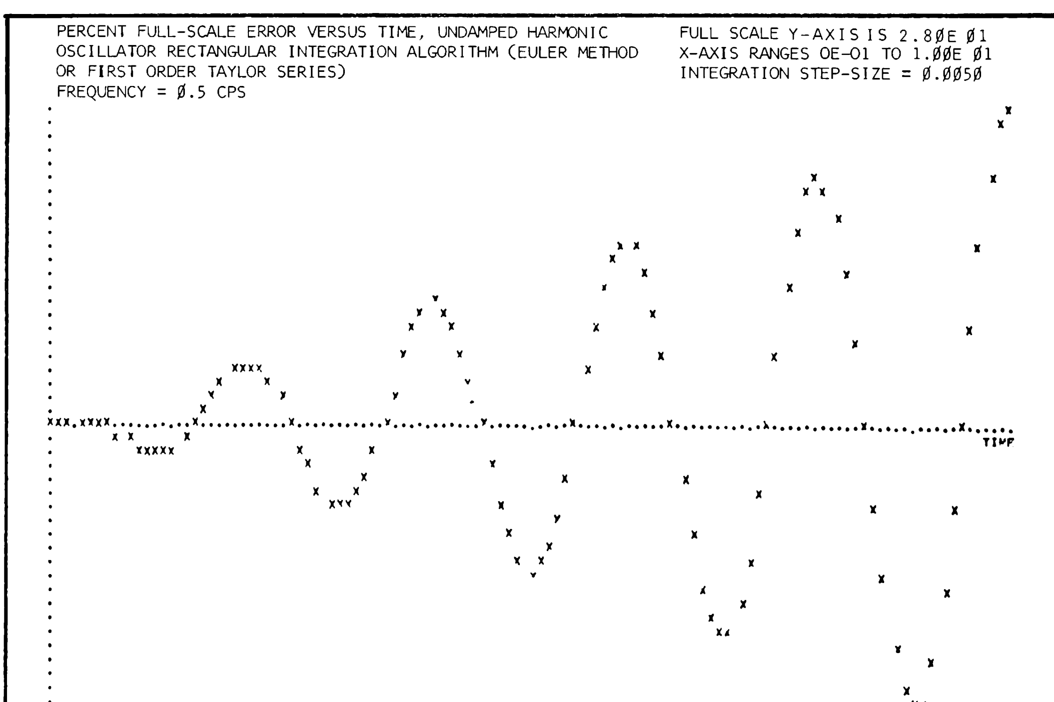


Figure 4.

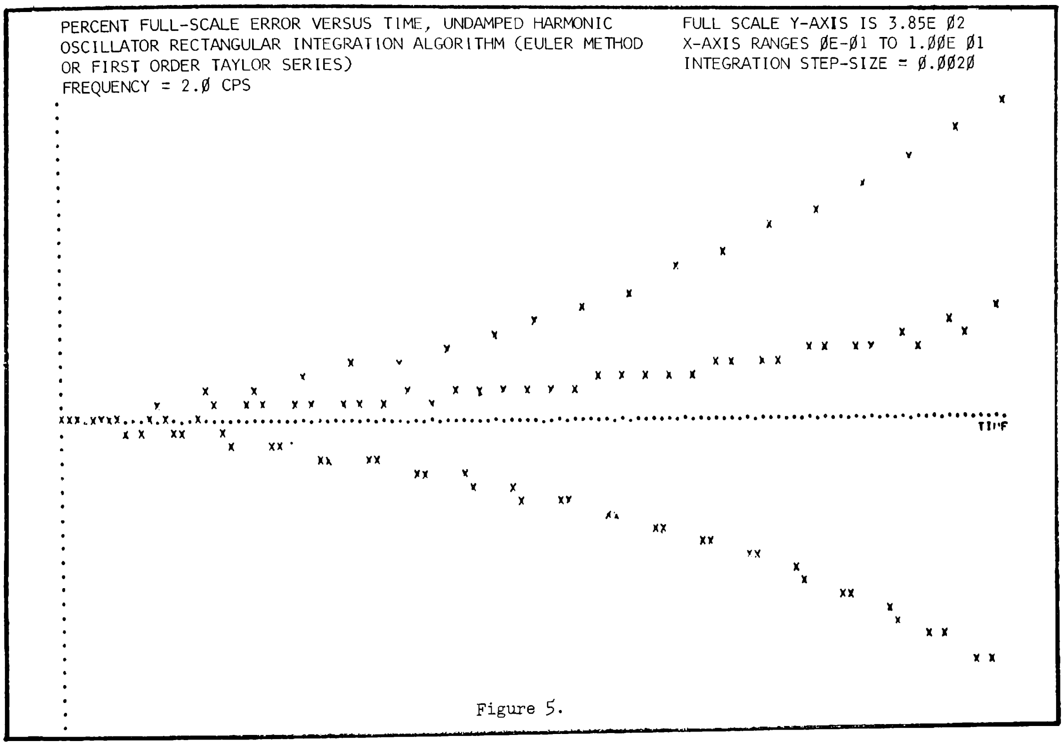


Figure 5.

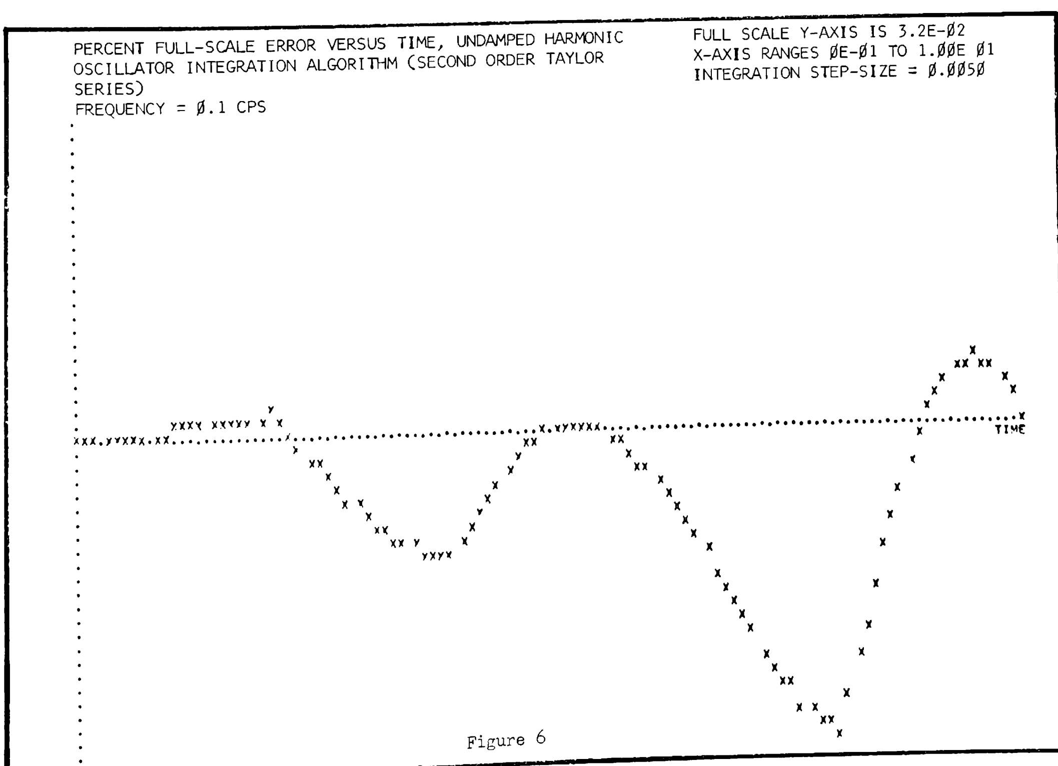


Figure 6

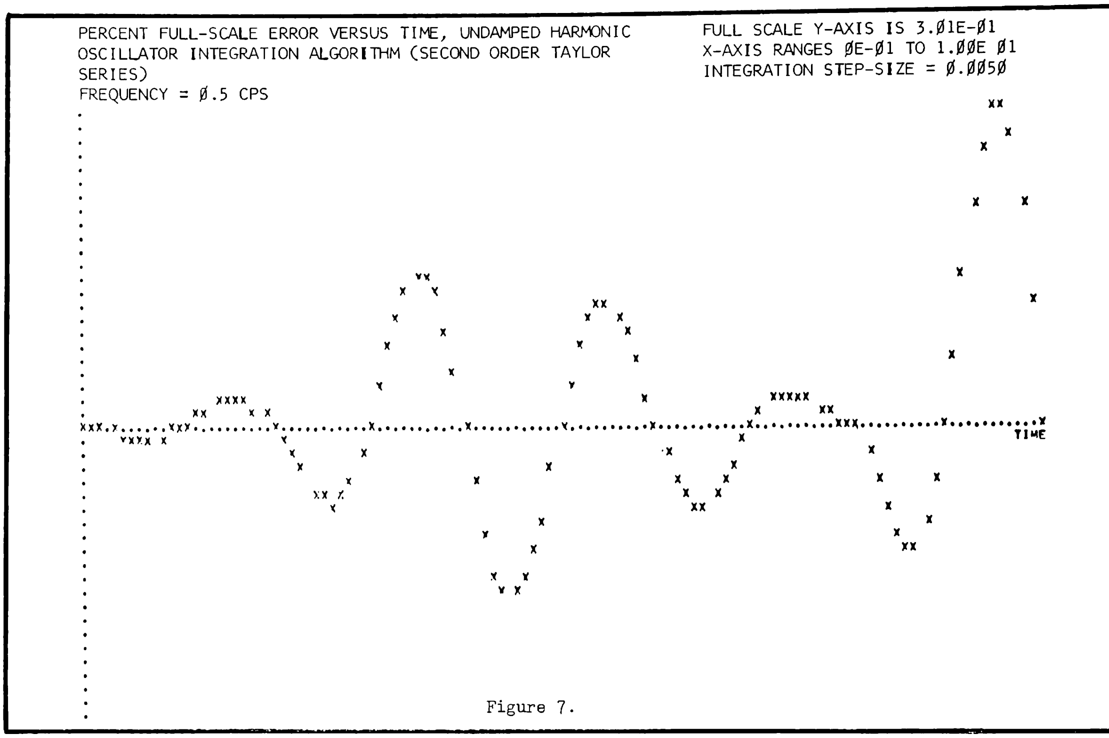


Figure 7.

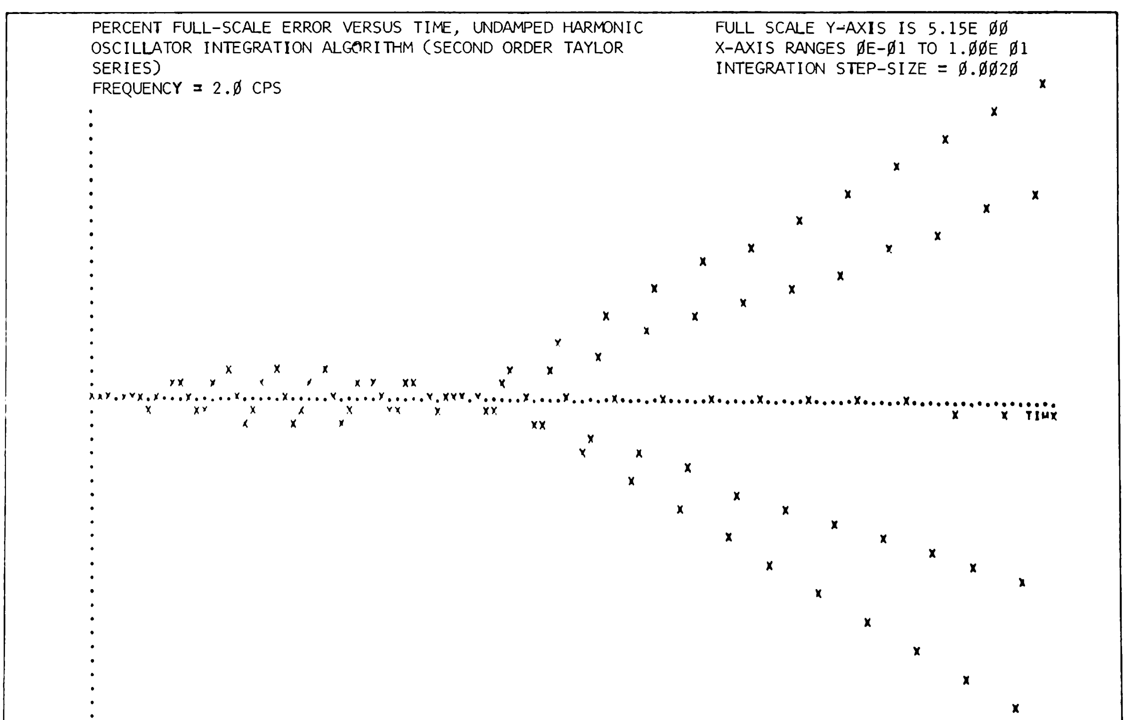


Figure 8.

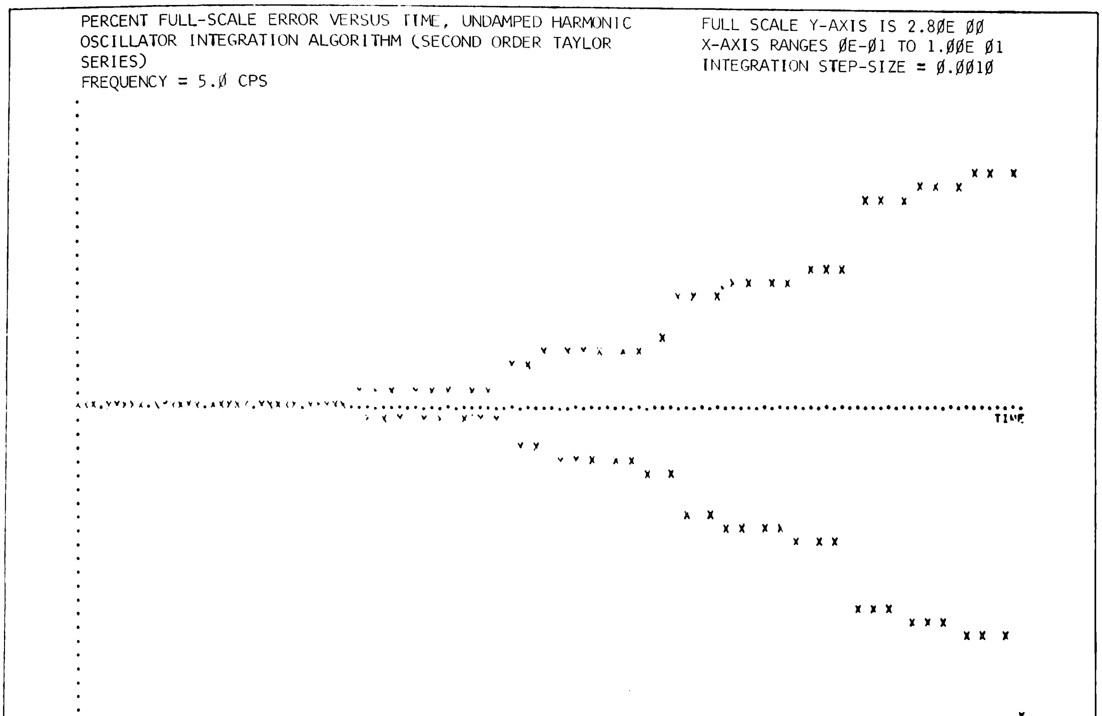


Figure 9.

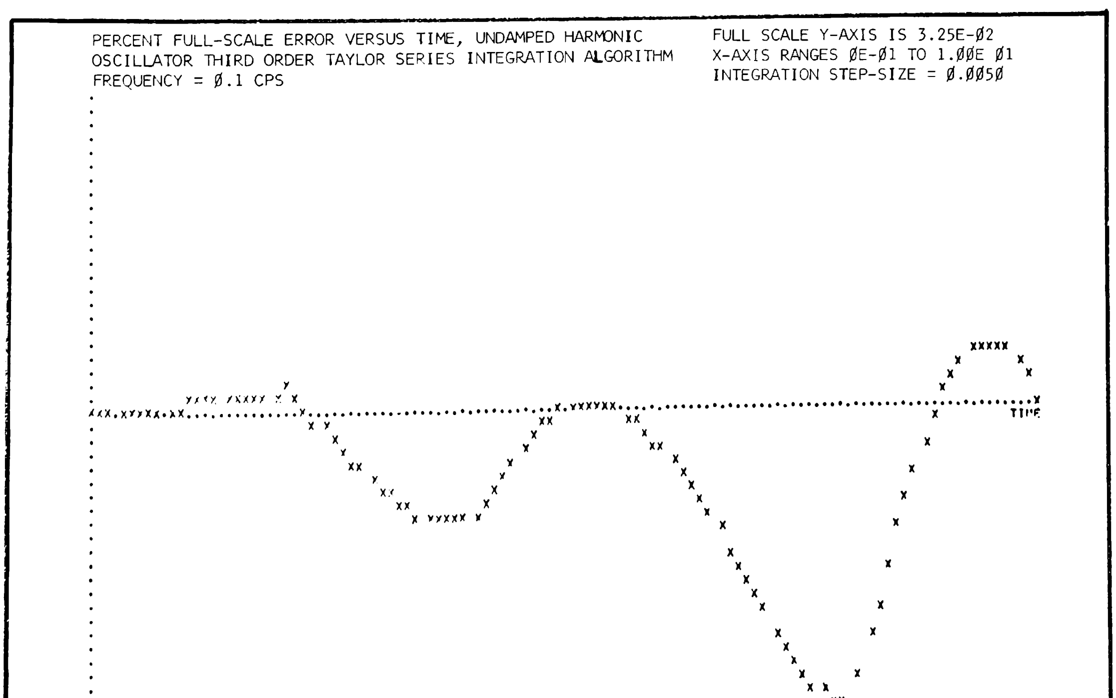


Figure 10.

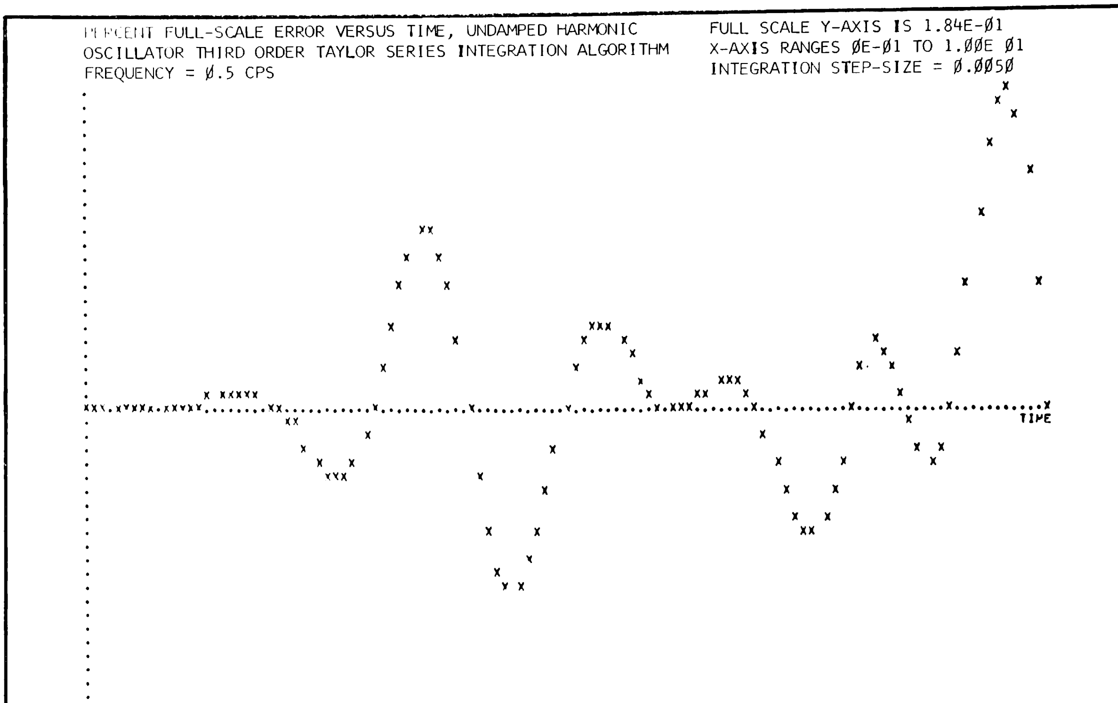


Figure 11.

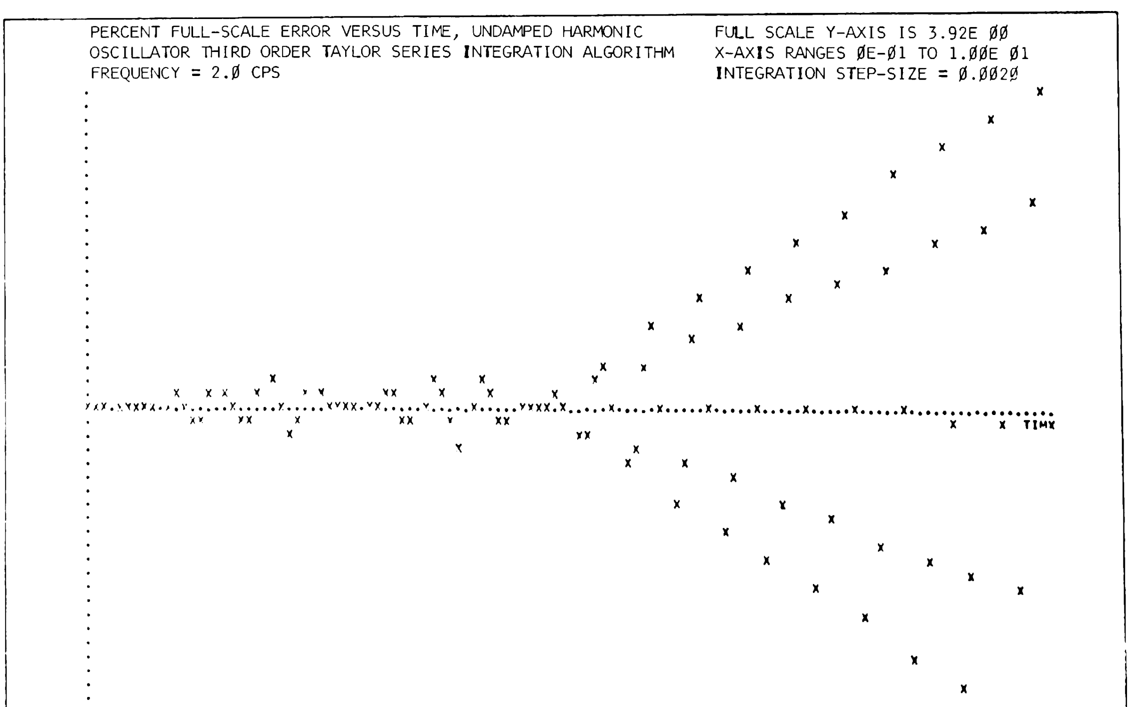


Figure 12.

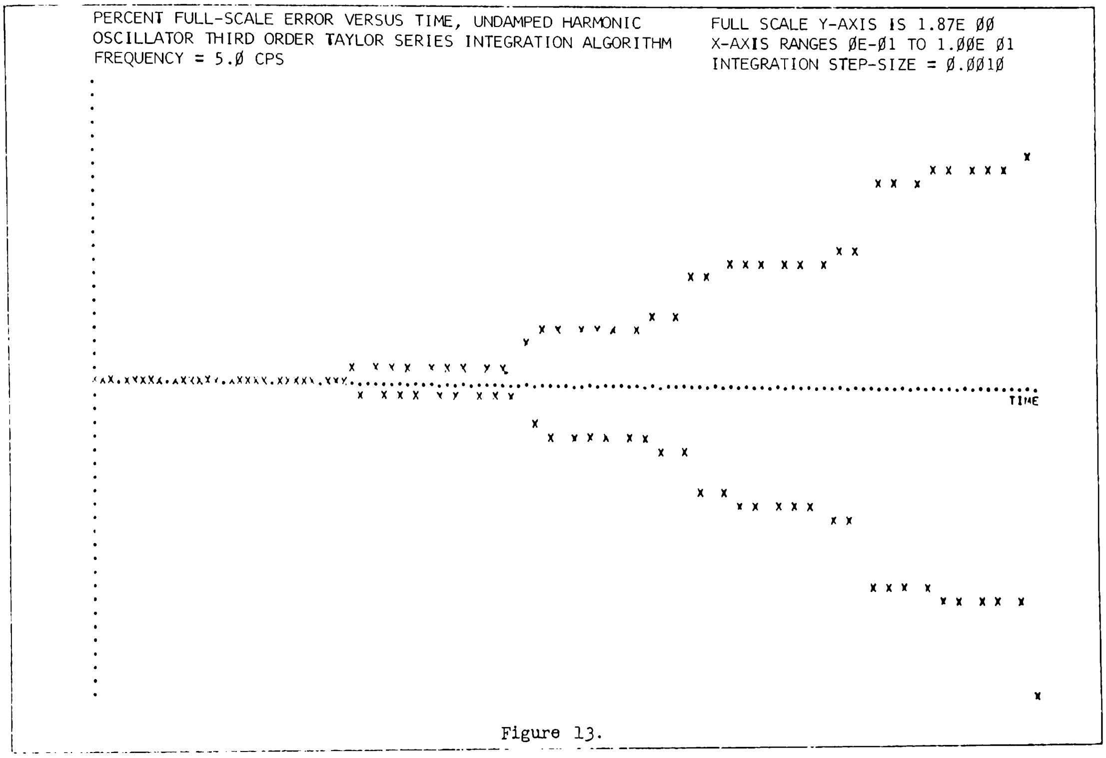


Figure 13.

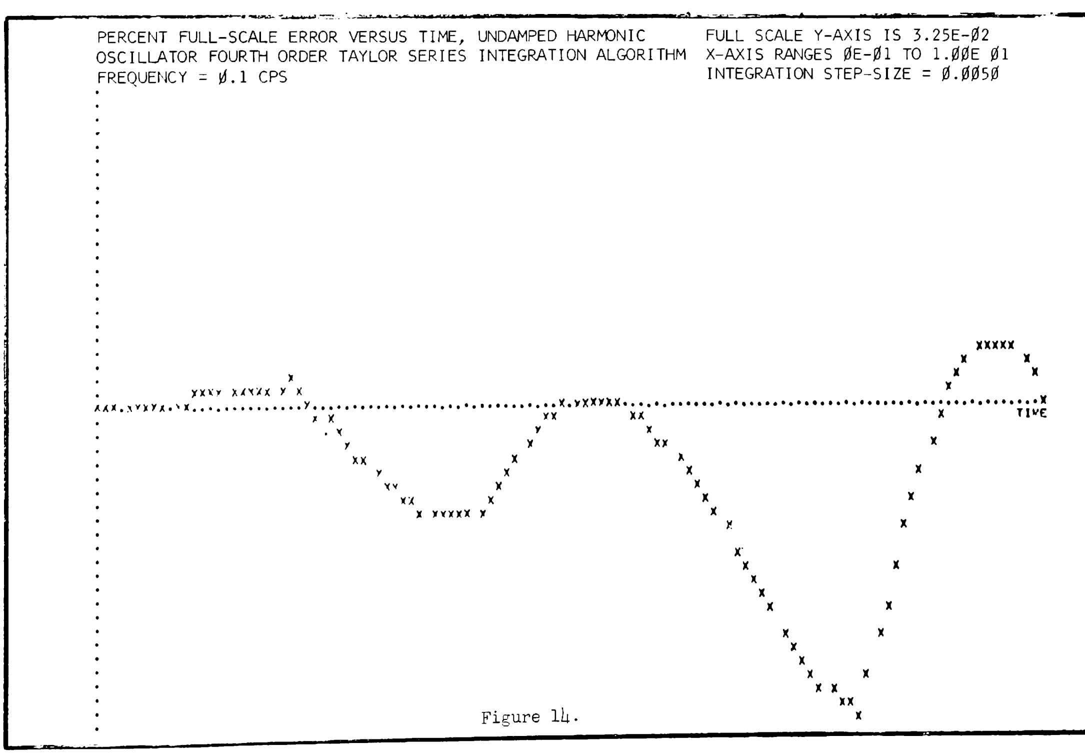


Figure 14.

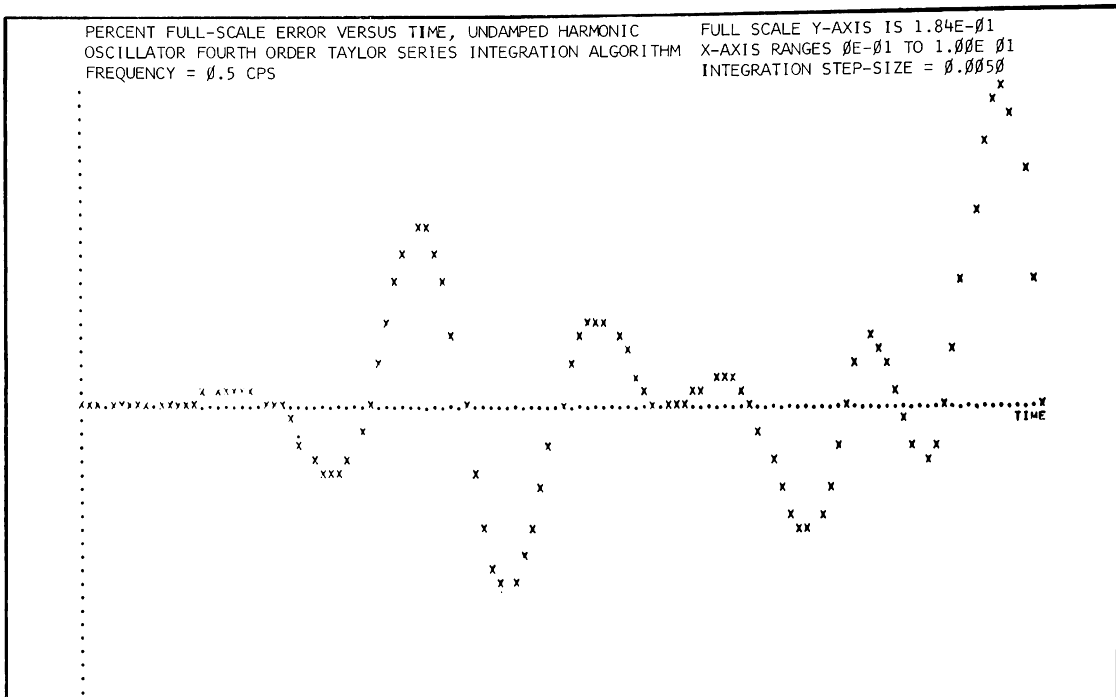
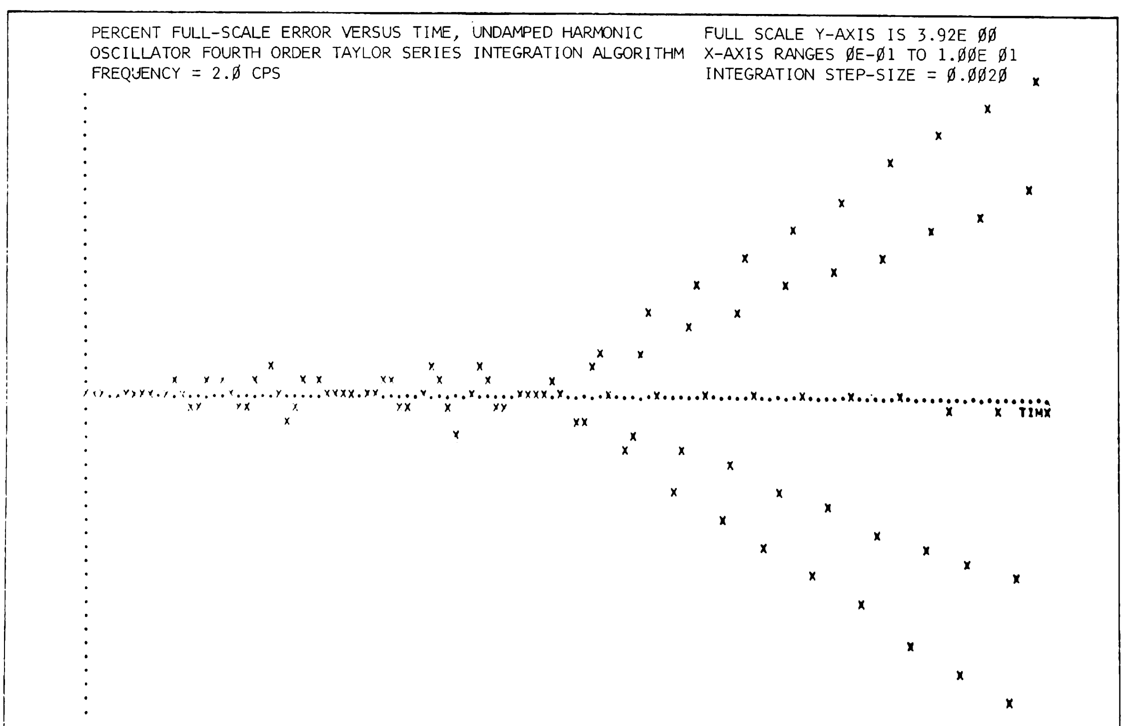


Figure 15.



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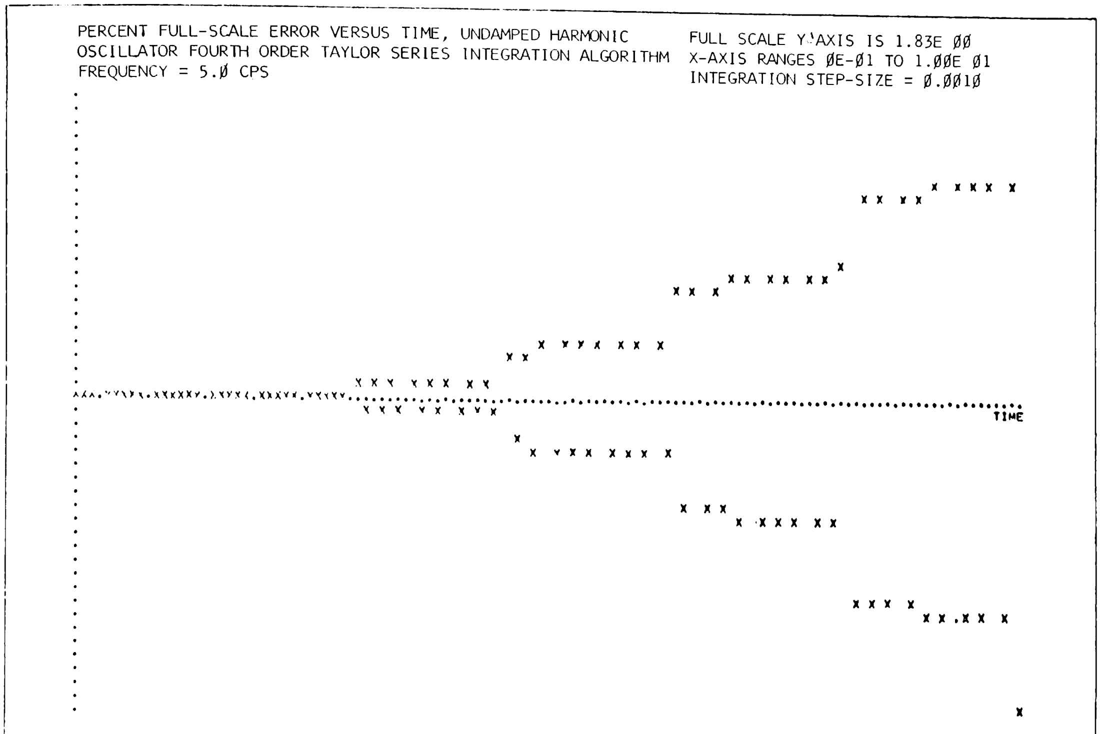

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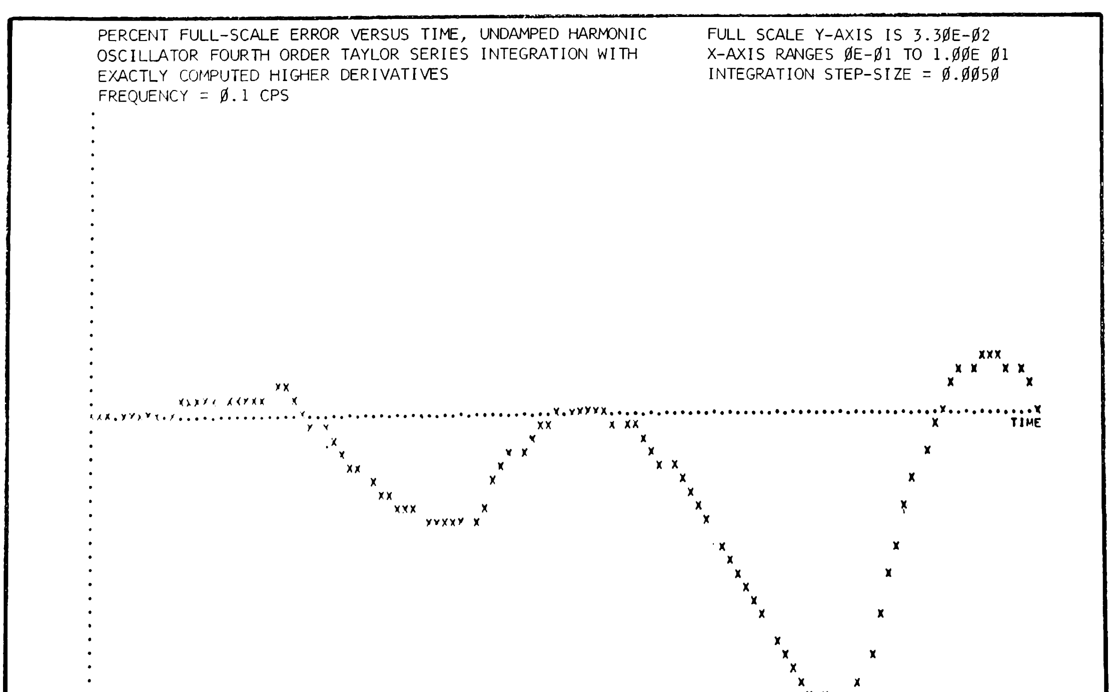

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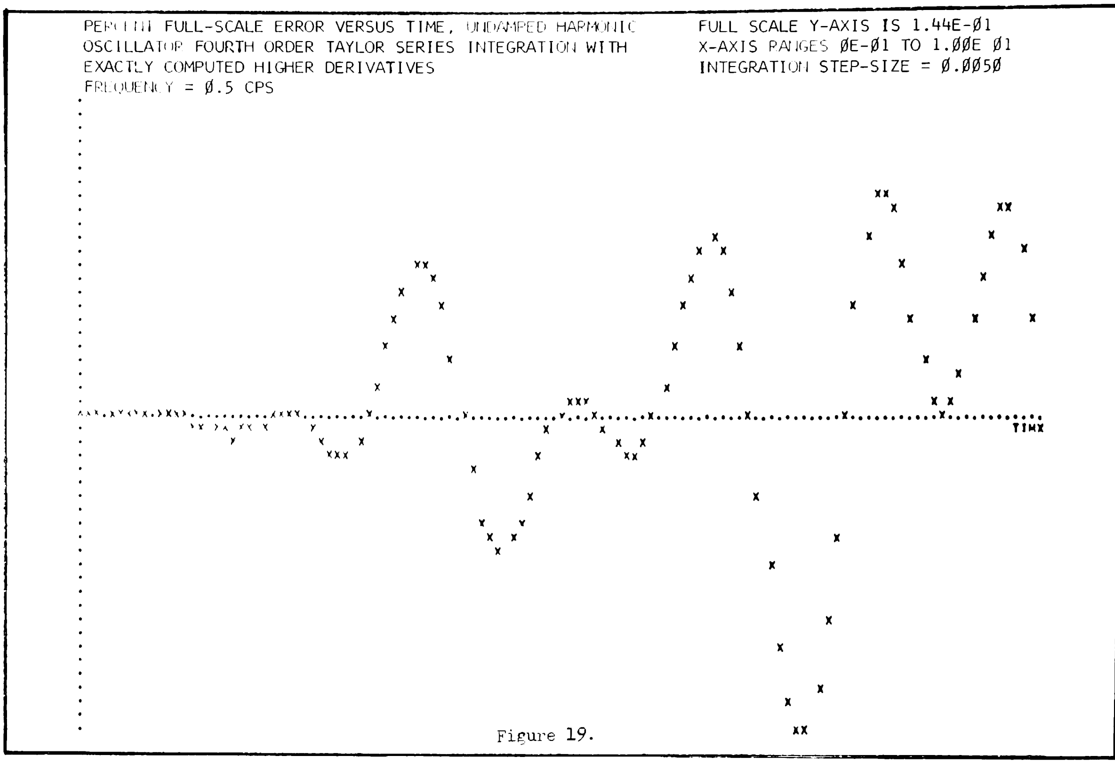

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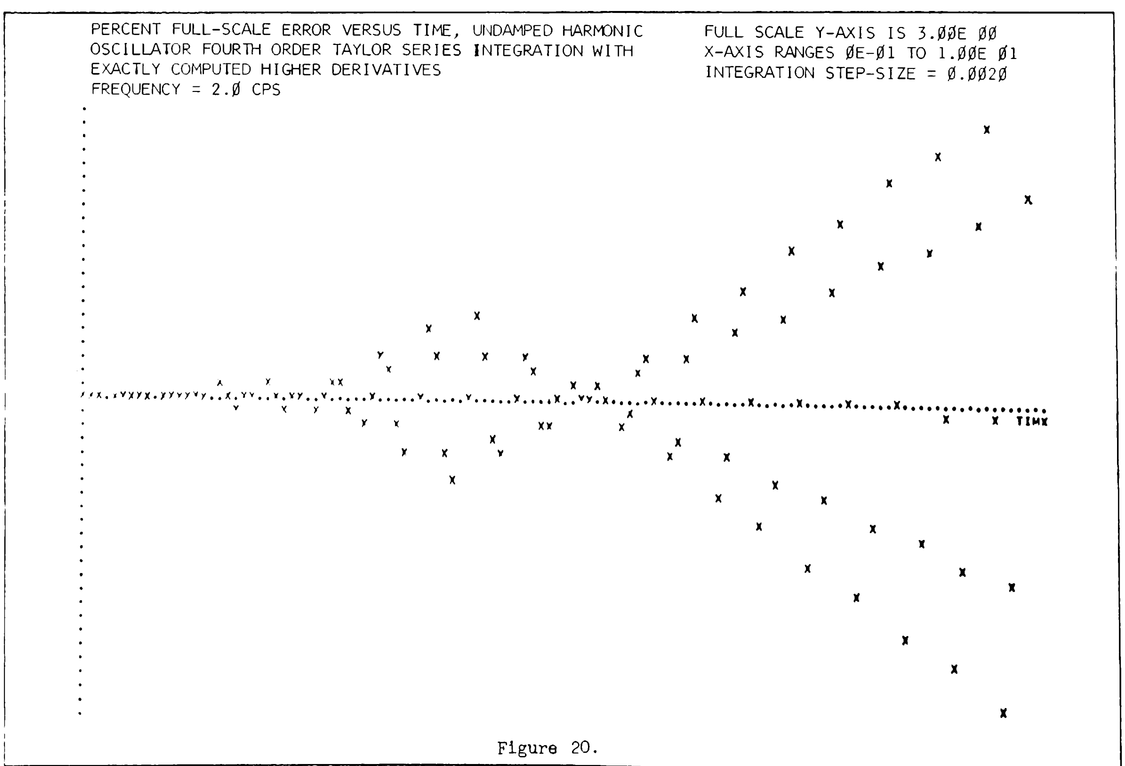

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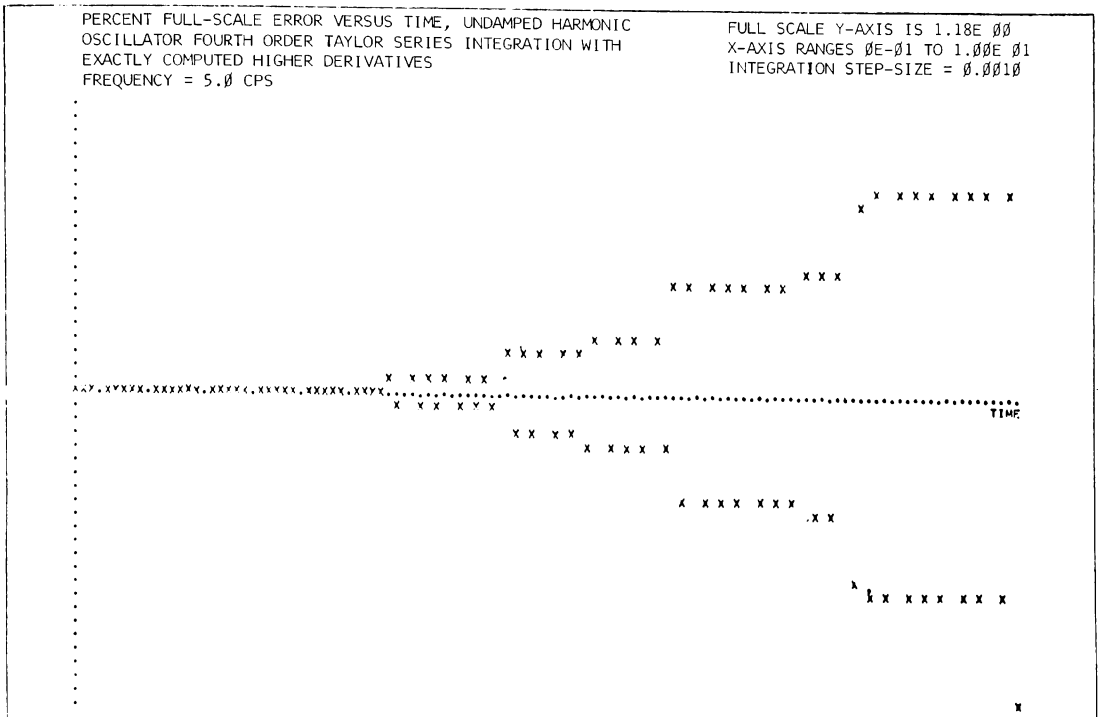

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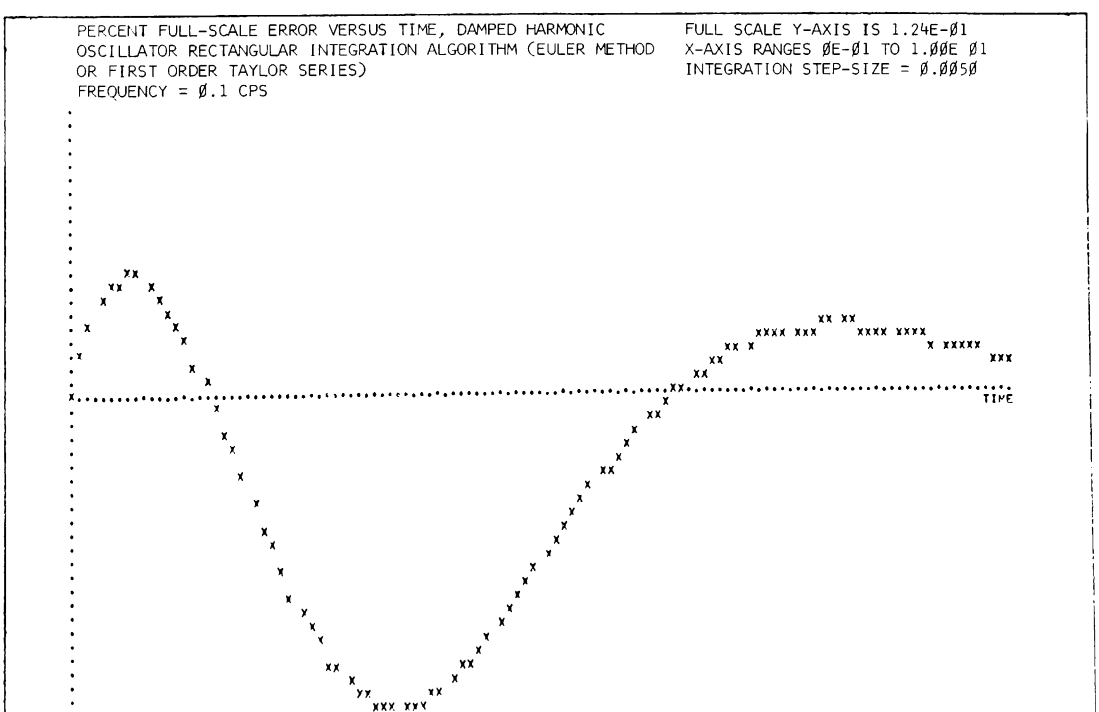

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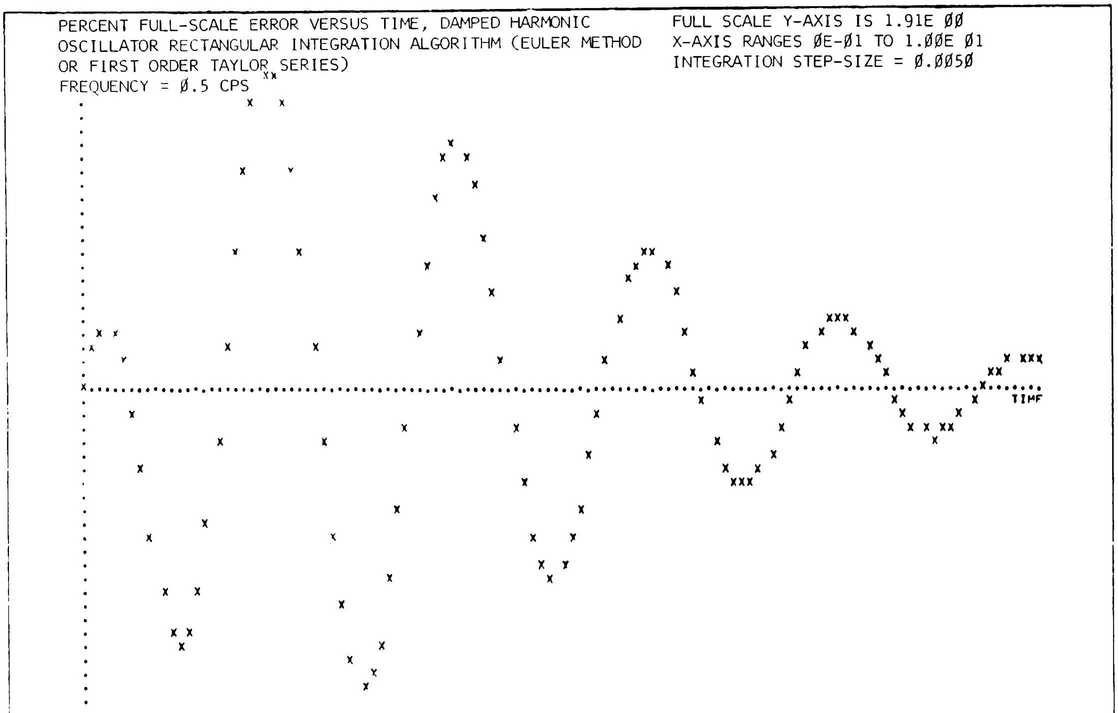

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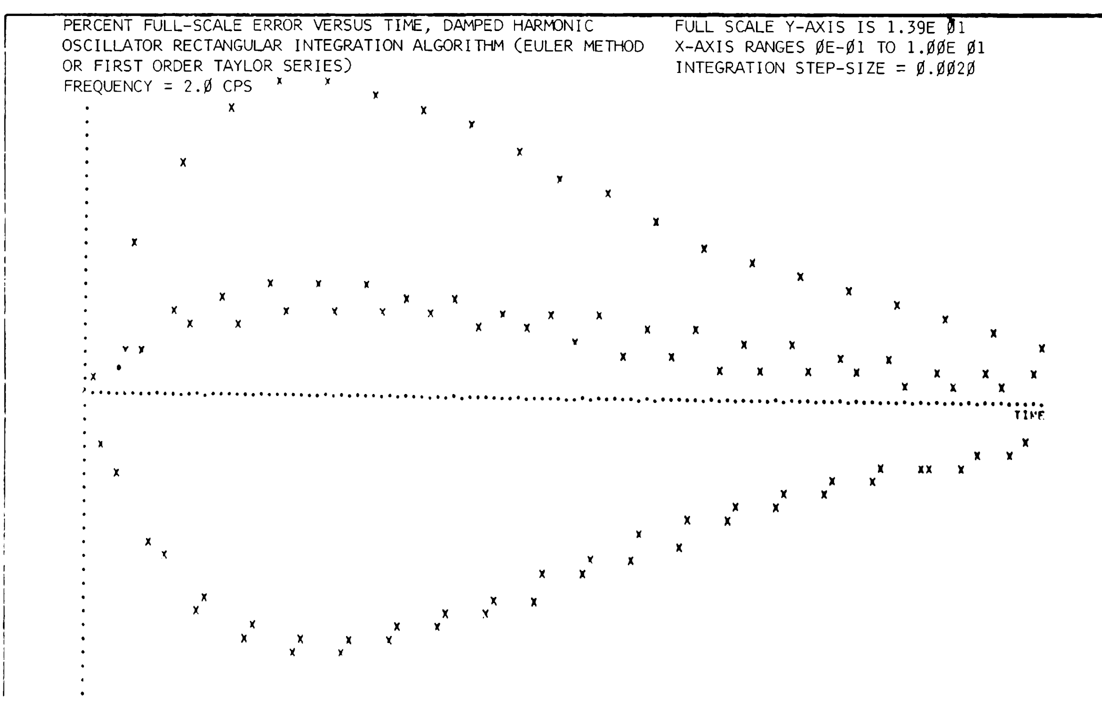

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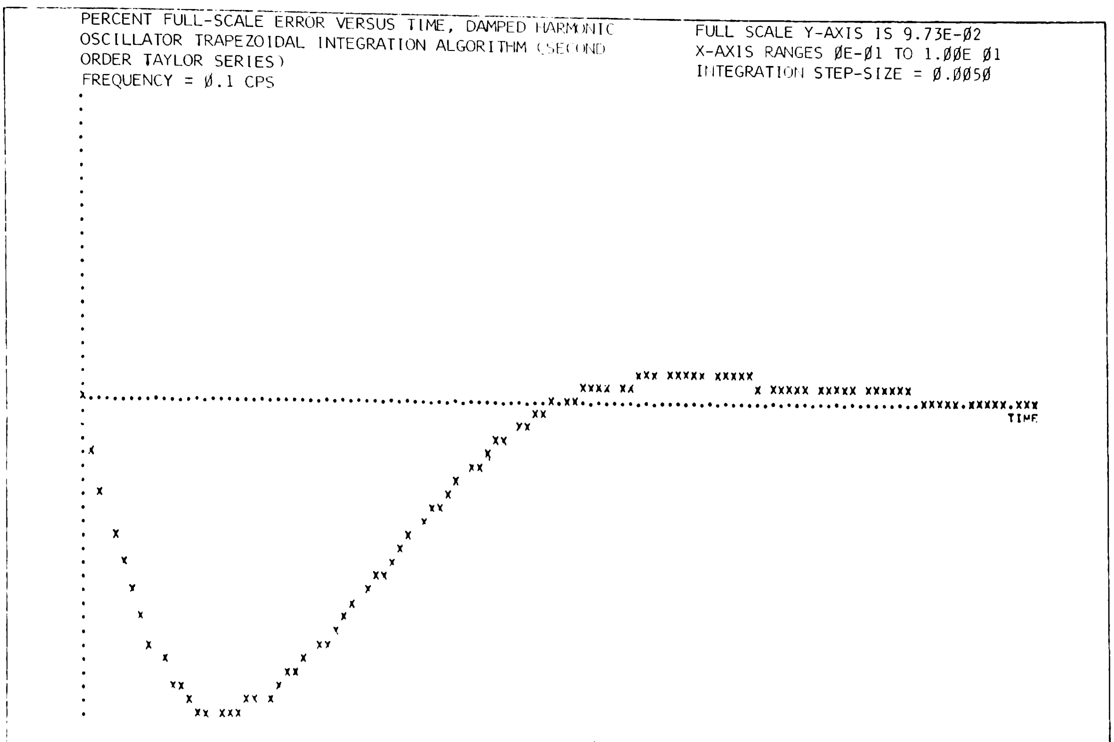

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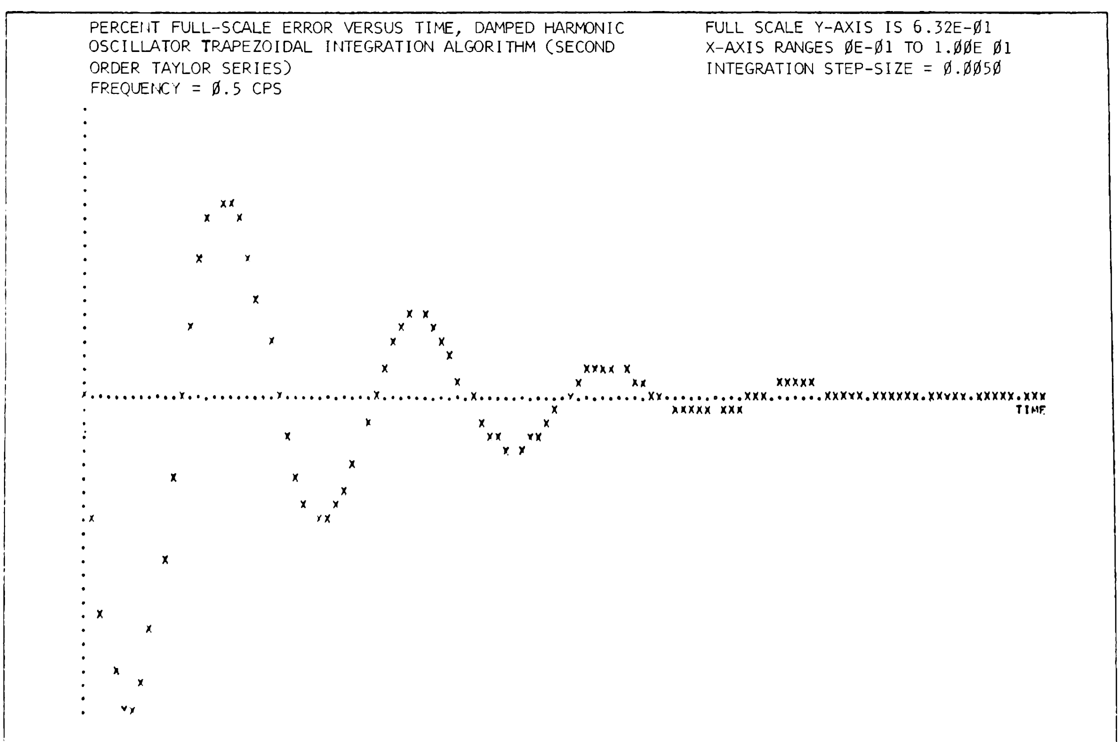

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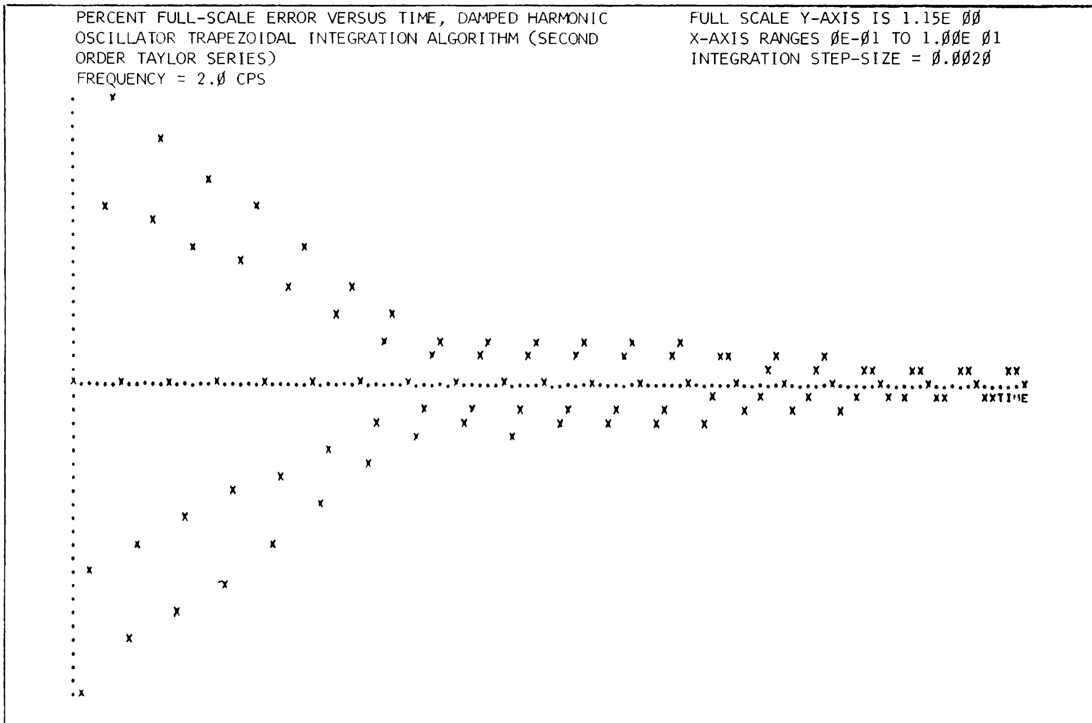

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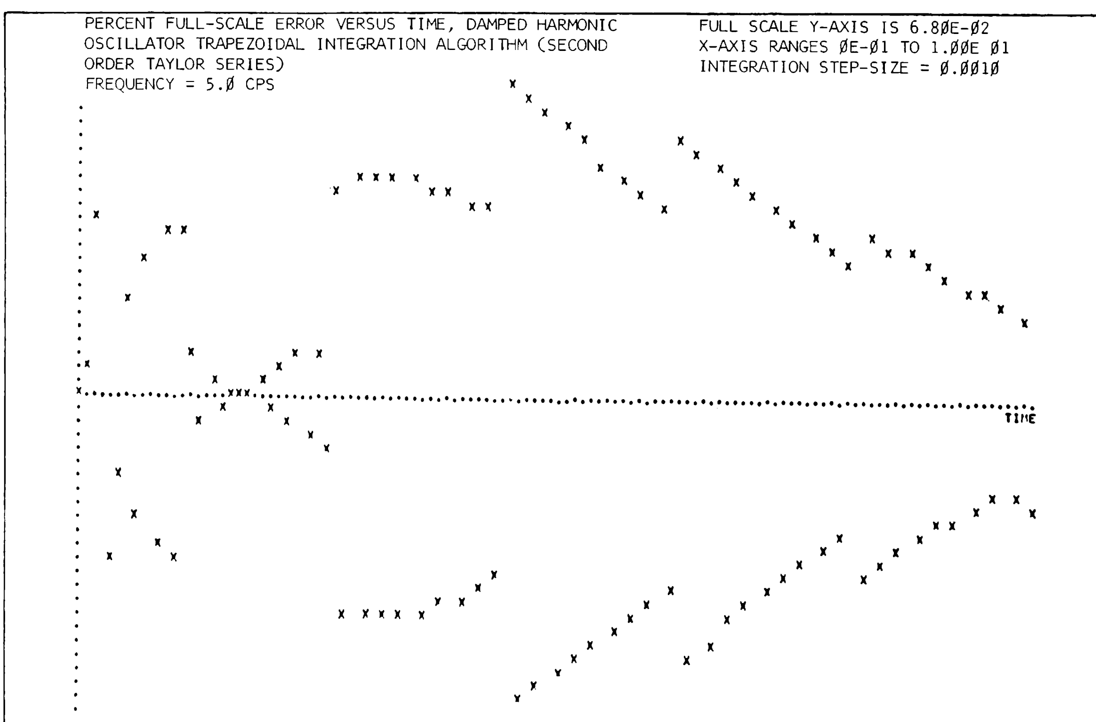

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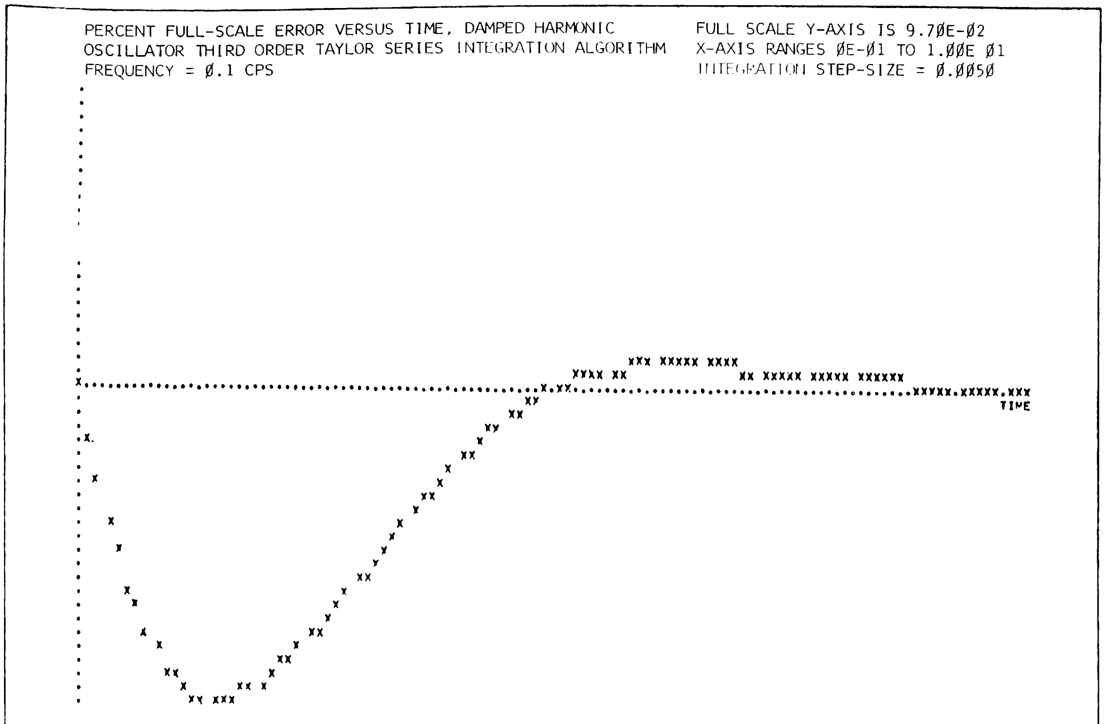

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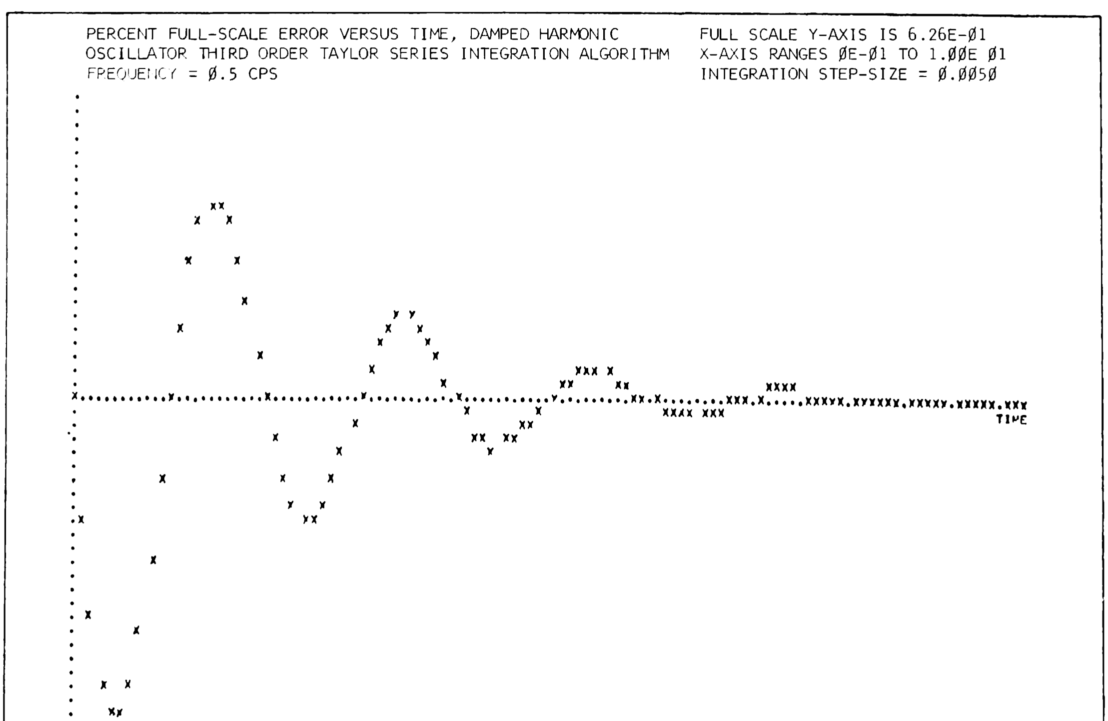

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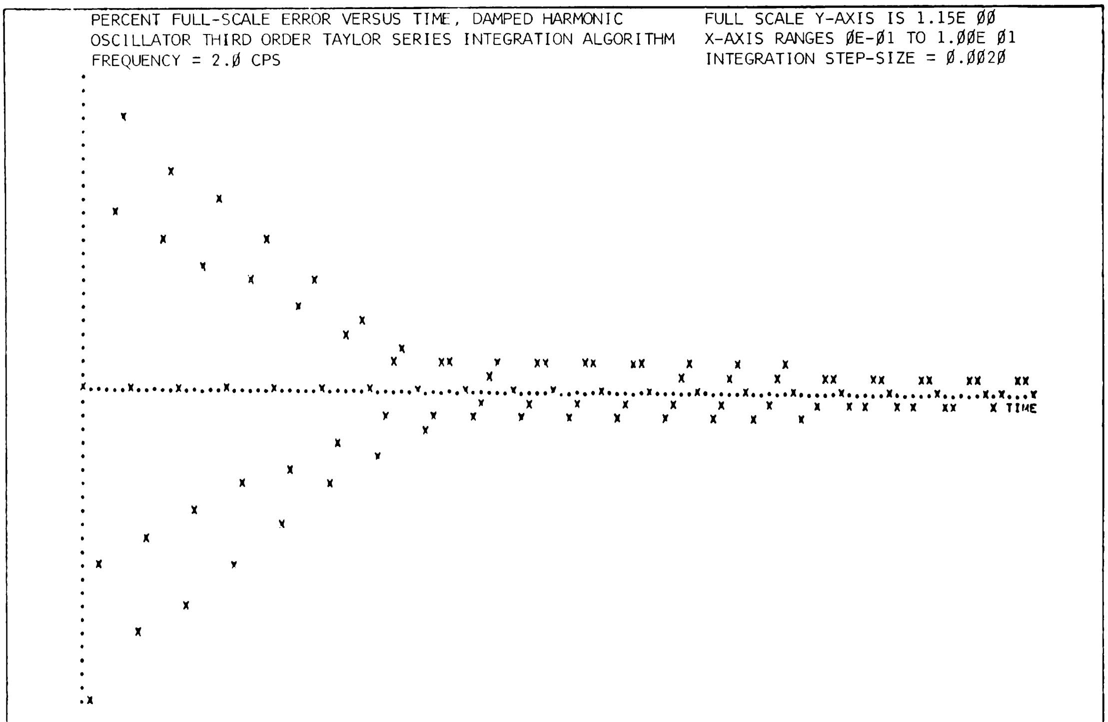

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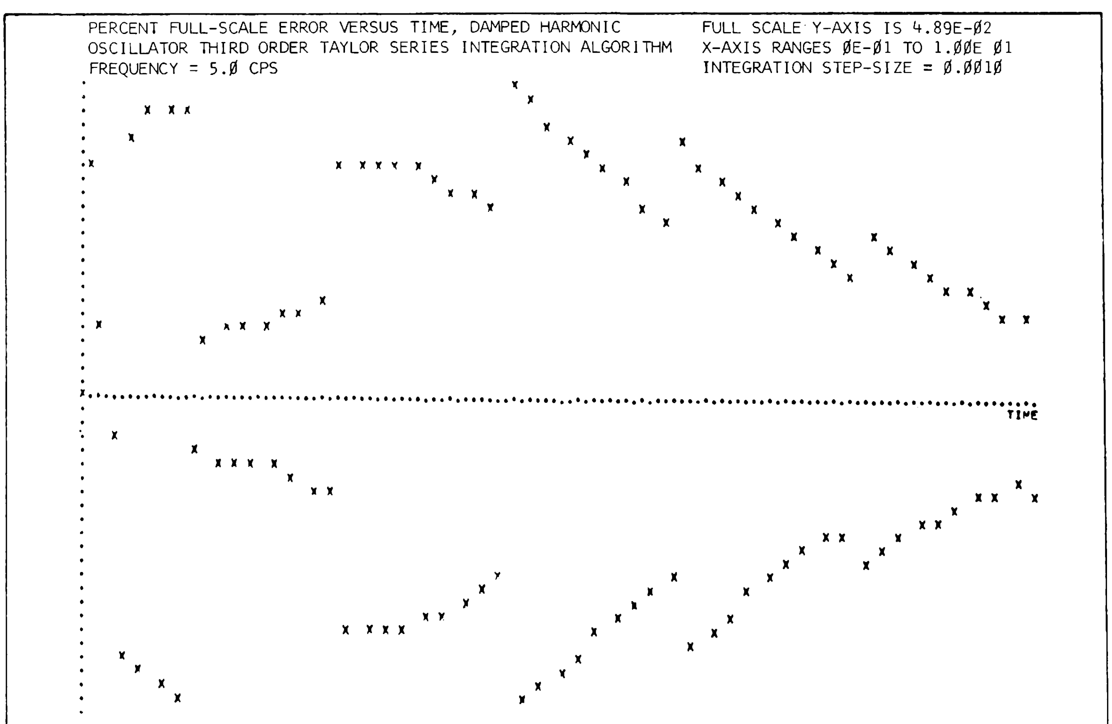

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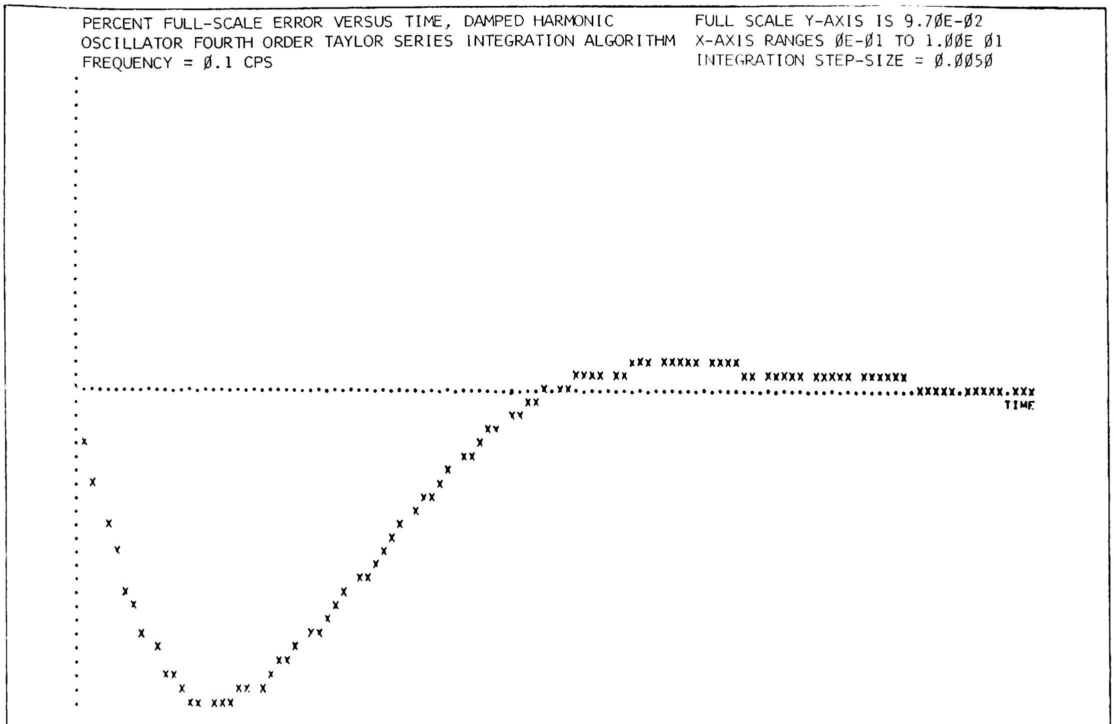

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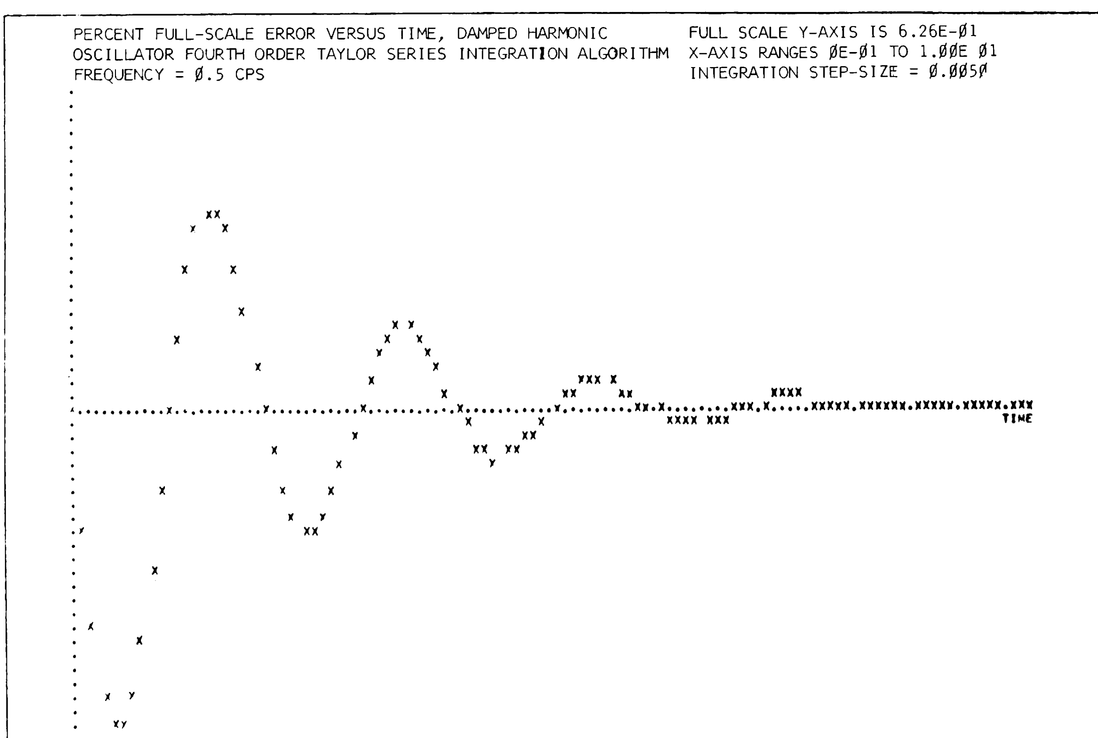

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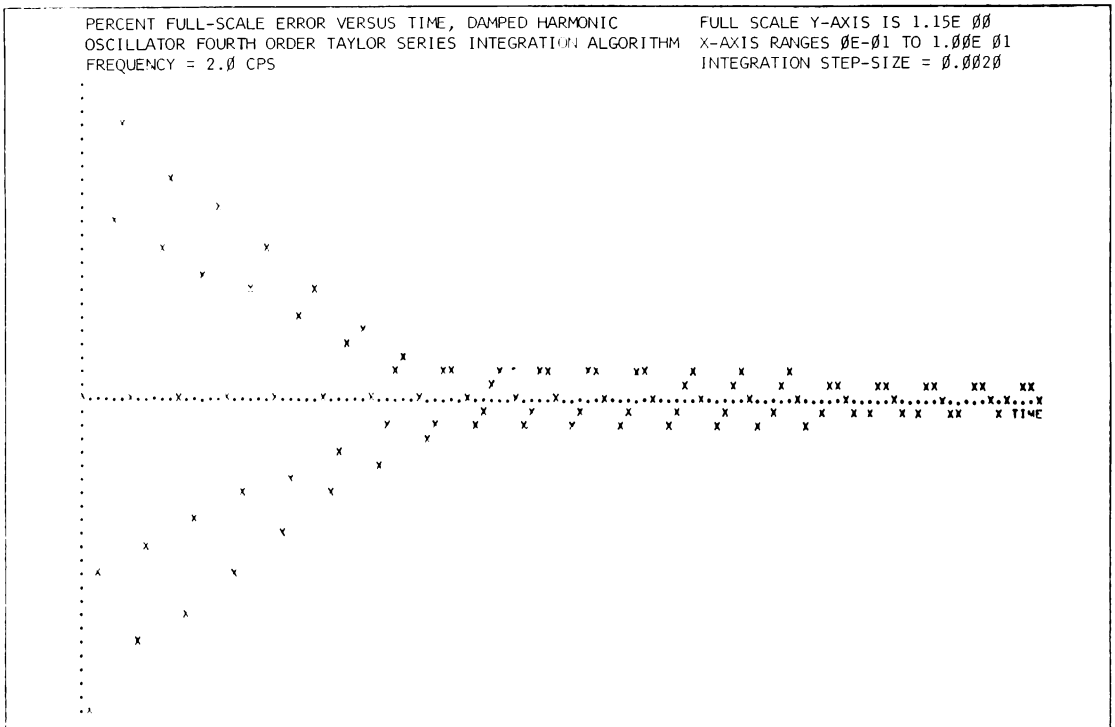

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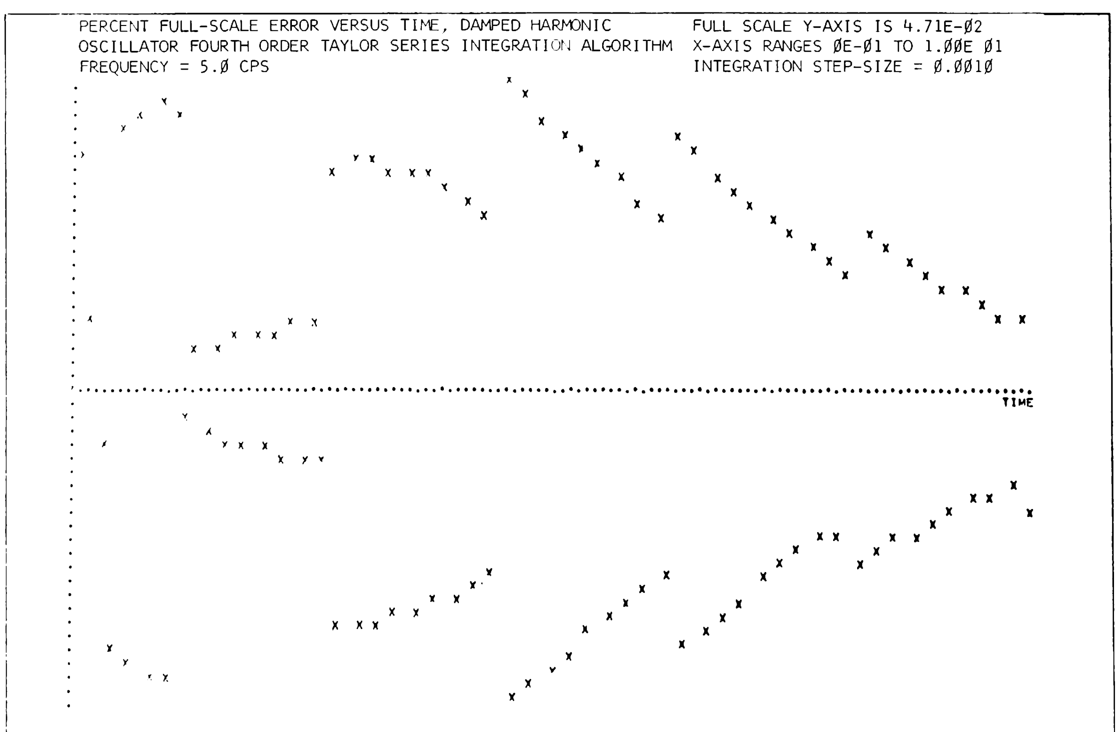

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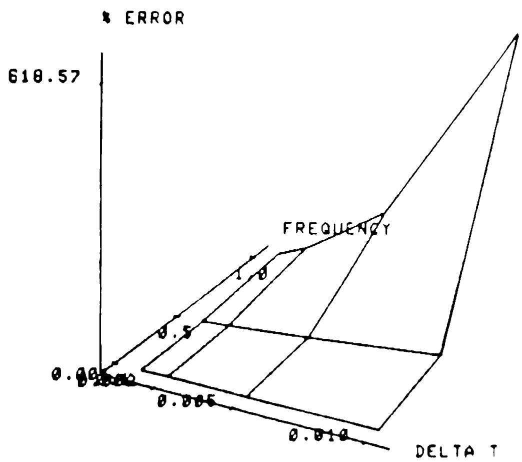

Figure 37. Euler Integration Error, Undamped Solutions.

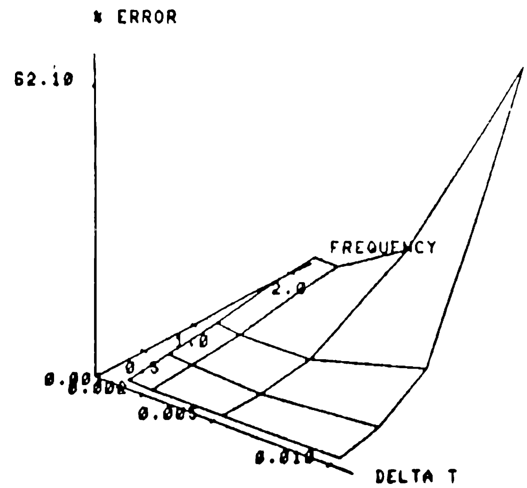

Figure 38. Second Order Taylor Integration Error, Undamped Solutions.

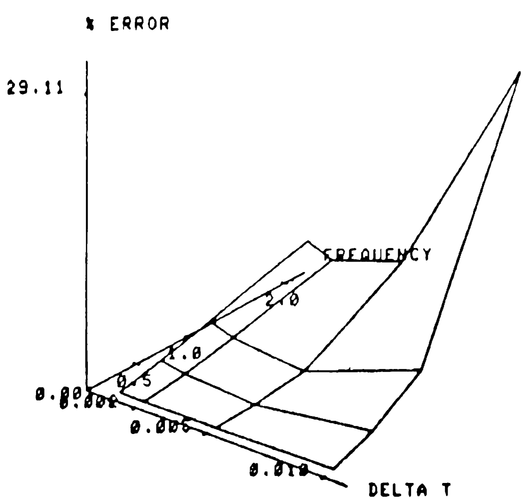

Figure 39. Third Order Taylor Integration Error, Undamped Solutions.

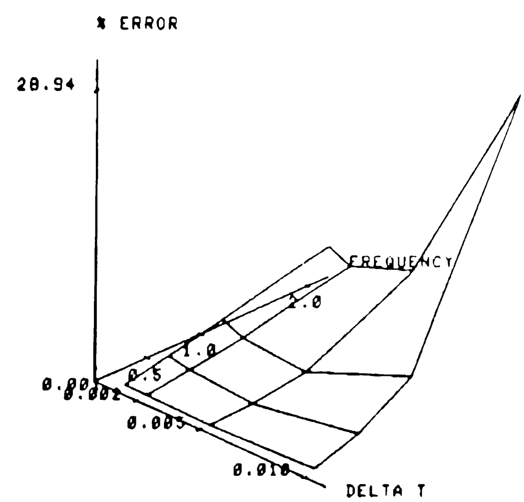

Figure 40. Fourth Order Taylor Integration Error, Undamped Solutions.

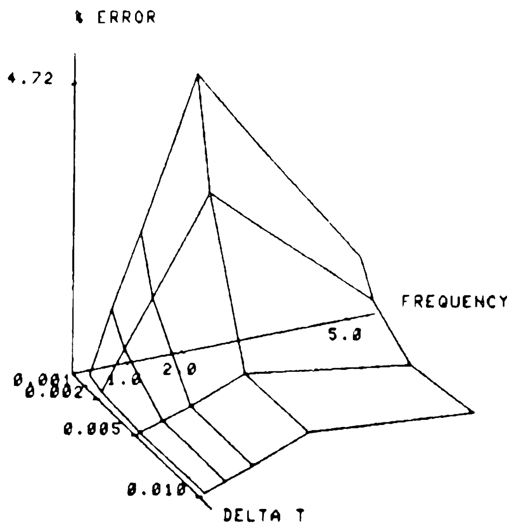

Figure 41. Fourth Order Taylor Integration with Exact Derivatives, Undamped Solutions. (Low Frequency Detail).

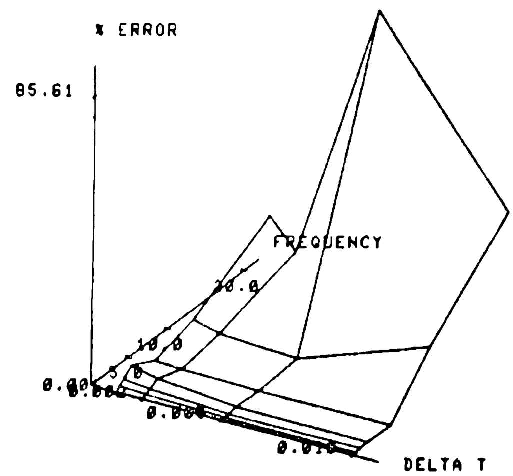

Figure 42. Fourth Order Taylor Integration with Exact Derivatives, Undamped Solutions.

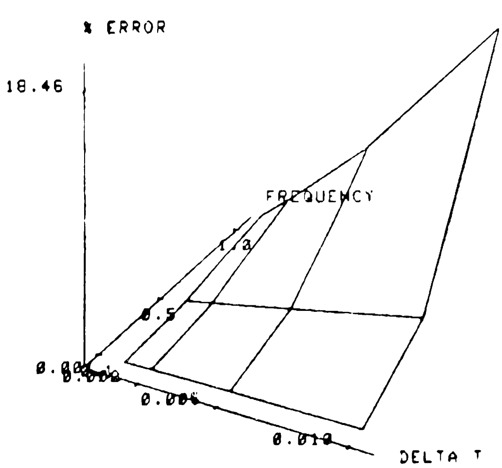

Figure 43. Euler Integration Error, Damped Solutions.

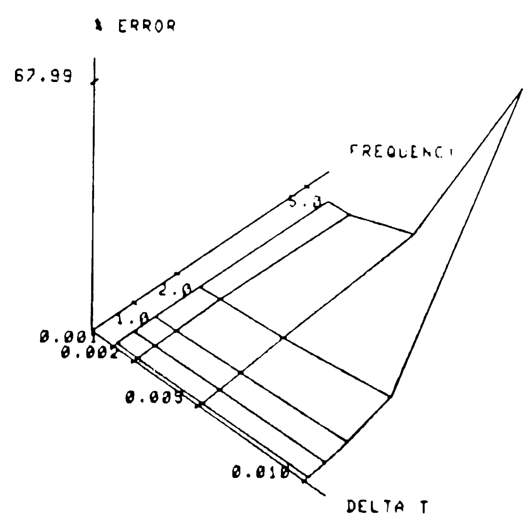

Figure 44. Second Order Taylor Integration Error, Damped Solutions.

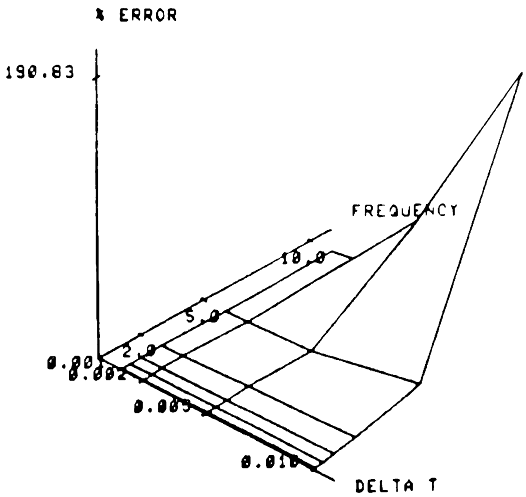

Figure 45. Third Order Taylor Integration Error, Damped Solutions.

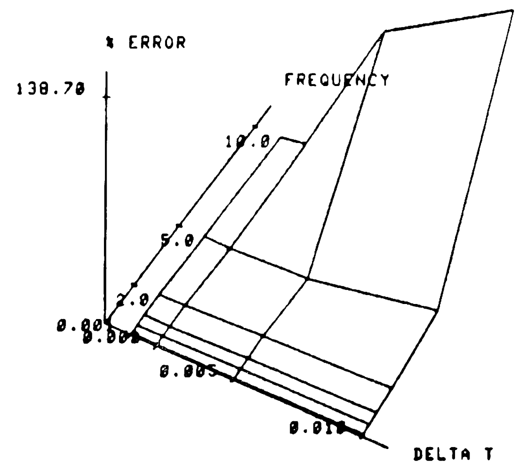

Figure 46. Fourth Order Taylor Integration Error, Damped Solutions.

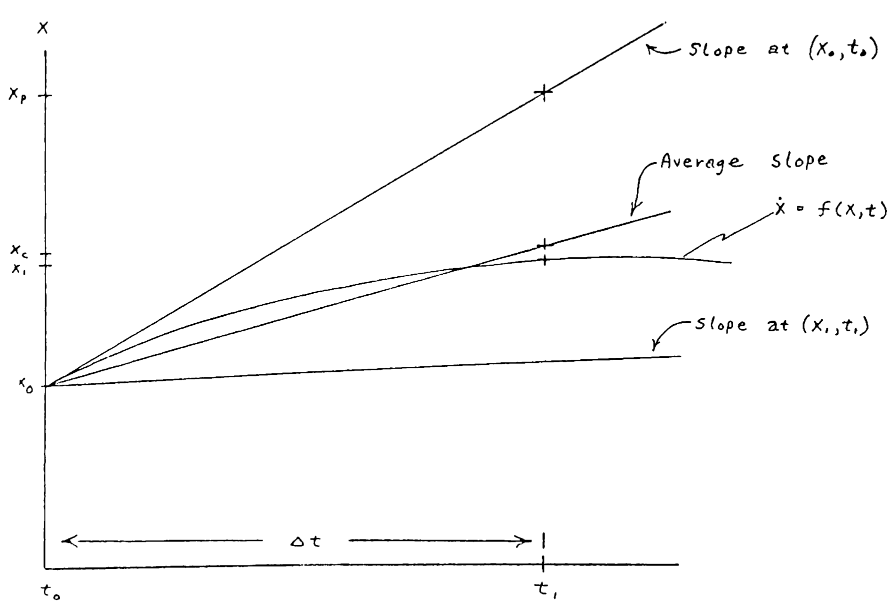

Figure 47. Corrected Euler Method