PROCEDURES FOR GENERATING GAMMA VARIATES WITH NON-INTEGER PARAMETER SETS

Don T. Phillips
School of Industrial Engineering
Purdue University
Lafayette, Indiana

Charles Beightler
Mechanical Engineering Department
University of Texas
Austin, Texas

One of the most useful distributions in stochastic modeling is the two parameter gamma distribution. This paper presents a technique for generating random gamma variates from any two parameter gamma form with either integer or non-integer parameters. Only two techniques are presently available for generating gamma variates with a non-integer shape parameter, those being a variate averaging technique proposed by Naylor, et al. and a rejection scheme proposed by Johnk. The three generation schemes are compared with respect to (1) statistical goodness of fit, (2) computer running time and (3) random number calls. The author's scheme is shown to be statistically comparable with respect to goodness of fit, and generally superior relevant to computer running times and random number calls.

In most engineering and economic studies the experimental data to be dealt with is often non-negative and can be considered as unimodal over an arbitrary range of values. The most common question that I have encountered in working with the layman engineer is, "What sort of distribution can I use to represent this data?" More often than not the answer is to assume normality and proceed. In many applications this is probably justified but in many more, this assumption might not be valid -- and may yield questionable results even if a goodness of fit test accepts normality at marginal decisions. If a set of data is strictly non-negative and unimodal, the gamma distribution can provide an excellent representation for a wide variety of functional shapes. The distribution function is given by

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} & \infty \geq x \geq 0 \\ & \alpha,\beta \text{ constants} \\ 0 & \text{elsewhere} \end{cases} \qquad (1)$$

where: $\mu = \alpha\beta$

$\sigma^2 = \alpha\beta^2$

The above distribution function has been successfully used in many real-world applications, including inventory, queueing, bidding strategies, and income distribution analysis. Special forms include the exponential decay distribution when $\alpha=1$ and a normal form as α becomes increasingly large. In general, the distribution is completely described by the value of α and β, which are the shape and scale parameters respectively. In dealing with real-world data, a primary advantage is the ease by which these two parameters can be estimated using the method of moments. In particular, $\tilde{\alpha} = E[x]^2/\text{var}[x]$ and $\tilde{\beta} = \text{var}[x]/E[x]$, where we define $E[\tilde{\alpha}] = \alpha$, $E[\tilde{\beta}] = \beta$. This paper will be exclusively concerned with the use of the gamma distribution in digital simulation analysis. A case has been presented for the extensive use of the gamma distribution function; it now remains to discuss the merits of using this function.

Since the gamma forms are easily obtained from raw data and can approximate a wide variety of functional shapes, it could play a major role in digital simulation studies. The primary difficulty is that there are no efficient techniques for generating random gamma variates. This difficulty arises because the cumulative distribution function for the gamma distribution cannot be obtained in closed form. The purpose of this paper is to present a new technique for generating random gamma variates which avoids this problem, and to compare this technique to existing gamma generation procedures.

Algorithms

In digital simulation analysis, there are only two techniques presently being used to generate random gamma variates with arbitrary parameters; these being the techniques of composition and rejection. Of the two, the composition technique is the more widely used, primarily due to its popularization by Naylor, et al.[9] An alternate technique using rejection has only recently been presented by Johnk[5, 8] which depends upon mathematical convolutions of relevant random variables. These techniques will subsequently be referred to as simply Naylor's Technique or Johnk's Technique.

The principal thrusts of this paper will be to present (1) a new technique for generating random gamma variates with arbitrary parameters; (2) comparison of this technique to those of Naylor and Johnk.

A New Algorithm

The author has developed a gamma deviate generator capable of producing random deviates from distributions characterized by either integer or non-integer parameters. The primary objective was to develop a gamma generator which would be both statistically sound with respect to distribution representation, and not use excessive computer time in producing a continuing sequence of deviates. Both of the previously mentioned techniques fail to simultaneously satisfy this criteria, as will subsequently be illustrated.

Normally an efficient method for generating a random deviate is through the application of the Inverse Transform[7]. Unfortunately, this technique can only be applied to a limited number of statistical distributions since one must obtain the cumulative distribution in closed form. As previously mentioned, this cannot be obtained for the gamma distribution. A primary difficulty is in the evaluation of the complete gamma function for non-integer valued parameters. The procedure here was inspired by Hastings[7], who employed a numerical approximation procedure in developing a normal density random deviate generator. Since the complete gamma function is a fairly well behaved function, it was believed that a similar technique could be successfully employed to generate generalized gamma variates. The original density function was scaled by choosing constants A and B in such a manner as to permit an inverse transform to be accomplished. (Actually three different representations are used; one for the intermediate ranges and two others for lower and upper regions.) These two factors were empirically determined through the use of a stepwise polynomial regression procedure. For increased accuracy, three different approximating transformations were derived based upon the value of α, the shape parameter. Using this procedure, the following FORTRAN IV machine independent function was written to generate random gamma deviates from any two parameter density with arbitrary parameters α and β. The algorithm functions in two distince phases. The first time the generator is called a scaling routine is implemented to compute appropriate scaling factors. In all succeeding calls, only three elementary operations need be performed to generate a random gamma variate. Hence, the entire routine is increasingly efficient as the number of desired deviates becomes larger. The algorithm is given in Figure 1.

Extensive use of the generator has indicated that the generator will produce random deviates which closely approximate the desired gamma form. Mathematically, the procedure was tested by subjecting the generator to a series of Kolmogorov-Smirnov goodness of fit tests. The results are shown in Figure 2, and it is statistically significant to note that in no case was the hypothesis of goodness of fit rejected, even for samples as small as n=10. The question now to be answered is how well the generator performs with regard to the existing generation techniques. Both Naylor's and Johnk's technique will subsequently be justified, and then a statistical comparison will be made in an effort to establish the best available technique with respect to functional representation and computer running times.

Figure 1
The Algorithm

```
      FUNCTION GAM(ALPHA,BETA,START    )
NON INTEGER GAMMA GENERATOR
      IF(START.GT.1.5) GO TO 50
      X3=1.0
      IF(ALPHA.LE.2.0) GO TO 1
      IF(ALPHA.LE.5.0) GO TO 2
      GO TO 3
    1 B=0.2177+(2.10299768*ALPHA)-(4.34611961*
     1ALPHA**2)+(6.00914764*ALPHA**3)-(3.95097728*
     2ALPHA**4)+(0.97279251*ALPHA**5)
      GO TO 4
    2 B=0.64350+(0.45839602*ALPHA)-(0.02952801*
     1ALPHA**2)+(0.00172718*ALPHA**3)-(0.00005810*
     2ALPHA**4)+(0.00000082*ALPHA**5)
      GO TO 4
    3 B=1.33408+(0.22499991*ALPHA)-(0.00230695*
     1ALPHA**2)+(0.00001623*ALPHA**3)-(0.0000000*
     2ALPHA**4)
    4 X=1.0+(1.0/B)
      START=5.0
   12 IF(X-2.0) 110,50,15
   15 X=X-1.0
      X3=X3*X
      GO TO 12
  110 Y=X-1.0
      GY=1.0+Y*(-0.5771017+Y*(0.985854+Y*(-0.8764218
     1+Y*(0.8328212+Y*(-0.5684729+Y*(0.2548205+Y*
     2(-0.05149930)))))))
      X3=X3*GY
      A=(X3/(ALPHA*BETA))**B
      B=1.0/B
   50 CONTINUE
      RN=RANF(0)
      GAM=(-(1.0/A)*ALOG(RN))**B
      RETURN
      END
```

Legend: ALPHA = α BETA = β

START = any real integer initially less than 1.5

GAM = random gamma deviate

RANF(0) = Random number generator

Figure 2
Kolmogorov-Smirnov Goodness of Fit Tests
for Gamma Generator; $\beta = 2.0$

α	n=10	n=50	n=100
0.50	0.196	0.093	0.063
0.70	0.134	0.134	0.092
1.20	0.235	0.076	0.117
1.70	0.217	0.105	.059
2.30	0.262	0.056	.078
2.70	0.334	0.169	.064
3.60	0.298	0.164	.056
4.00	0.271	0.087	.067
4.60	0.211	0.086	.059
5.40	0.114	0.149	.097
6.10	0.240	0.066	.092
6.80	0.176	0.161	.114
7.00	0.237	0.098	.114
10.00	0.287	0.066	.086
15.00	0.354	0.118	.054
20.00	0.255	0.191	.083
25.00	0.137	0.131	.076

CRITICAL VALUES

	c = .05	c = .01
n=10	.410	.490
n=50	.193	.230
n=100	.136	.163

Gamma Variate Averaging (Naylor's Technique

If the gamma shape parameter α is a non-negative integer, then Equation (1) takes the following form and is commonly known as an Erlang distribution.

$$f(x) = \frac{\beta^{-\alpha}}{(\alpha-1)!} x^{\alpha-1} e^{-x/\beta} \qquad \infty \geq x \geq 0 \qquad (2)$$

It is easily shown that an Erlang variate can be produced by summing α exponential variates, each with expected value α.[9] Hence, a gamma variate with integer shape parameter is readily obtained. For a non-integer shape parameter α, define $\alpha = \alpha_1 + P$; where α_1 is the integer part of α and P is a number between zero and one. Denote α_2 as $\alpha_1 + 1$. Since $\alpha_2 - \alpha = 1-P$ then an appropriate choice for a stochastic variate would be to choose in repeated sampling trials the gamma variate with parameter α_1 [1-P](100) percent of the time and one with parameter α_2[P](100) percent of the time. The statistical logic here is that in many sampling trials the expected value and variance of all variates should approach that of the original distribution. This logic is further supported by the fact that the first three moments of a gamma distribution function are linear in α. Of course, the technique fails for values of $\alpha < 1$.

A Sampling Rejection Technique (Johnk's Method)

A recently developed technique for generating stochastic gamma variates is one proposed by Johnk.[5] Establish the following fundamental theorems.

Theorem I: The sum of two gamma random variables with parameters $G_1(\alpha_1,\beta)$ and $G_2(\alpha_2,\beta)$ is also gamma with parameters $\tilde{G}(\alpha_1+\alpha_2,\beta)$.

Theorem II: If U_1 and U_2 are continuous uniform random variables described by

$$f(U_i) = 1 \quad i=1,2 \quad 1 \geq U_i \geq 0 \qquad (a)$$

and

$$x = U_1^{1/A} \quad y = U_2^{1/B} \qquad (b)$$

Then if

$$x+y \leq 1 \qquad (c)$$

$$z = \frac{x}{x+y} \qquad (d)$$

is beta distributed with distribution function

$$f(z) = \frac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)} z^{A-1}(1-z)^{B-1} \quad 1 \geq z \geq 0$$

Proof: Define $z = \frac{1}{1+y/x}$

Let us proceed to find the distribution of y/x provided that $x+y \leq 1$. Further, define $V = y[1/x]$ so that we are dealing with a product function.

Recall that if we define the Mellin transform of a distribution function, f(x), to be

$$M[f(x)|s] = \int x^{s-1} f(x)dx \qquad (e)$$

Then the Mellin transform of the distribution of P=RQ is defined to be

$$M[f(P)|s] = M[f(R)|s]\ M[f(Q)|s] \qquad (f)$$

provided R and Q are independent.

Like the moment generating function for sums of random variables, the Mellin transform for products of random variables possesses the uniqueness property. Our purpose is to prove that f(z) is B(A,B). In finding f(z) we must first define f(x) and f(y). By (1) and (2), $U_1 = x^A$ and $U_2 = y^B$, hence

$$f(x) = (1)|J_1| \text{ and } f(y) = (1)|J_2|$$

where $|J_1|$ and $|J_2|$ are the Jacobians of the transformation, or

$$f(x) = Ax^{A-1} \quad 1 \geq x \geq 0 \qquad |J_1| = Ax^{A-1}$$

since

$$f(y) = By^{B-1} \quad 1-x \geq y \geq 0 \qquad |J_2| = By^{B-1}$$

Now, notice that we can't use (b) because the range of y depends upon x (or vice versa). Hence we must form the joint Mellin transform of f(y) and f(1/x) since we define $V = y[1/x]$. To do this, we must first form the Mellin transform of $f_1(x)$ and $f_1(y)$, replacing the argument s in the integral involving x by (2-s), since the Mellin transform of f(1/x) is equivalent to the Mellin transform of $f_1(x)$ with the argument s replaced by (2-s). In other words,

$$M[f(1/x)|s] = M[f(x)|2-s].$$

Hence, in the case of independent random variables x and y

$$M[f(V)|s] = M[f(y)|s]\ M[f(x)|2-s].$$

But since the range of y depends upon x

$$M[f(V)|s] = \int_0^1 x^{(2-s)-1} Ax^{A-1} \left[\int_0^{1-x} y^{s-1} By^{B-1} dy\right] dx$$

$$= \int_0^1 Ax^{A-s} \left[B \int_0^{1-x} y^{s+B-2} dy\right] dx$$

$$= \int_0^1 Ax^{A-s} \left[\frac{B(1-x)^{B+s-1}}{B+s-1}\right] dx$$

$$M[f(V)|s] = \frac{AB}{B+s-1} B(A-s+1, B+s)$$

And since $B(A+s+1, B+s) = \frac{\Gamma(A-s+1)\Gamma(B+s)}{\Gamma(A+B+1)}$

then

$$M[f(V)|s] = \frac{\Gamma(s+B-1)\Gamma(A-s+1)}{\Gamma(A)\Gamma(B)}$$

Now if we define

$$B = \alpha + 1 \quad \Rightarrow A+B = \beta$$
$$A = \beta - \alpha - 1$$

Then $$M[f(V)|s] = \frac{\Gamma(\alpha+s)\Gamma(\beta-\alpha-s)}{\Gamma(\beta-\alpha-1)\Gamma(\alpha+1)} . \qquad (g)$$

But (g) is simply the Mellin transform of a beta variable of the second kind; that is, by the uniqueness property

$$f(V) = \frac{\Gamma(\beta)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)} \frac{V^{\alpha}}{(1+V)^{\beta}} \qquad \infty \geq V \geq 0$$

or

$$f(V) = \frac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)} \frac{V^{B-1}}{(1+V)^{A+B}} \qquad \infty \geq V \geq 0.$$

Now, we are not interested in the distribution of V per se but rather in the distribution of

$$z = \frac{x}{x+y} = \frac{1}{1+y/x} = \frac{1}{1+V} .$$

Hence, $V = 1-z/z$ $\quad dV = dz/z^2 = |J|$. Therefore,

$$g(z) = \frac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)} \frac{\left[\frac{1-z}{z}\right]^{B-1}}{[1/z]^{A+B}} |sz+z^2|$$

$$\Rightarrow g(z) = \frac{1}{B(A,B)} (1-z)^{B-1} z^{A-1} \qquad 1 \geq z \geq 0$$

- Q.E.D. -

Theorem III: If x is a random variable gamma distributed with $\alpha = n+k$, $\beta = 1$; and if y is beta distributed with parameters A=n and B=k; then z=xy is gamma distributed with $\alpha = n$, $\beta = 1$.

Proof: The Mellin transform of the gamma distribution is given by

$$M_1[f(x)|s] = \frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)} .$$

Define a beta distribution by

$$f(y) = \frac{1}{B(A,B)} y^{A-1} (1-y)^{B-1} \qquad 1 \geq y \geq 0$$

So that its Mellin transform is given by

$$M_2[f(y)|s] = \frac{\Gamma(A+s-1)\Gamma(A+B)}{\Gamma(A+s+B-1)\Gamma(A)} .$$

Now let $\alpha = n+k$
$A = n$
$B = k$

and defining $M[f(z)|s] = M[f(x)|s]\ M[f(y)|s]$ we obtain

$$M[f(z)|s] = \frac{\Gamma(n+k+s-1)\Gamma(n+s-1)\Gamma(n+k)}{\Gamma(n+k)\Gamma(n)\Gamma(n+s+k-1)}$$

or $M[f(z)|s] = \Gamma(n+s-1)/\Gamma(n)$ which is simply the Mellin transform of a gamma distribution function with $\alpha = n$.

- Q.E.D. -

Efficiency: The entire procedure depends upon the ability to produce a random beta deviate using Theorem II, which is a rejection technique. In general, we can state

Lemma 1: Given a beta distribution function with parameters A and B, if we define $x = u_1^{1/A}$ and $y = u_2^{1/B}$, then the probability that $x+y \leq 1$ is given by

$$Pr\{x+y \leq 1\} = \frac{AB\ \Gamma(A)\Gamma(B)}{\Gamma(A+B)(A+B)} .$$

Proof:

Define $x = u_1^{1/A}$
$y = u_2^{1/B}$
$f(u) = \begin{cases} 1 & 1 \geq u_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Then $g(x) = Ax^{A-1} \qquad 1 \geq x \geq 0$

$g(y) = By^{B-1} \qquad 1 \geq y \geq 0$

Assuming x and y are independent random variables, then

$$f(x,y) = ABx^{A-1}y^{B-1} \qquad \begin{matrix} 1 \geq x \geq 0 \\ 1 \geq y \geq 0 \end{matrix}$$

Now let $z = x+y$ so that

$$f(z,y) = AB(z-y)^{A-1}(y)^{B-1} \qquad \begin{matrix} 2 \geq z \geq 0 \\ 1 \geq y \geq 0 \end{matrix}$$

Hence,

$$f(z) = \int_{y=0}^{z} AB(z-y)^{A-1}(y)^{B-1} dy$$

If we let $u = y/z$ then

$$f(z) = AB \int_{u=0}^{1} (z-y)^{A-1}(uz)^{B-1} z du = \beta(A,B)ABz^{A+B-1}.$$

Hence, $Pr\{z \leq 1\} = \beta(A,B)AB \int_{z=0}^{1} z^{A+B-1} dz$

$$= \beta(A,B) \frac{AB}{A+B} ; \quad \beta = \frac{\Gamma(A)\Gamma(B)}{\Gamma(A+B)}$$

which is the desired result.

- Q.E.D. -

If we define $P = [\beta(A,B)AB/A+B]^{-1}$ then an expression is obtained for the expected number of rejections in obtaining one beta variate. Various combinations of A and B are given below:

B \ A	1	3	5
1	1	4	6
3	4	30	56
5	6	56	252

A general conclusion is that if Johnk's technique is used in beta generations per se, the time required to obtain a statistical deviate could prove prohibitive except for selected combinations of A and B. At this point let us consider the use of this technique to generate gamma variates via Theorem III. Suppose that we desire to generate a gamma distributed random variable with non-integer shape parameter, α. Now define $n = \alpha - [\alpha]$, where $[\alpha]$ is the largest truncated integer, and similarly define $K = 1 - \alpha + [\alpha]$. By Theorem III let A=n and B=k. Under these rules, it is necessary that $\alpha \equiv n+k \equiv 1$. It follows that if a beta variate is generated following the parameters A and B, and multiplied by a gamma variate with parameters $\alpha = 1$ and $\beta = 1$, then a gamma variate with $\alpha = n$ and $\beta = 1$ will be produced. Two facts are now significant and should be noted. (1) A gamma distribution with parameters $\alpha = 1$, $\beta = 1$ is given by

$$f(x) = e^{x} \qquad \infty \geq x \geq 0$$

from which a variate is easily produced using an inverse transform. (2) Since A=n and B=k, A and B will always be less than one. By Lemma 1, the expected number of rejections in the beta generation phase would be

$$R = [AB\Gamma(A)\Gamma(B)]^{-1} = [(1-B)B\Gamma(1-B)\Gamma(B)]^{-1}$$

$$\approx \left[\frac{(1-B)B\Pi}{\sin n\Pi}\right]^{-1},$$

or expected number of rejections is approximated by $1.33 \geq R \geq 1$.

The fundamental ingredients have now been established to construct a random deviate generator for a gamma density with arbitrary non-integer shape parameter α and scale parameter β. The algorithm is as follows:

Define: i) α is a non-integer shape parameter

ii) $\alpha_1 = [\alpha]$ is the truncated integer root of α

iii) u_i is the i^{th} random number $1 \geq u_i \geq 0$.

1. Let $x = -\ell n \prod_{i=1}^{\alpha_1} u_i$

2. Set $A = \alpha - \alpha_1$; $B = 1 - A$

   a) Set j=1.

   b) Generate a random number, u_j, and set $y_1 = (u_j)^{1/A}$.

   c) Generate a random number, u_{j+1}, and set $y_2 = (u_{j+1})^{1/B}$.

   d) If $y_1 + y_2 \leq 1$, go to (f).

   e) Set j=j+2; go to (b).

   f) Let $z = y_1/(y_1 + y_2)$ so that z is a beta variable with parameters A and B by Theorem II.

3. Generate a random number, u_N, and let $Q = -\ell n (u_N)$.

4. The desired deviate is $D = (x+zQ)\beta$ by Theorem I and Theorem III.

Algorithm Comparisons

An attempt was made to compare the three algorithms previously presented with respect to (1) random number calls; (2) Kolmogorov-Smirnov goodness of fit values for n=10,50, and 100 variates; (3) generation run times for n=100,500, and 1000 variates.

Random Number Calls

The quantity of random numbers required for each gamma random deviate is easily evaluated.

a) Phillips Technique: Since the generation technique proposed by the author is based on an approximated inverse transformation, only one random number is required per gamma deviate.

b) Naylor's Technique: As previously described Naylor's technique is based upon random deviate averaging, and therefore requires on the average $(\alpha+1)/2$ random numbers per random gamma deviate.

c) Johnk's Technique: It has already been established that a variate from a gamma form with parameter α will require approximately between 1.0 and 1.33 rejections per beta variate, plus $\alpha_1 = [\alpha]+1$ random exponential variates. Hence, one would require between $(3+\alpha_1)$ and $(3.67+\alpha_1)$ random numbers, or approximately $(\alpha_1+3.34)$ random numbers per gamma deviate.

The conclusion is that in terms of random numbers required, Phillips method would be the most efficient followed by Naylor and Johnk's techniques respectively.

Goodness of Fit Analysis

A computer program was written to facilitate a Kolmogorov-Smirnov (K & S) comparison for the three algorithms. Levels of confidence of c = 0.01 and 0.05 were chosen, along with sample sizes of n=10, 50, and 100.

	c = 0.05	c = 0.01
n=10	0.410	0.490
n=50	0.193	0.230
n=100	0.136	0.163

Critical Values for The K & S Test

Goodness of fit values for $24.80 \geq \alpha \geq 0.50$, n=50, and $\beta = 2.0$ are given in Table 1. (Since β is the gamma scale parameter, it was fixed at 2.0 for comparative purposes.) Table II gives a summary of values for n=10, n=50, and n=100. Four indicators of relative performance are given for each sample size: (a) The "best" test result with respect to the lowest K&S value; (b) The "worst" test result with respect to the highest K&S value; (c) The number of goodness of fit rejections for c=0.05 and c=0.01. Each tabulated value is the arithmetic average of two different K&S tests starting with different random number seeds. The conclusion was that based on these results, the overall performance of each test was evenly balanced, with Johnk's test showing a tendency to produce a wider range of K&S values.

All tests produced statistically sound results from a goodness of fit standpoint for values of $\alpha \geq 1.20$.

For lower range values, the only technique which consistently passed the K&S tests was that of Phillips. In order to further investigate the lower range of α parameters, Table III was constructed for ten values of $1.00 \geq \alpha \geq 0.10$, starting with $\alpha = 0.10$ and ending with $\alpha = 1.00$ in increments of 0.10. A summary of these results is given by Table III. It is believed that based on these sampling experiments, Phillips' technique is consistently better than the Johnk procedure.

Table I. K&S Goodness of Fit Values for n=50, and $24.80 \geq \alpha \geq 0.50$

α, β=2.0	K&S Values, n=50		
	Phillips	Johnk	Naylor
0.50	0.093	0.540*	**
0.70	0.134	0.488*	**
1.20	0.076	0.056	0.176
1.70	0.105	0.089	0.048
2.30	0.056	0.114	0.099
2.70	0.169	0.111	0.089
3.60	0.165	0.175	0.092
4.00	0.087	0.105	0.081
4.60	0.086	0.085	0.107
5.40	0.149	0.067	0.104
6.10	0.112	0.083	0.126
6.80	0.161	0.084	0.151
7.25	0.105	0.071	0.122
9.75	0.138	0.110	0.054
14.62	0.060	0.126	0.066
19.91	0.076	0.094	0.084
24.80	0.070	0.073	0.071

α

* K&S value exceeded critical value(s)
** Technique is not applicable

Table II. K&S Summary for Table I, $\alpha > 1.00$

				(K&S)OBS > (K&S)CRIT	
		Best	Worse	0.05	0.01
n=10	P*	6	5	0	0
	J*	6	6	0	0
	N*	3	4	0	0
n=50	P	5	4	0	0
	J	6	6	0	0
	N	4	5	0	0
n=100	P	5	3	0	0
	J	5	4	1	1
	N	5	8	0	0

* P = Phillips; J = Johnk; N = Naylor

Table III. K&S Summary

				(K&S)OBS > (K&S)CRIT	
		Best	Worse	0.05	0.01
n=10	P*	10	0	0	0
	J*	0	10	7	7
n=50	P	10	0	2	1
	J	0	10	9	9
n=100	P	10	0	3	3
	J	0	10	10	9

* P = Phillips; J = Johnk; N = Naylor

In order to complement the overall goodness of fit results, additional statistics were compiled relating the sampled mean and variance to the theoretical mean and variance for n=10, 50, and 100. The results were that all three techniques perform quite well with respect to sampled means and variances for $\alpha > 1.00$, even for samples as low as n=10. Two performance measures were calculated from these results, those being the total variation of sampled expected values from theoretical expected values and the average deviation for each sample size. The results are given below:

	Total Deviation		
	Phillips	Johnk	Naylor
n=10	21.20	17.96	18.93
n=50	6.66	7.20	9.45
n=100	5.15	7.32	6.10
	Mean Deviation		
	Phillips	Johnk	Naylor
n=10	1.44	1.19	1.18
n=50	0.45	0.48	0.63
n=100	0.35	0.49	0.41

As previously indicated by the K & S tests, each technique performs quite well with respect to expected value representation.

Based on these experimental results, general rules as to which test to choose with respect to goodness of fit would be as follows:

1) Choose the Phillips technique for values of $1.0 \geq \alpha \geq 0.10$.

2) Choose any one of the three tests for values of $\alpha > 1.0$.

3) No technique is recommended for values of $\alpha < 0.10$ If α is found to be in this range data scaling is recommended.

Computer Running Time

Certainly one of the primary considerations in choosing a random deviate generator would not only be its statistical reliability, but also the speed at which random deviates can be produced. In an effort to determine which of the three techniques was more efficient, random deviates were generated for n=100, 500, and 1000. For each value of α and n the actual DCD 6500 computation times were independently recorded. Since Phillips' technique is an inverse transform approximation, the generation time for a fixed sample size is independent of α, hence, a constant generation time for n=100, 500, and 1000 was obtained. Conversely, the Naylor technique exhibited increasing generation times for increasing values of α since at least $[\alpha]$ random number calls are always required for each random deviate. Johnk's technique required generation times which are directly dependent upon the two parameters α and β. In general, the rejection technique will be less efficient as α and β increase, hence, computation times will increase proportionately. Over the range of values studied, Johnk's technique was considerably better than Naylor's technique in the midranges, but becomes less efficient for values of $\alpha \geq 7.50$. Phillips' technique was always faster than Naylor's technique and for values of $\alpha \geq 2.70$ became more efficient than Johnk's technique. Again, Naylor's technique is not applicable for $\alpha < 1.0$.

Summary and Conclusions

There are presently three techniques for generating random gamma variates from a gamma distribution with arbitrary parameters α and β; a variate averaging technique developed by Naylor, et al., a rejection scheme first proposed by Johnk, and an inverse transform approximation presented in this paper. With regard to statistical goodness of fit, it appears based on limited experiments that all three methods are capable of generating random gamma deviates closely approximating the desired gamma density for values of $\alpha > 1.0$. For lower values of the scale parameter α, the method presented in this paper appears to be superior. With regard to compute generation times, Johnk's method is recommended for values of $\alpha < 2.5$ and Phillips' method appears to be superior for $\alpha > 2.50$. Naylor's method seems to function quite well for values of $\alpha > 1.0$ and is perhaps the easiest method to program, but will consistently require more computer running time.

The primary contribution of this paper was to present a new technique for generating random gamma deviates using arbitrary parameters α and β. An attempt was also made to compare this technique to the two known methods presently being used. Hopefully, this report will stimulate the use of this technique and encourage further investigations of the statistical properties of each gamma generator.

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