STOPPING RULES FOR QUEUEING SIMULATIONS: NON INDEPENDENT TOURS

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Summary

Expressions have been given for the approximate calculation of the mean and variance of the estimate of interest for use in queueing simulations. The results contained herein are an extension of a case previously solved where the estimates of interest were independent because of independent tours.

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Introduction:

An ever present question in all simulations is "Have enough trials been processed by the simulator?" A usual stopping rule is: stop the simulation when the (theoretical) variance of the statistic that estimates the quantity of interest is within limits. However, there is an operational difficulty in the administration of this measure, i.e., because of the correlation of successive trials one cannot easily calculate the required variance. This is discussed in some detail by Hauser, et.al. [1].

In a previously published paper [2] a stopping rule for queueing simulations was presented. Therein, expressions for the calculation of the mean and variance of the estimate of interest were given. The expressions were based upon independent estimates of the quantities of interest. In this paper similar expressions are furnished for the case of non-independent estimates of the quantities of interest.

The state of the system is defined as the number of attempts in the system either being served or waiting to be served. When an event (an arrival or a departure) causes the system to be in state j, we begin a tour. A departure from state j and a subsequent return to j, of the same type (arrival or departure) that began the tour completes the tour. Referring to Figure 1, all tours illustrated for state C are [1,3), [3,7), [7,10), [2,4) and [2,4) and [4,8). Note that [1,2) is not a tour.

Bemoulli Type Estimates and Queueing Simulations

One is often interested in the probability of blocking (not being served immediately) that attempts experience. In such cases one simply divides the blocked attempts by the total number of attempts and obtains an estimate of the probability directly. The question of how to handle the delay distribution is now discussed. It is suggested that the estimate of the cumulative distribution function of the delays be obtained in a Bemoulli manner at several points. For example, say we are interested in four points of the delay distribution: we want to know what proportion of the delayed requestes are served within 0.1, 0.5, 1.0 and 2.0 average service times. From each tour we obtain four estimates, q_j, $j=1(1)4$*, which chacterize the conditional delay distribution. Each of these estimates are to be treated separately to obtain four variances. When all of the variances are within limits we may stop the simulation. Although it is recognized that the q_j are not independent and hence the respective variances are also not independent this is asserted to be a reasonable operational stopping rule.

* That is j=1,2,3,4

Calculation of the Variance

We will now indicate specifically how we may calculate the variance of the estimate that interests us. Let

N = the random number of arrivals during each tour

N_i = the random number of arrivals during tour i

n_i = the sample number of arrivals during tour i

We will designate as N', N_i' and n_i' the arrivals interest for the aforementioned tours; they may represent the number of attempts blocked, or perhaps the number of attempts served within two minutes. The statistic of interest for tour i is

$$p_i = (N_i')/N_i \qquad i = 1(1)M \tag{1}$$

where M equals the fixed number of tours under consideration. It is suggested that the weighted average over all tours be used for the grand estimate p, namely

$$p = \left(\sum_{i=1}^{M} N_i'\right)/\left(\sum_{i=1}^{M} N_i\right) \tag{2}$$

It was demonstrated [2] that for two simple queueing systems with independent tours the grand estimate p yields asymptocally unbiased result for the probability of blocking.

We are interested in the variance of p, viz.

$$V(p) = V\left[\left(\sum_{i=1}^{M} N_i'\right)/\left(\sum_{i=1}^{M} N_i\right)\right] \tag{3}$$

An approximate expression for the variance of a ratio can be obtained by expanding the ratio in a Taylor series about the mean of the numerator and then taking expected values. The result, given in [3], is

$$V(A/B) \approx E^2(A)V(B)+V(A)/E^2(B) -2E(A)\mathrm{Cov}(A,B)/E^{3}(B) \tag{4}$$

For our case we have

$$A = \sum_{i=1}^{M} N_i' \tag{5}$$

$$B = \sum_{i=1}^{M} N_i \tag{6}$$

The moments of (5) and (6) are, in general, not known; sample estimates will be used in the dexter (right hand side) of (4).

Let us assume stationarity in first and second moments for each tour*. We therefore have

$$E(A) = ME(N') = \sum_{i=1}^{M} N_i' \tag{7}$$

and when using sample values

$$E(A) \approx \sum_{i=1}^{M} n_i' \tag{8}$$

and similarly for the denominator

$$E(B) \approx \sum_{i=1}^{M} n_i \tag{9}$$

*Actually it may not be unreasonable to assume that the statistics from each tour are identically but not independently distributed.

Now for the variance of the numerator we have

$$V(A) = \sum_{j=1}^{M} V(N_j) + \sum_{i<j}^{M} \mathrm{Cov}(N_i', N_j) \qquad (10)$$

because of stationarity

$$\sum_{j=1}^{M} V(N_j) = MV(N') \qquad (11)$$

and when using sample values

$$\sum_{j=1}^{M} V(N_j') \approx \frac{M}{M-1}\left[\sum_{j=1}^{M}(n_j')^2 - \frac{(\sum_{j=1}^{M} n_j')^2}{M}\right] \qquad (12)$$

For the covariance term we have

$$2\sum_{i<j}\mathrm{Cov}(N_i', N_j') = 2\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\mathrm{Cov}(N_i', N_j') \qquad (13)$$

because of the stationarity of N_i' and N_j' only the lag, i.e. j-i, need be considered and hence

$$2\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\mathrm{Cov}(N_i', N_j') = 2\sum_{i=1}^{M-1}(M-i)\mathrm{Cov}(N_t', N_{t+i}') \qquad (14)$$

By definition

$$\mathrm{Cov}(N_t', N_{t+i}') = E(N_t', N_{t+i}') - E(N_{t+i}')E(N_t') \qquad (15)$$

and when using sample values

$$\mathrm{Cov}(N_t', N_{t+i}') \approx \frac{1}{M-i}\sum_{t=1}^{M-i} n_t' n_{t+i}' - \frac{1}{M}\sum_{t=1}^{M} n_t' \frac{1}{M-i}\sum_{t=1}^{M-i} n_{t+i}' \qquad (16)$$

The variance of the numerator, V(A), can then be approximated by

$$V(A) \approx \frac{M}{M-1}\left[\sum_{j=1}^{M}(n_j')^2 - \frac{(\sum_{j=1}^{M} n_j')^2}{M}\right] + 2\sum_{i=1}^{M-1}\left[\sum_{t=1}^{M-i} n_t' n_{t+i}' - \frac{1}{M}\sum_{t=1}^{M} n_t' \sum_{t=1}^{M-i} n_{t+i}'\right] \qquad (17)$$

Similarly with V(B)

$$V(B) \approx \frac{M}{M-1}\left[\sum_{j=1}^{M} n_j^2 - \frac{(\sum_{j=1}^{M} n_j)^2}{M}\right] + 2\sum_{i=1}^{M-i} n_t n_{t+i} - \frac{1}{M}\sum_{t=1}^{M} n_t \sum_{t=1}^{M-i} n_{t+i}\Big] \qquad (18)$$

The covariance of the numerator and the denominator Cov(A,B) is written as

$$\mathrm{Cov}(A,B) = E(AB) - E(A)E(B) \qquad (19)$$

where expressions for E(A) and E(B) have been given previously in (8) and (9) respectively.

We now have

$$E(AB) = E\left(\sum_{i=1}^{M} N_i' \sum_{j=1}^{M} N_j\right) = E\left[\sum_{i=1}^{M} N_i' N_i + 2\sum_{i<j} N_i' N_j\right] \qquad (20)$$

because of stationarity, we have, in some detail,

$$E(AB) = E\left[\sum_{i=1}^{M} N_i' N_i + 2\sum_{i=1}^{M-1}(M-i)\, N_k' N_{k+i}\right]$$

$$E(AB) = ME(N_i' N_i) + 2\sum_{i=1}^{M-1}(M-i)E(N_k' N_{k+i})$$

and using sample values

$$E(AB) \approx M\,\frac{1}{M}\sum_{i=1}^{M} n_i' n_i + 2\sum_{i=1}^{M-1}(M-i)\frac{1}{M-i}\sum_{k=1}^{M-i} n_k' n_{k+i}$$

and finally

$$E(AB) \approx \sum_{i=1}^{M} n_i' n_i + 2\sum_{i=1}^{M-1}\sum_{k=1}^{M-i} n_k' n_{k+i}$$

References

[1] N. Hauser, N. N. Barish and S. Ehrenfeld, "Design Problems in a Process Control Simulation", Journal of Industrial Engineering, Vol. 17 (1966)

[2] I. W. Kabak, "Stopping Rules for Queueing Simulations", Operations Research, Vol. 16 N2, March-April 1968, pp.431

[3] M. G. Kendall and A. Stuart, The Advanced Theory of Statistics, Vol. I, Griffin, London, 1958, p. 232

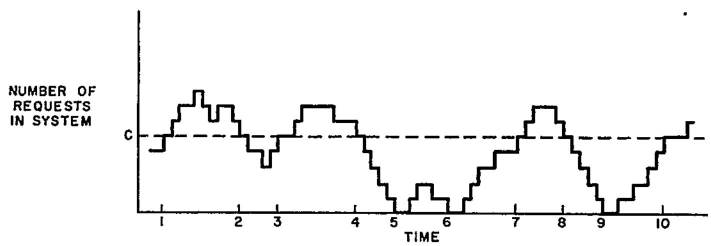

Fig. 1