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CONSTRUCTING CONFIDENCE INTERVALS FOR VALUE-AT-RISK VIA NESTED SIMULATION

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ABSTRACT

Nested simulation is a powerful tool for estimating widely-used risk measures, such as Value-at-Risk (VaR). While point estimation of VaR has been extensively studied in the literature, the topic of interval estimation remains comparatively underexplored. In this paper, we present a novel nested simulation procedure for constructing confidence intervals (CIs) for VaR with statistical guarantees. The proposed procedure begins by generating a set of outer scenarios, followed by a screening process that retains only a small subset of scenarios likely to result in significant portfolio losses. For each of these retained scenarios, inner samples are drawn, and the minimum and maximum means from these scenarios are used to construct the CI. Theoretical analysis confirms the asymptotic coverage probability of the resulting CI, ensuring its reliability. Numerical experiments validate the method, demonstrating its high effectiveness in practice.

1 A PROCEDURE FOR VAR CONFIDENCE INTERVALS

VaR interval estimation remains relatively underexplored. A notable work in this area is Zhang et al. (2022), who established a central limit theorem for VaR point estimators and designed a budget allocation rule for constructing CIs. In this paper, we propose a new procedure to construct CIs for VaR. Let the risk scenario $\mathbf{Z} = (Z_1, \dots, Z_d)^\top \in \mathbb{R}^d$ be generated independently from the distribution of $\mathbf{Z}$. Further, the portfolio loss $X \in \mathbb{R}$ is generated independently from conditional distribution of $X \mid \mathbf{Z}$. Define the conditional expectation as $V(\mathbf{Z}) = \mathbb{E}[X \mid \mathbf{Z}]$, which represents the portfolio loss function given a risk scenario $\mathbf{Z}$ with the expectation taken under the risk-neutral pricing measure. Our objective is to construct a CI for VaR at $1-p$ level of $V(\mathbf{Z})$ with $p \in (0,1)$, defined as $\mathrm{VaR}_{1-p}(V(\mathbf{Z})) = \inf\{x \in \mathbb{R} : \mathbb{P}(V(\mathbf{Z}) \le x) \ge 1-p\}$. Our procedure takes as input the total simulation budget Γ, the VaR confidence level $1-p$, and the CI significance level $1-\alpha$, where $\alpha \in (0,1)$, and returns the confidence interval estimation. Let $\alpha = \alpha_{\text{out}} + \alpha_{\text{screen}} + \alpha_{\text{est}}$, and the proposed procedure is outlined as follows.

1. **Outer Scenario Generation.** Generate n scenarios $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n$.
2. **First-Stage Inner Sampling.** For each scenario $\mathbf{Z}_i$, $i = 1, \cdots, n$, simulate m' loss samples $X_{i,1}, X_{i,2}, \dots, X_{i,m'}$ using Common Random Number techniques, and calculate the sample mean $\overline{X}_{i,m'} = \sum_{j=1}^{m'} X_{i,j}/m'$ and the sample variance $S^2_{i,m'} = \sum_{j=1}^{m'} \left(X_{i,j} - \overline{X}_{i,m'}\right)^2/(m'-1)$.
3. **Scenario Screening.**
(1) For each scenario pair $(\mathbf{Z}_i, \mathbf{Z}_j)$, $i, j = 1, \cdots, n$, the pair difference is defined as $D_{i,j} = V(\mathbf{Z}_i) - V(\mathbf{Z}_j)$. Samples of the paired differences are computed as $D_{i,j,l} = X_{i,l} - X_{j,l}$, $l = 1, \cdots, m'$. Calculate the sample mean $\overline{D}_{i,j} = \sum_{l=1}^{m'} D_{i,j,l}/m'$ and the sample variance $S^2_{D_{i,j}} = \sum_{l=1}^{m'} \left(D_{i,j,l} - \overline{D}_{i,j}\right)^2/(m'-1)$.
(2) Let $k_{\min}$ and $k_{\max}$ be the minimum and maximum elements of the set $\Big\{k : n^n\left((1-p)/k\right)^k (p/(n-k))^{n-k} \ge \exp\left(-\chi^2_{(1),1-\alpha_{\text{out}}}/2\right)\Big\}$. For $i, j = 1, \cdots, n$, define the test statistic as $T_{i,j} = \sqrt{m'}\,\overline{D}_{i,j}/S_{D_{i,j}}$, $B_i = \sum_{j=1, j\neq i}^{n} \mathbf{1}\{T_{i,j} > d_1\}$ and $B_i' = \sum_{j=1, j\neq i}^{n} \mathbf{1}\{T_{i,j} < -d_2\}$, where $\mathbf{1}\{\cdot\}$ is an indicator function. Define $l_1 = (k_{\max}+1)(n-k_{\max}-1)$, $l_2 = (k_{\min}-1)(n-k_{\min}+1)$, $\alpha_{\text{screen1}} = \alpha_{\text{screen}}\, l_1/(l_1+l_2)$, and $\alpha_{\text{screen2}} = \alpha_{\text{screen}}\, l_2/(l_1+l_2)$, then $d_1 = t_{m'-1,1-\alpha_{\text{screen1}}/l_1}$ and $d_2 = t_{m'-1,1-\alpha_{\text{screen2}}/l_2}$, where $t_{m'-1,1-\beta}$ is the $1-\beta$ quantile of the t-distribution with $m'-1$ degrees of freedom.

(3) For $i = 1, \cdots, n$, if $B_i < k_{\max} + 1$ and $B_i' < n - k_{\min} + 1$, then include index i in the set of surviving scenario indices, denoted by $\mathcal{I}$. We denote $c = \mathrm{card}(\mathcal{I})$, where $\mathrm{card}(\cdot)$ is the cardinality of a set.

4. **Restarting and Second-Stage Inner Sampling.** Define $\mathbf{m} = (m_i)^\top \in \mathbb{R}^c$, $i \in \mathcal{I}$, where m_i denotes the number of second-stage inner samples for scenario i as $m_i = \left\lfloor (\Gamma - nm') S_{i,m'}^2 / \sum_{j \in \mathcal{I}} S_{j,m'}^2 \right\rfloor$. For each surviving scenario $\mathbf{Z}_i$, $i \in \mathcal{I}$, simulate m_i new loss samples $X_{i,1}, \ldots, X_{i,m_i}$, and calculate the sample mean $\overline{X}_{i,m_i}$ and the sample variance $S_{i,m_i}^2 = \sum_{j=1}^{m_i} \left(X_{i,j} - \overline{X}_{i,m_i}\right)^2 / (m_i - 1)$.
5. **Constructing VaR Confidence Interval.** We introduce a permutation π_1 mapping $\{1, \cdots, c\}$ to $\mathcal{I}$, which orders indices according to ascending values of their sample means. Specifically, the permutation satisfies $\overline{X}_{\pi_1(1), m_{\pi_1(1)}} \le \cdots \le \overline{X}_{\pi_1(c), m_{\pi_1(c)}}$. Calculate

$$\hat{L}_{n,\mathbf{m}} = \overline{X}_{\pi_1(1), m_{\pi_1(1)}} - z_{1-\alpha_{\mathrm{est}}/2} S_{\pi_1(1), m_{\pi_1(1)}} / \sqrt{m_{\pi_1(1)}}, \; \hat{U}_{n,\mathbf{m}} = \overline{X}_{\pi_1(c), m_{\pi_1(c)}} + z_{1-\alpha_{\mathrm{est}}/2} S_{\pi_1(c), m_{\pi_1(c)}} / \sqrt{m_{\pi_1(c)}},$$

where $z_{1-\alpha_{\mathrm{est}}/2}$ is the $1 - \alpha_{\mathrm{est}}/2$ quantile of the standard normal distribution, and return $\left[\hat{L}_{n,\mathbf{m}}, \hat{U}_{n,\mathbf{m}}\right]$.

2 THEORETICAL GUARANTEES AND NUMERICAL EXPERIMENTS

We denote $V(\mathbf{Z})$ as V, $V(\mathbf{Z}_i)$ as V_i, $\mathrm{VaR}_{1-p}(V(\mathbf{Z}))$ as VaR_{1-p}. Let F_V be the distribution of V. The following theorem shows the proposed CI procedure achieves the asymptotic confidence level.

Theorem 1 Assume F_V is continuous and strictly increasing at VaR_{1-p}, and for any scenario $\mathbf{Z}_i, i = 1, \cdots, n$, loss samples $X_{i,j}$, $j = 1, \cdots, m'$, are normally distributed, suppose V is continuous,

$$\mathbb{P}\left(\mathrm{VaR}_{1-p} \in \left[\hat{L}_{n,\mathbf{m}}, \hat{U}_{n,\mathbf{m}}\right]\right) \ge 1 - \alpha, \text{ as } n \to \infty, \mathbf{m} \to \infty.$$

Proof. Introduce permutation π_2 which maps indices $\{1, \cdots, n\}$ to a permuted order where scenarios are sorted by their true means in ascending order. Assume F_V is continuous and strictly increasing at VaR_{1-p}, by empirical theorem for quantile, as $n \to \infty$, $\mathbb{P}\left(\mathrm{VaR}_{1-p} \in [V_{\pi_2(k_{\min})}, V_{\pi_2(k_{\max}+1)}]\right) \to 1 - \alpha_{\mathrm{out}}$. Let $\gamma = \{\pi_2(k_{\min}), \cdots, \pi_2(k_{\max}+1)\}$. Since $X_{i,j}$, $j = 1, \cdots, m'$, are normally distributed for each $\mathbf{Z}_i, i = 1, \cdots, n$, then, by ranking and selection theories, $\mathbb{P}(\gamma \subseteq \mathcal{I}) \ge 1 - \alpha_{\mathrm{screen}}$. Suppose V is a continuous random variable, $\mathbb{P}\left(V_i \in \left[\hat{L}_{n,\mathbf{m}}, \hat{U}_{n,\mathbf{m}}\right], \forall i \in \mathcal{I}\right) \ge 1 - \alpha_{\mathrm{est}}$, as $\mathbf{m} \to \infty$. By Bonferroni's inequality, the proof comletes. □

We consider a portfolio risk measurement example similar in Zhang et al. (2022), and set $\alpha = 0.1$. We compare our performance metrics to those from the bootstrap method introduced by Zhang et al. (2022), where a user-specified parameter $\varepsilon \in (0, 2/3)$ is required. From Table 1, our procedure achieves high coverage probability, which mainly arises from the use of Bonferroni's inequality in our analysis. But our procedure produces narrower and more stable CIs, which results from large values of $m_{\pi_1(1)}$ and $m_{\pi_1(c)}$, because a sufficient computational budget remains available for the inner-level estimation after screening.

Table 1: Performance comparison of our method and bootstrap method.

Budget	Method	Cov.Prob.	Ave.Wid.Ratio (Std.Dev.)	Ave.Low. (Std.Dev.)	Ave.Upp. (Std.Dev.)
5×10^5	Ours	1.000	0.093 (0.010)	19.668 (0.258)	21.594 (0.256)
	Bootstrap ($\varepsilon = 1/24$)	0.872	0.089 (0.032)	19.726 (0.873)	21.561 (0.696)
	Bootstrap ($\varepsilon = 1/12$)	0.894	0.123 (0.046)	19.425 (0.911)	21.951 (1.026)
	Bootstrap ($\varepsilon = 1/6$)	0.853	0.197 (0.085)	18.622 (1.348)	22.680 (1.784)
5×10^6	Ours	1.000	0.040 (0.003)	20.209 (0.114)	21.032 (0.122)
	Bootstrap ($\varepsilon = 1/24$)	0.856	0.045 (0.013)	20.225 (0.309)	21.156 (0.365)
	Bootstrap ($\varepsilon = 1/12$)	0.884	0.063 (0.019)	20.012 (0.439)	21.303 (0.461)
	Bootstrap ($\varepsilon = 1/6$)	0.862	0.117 (0.037)	19.515 (0.818)	21.920 (0.817)

REFERENCES

Zhang, K., G. Liu, and S. Wang. 2022. "Bootstrap-Based Budget Allocation for Nested Simulation". *Operations Research* 70(2):1128–1142.