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TWO-STAGE STOCHASTIC MULTI-OBJECTIVE LINEAR PROGRAMS: PROPERTIES AND ALGORITHMS

Akshita Gupta[1]

[1]Edwardson School of Industrial Engineering, Purdue University, West Lafayette, IN, USA

ABSTRACT

Consider a two-stage stochastic multi-objective linear program (TSSMOLP) which is a natural generalization of the well-studied two-stage stochastic linear program allowing modelers to specify multiple objectives in each stage. The second-stage recourse decision is governed by an uncertain multi-objective linear program (MOLP) whose solution maps to an uncertain second-stage nondominated set. The TSSMOLP then comprises the objective, which is the Minkowski sum of a linear term plus the expected value of the second-stage nondominated set, and the constraints, which are linear. We propose a novel formulation for TSSMOLPs on a general probability space. Further, to aid theoretical analysis, we propose reformulations to the original problem and study their properties to derive the main result that the global Pareto set is cone-convex. Finally, we leverage cone-convexity to demonstrate that solving a TSSMOLP under a sample average approximation framework is equivalent to solving a large scale MOLP.

1 INTRODUCTION

Let ξ be a real-valued random vector $\xi:\Omega\to\mathbb{R}^m$ defined on the complete probability space $(\Omega,\mathfrak{A},\mathbb{P})$. For the cost matrices $\mathsf{C}\in\mathbb{R}^{p\times q_1}$ and $\mathsf{D}(\xi)\in\mathbb{R}^{p\times q_2}$, we formulate the TSSMOLP for $p\geq 1$ objectives as

$$\text{minimize}\quad \{\mathsf{C}x+\mathsf{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)]\}\quad \text{s.t.}\quad x\in\mathcal{X}:=\left\{x\in\mathbb{R}^{q_1}\colon \mathsf{A}x=b,\, x\geqq 0\right\}, \tag{S1}$$

where the second-stage problem is a stochastic MOLP with nondominated set

$$\mathcal{V}_{\mathrm{N}}(x,\xi)=\min\quad \mathsf{D}(\xi)y\quad \text{s.t.}\quad y\in\mathcal{Y}(x,\xi):=\left\{y\in\mathbb{R}^{q_2}\colon \mathsf{T}(\xi)x+\mathsf{W}(\xi)y=h(\xi),\, y\geqq 0\right\}. \tag{S2}$$

The expected value of the second-stage nondominated set in (S1) is the selection expectation (Molchanov 2017). The constraints are specified by the matrix $\mathsf{A}\in\mathbb{R}^{r_1\times q_1}$ and vector $b\in\mathbb{R}^{r_1}$ in the first stage, and by (possibly random) matrices $\mathsf{T}(\xi)\in\mathbb{R}^{r_2\times q_1}$, $\mathsf{W}(\xi)\in\mathbb{R}^{r_2\times q_2}$, and vector $h(\xi)\in\mathbb{R}^{r_2}$ in the second stage. The image set for the second-stage MOLP in (S2) is a random polyhedral convex set denoted as $\mathcal{V}(x,\xi)=\{\mathsf{D}(\xi)y\colon y\in\mathcal{Y}(x,\xi)\}\subset\mathbb{R}^p$ and $\mathcal{Z}(x)=\{\mathsf{C}x+\mathsf{E}[\mathcal{V}_{\mathrm{N}}(x,\xi)]\}$ denotes the image set of (S1). Problem (S1) is a stochastic set-valued optimization problem. The solution to the TSSMOLP is the global efficient set, $\mathcal{X}_{\mathrm{E}}$, and its image set is the global Pareto set, $\mathcal{Z}_{\mathrm{P}}$,

$$\mathcal{Z}_{\mathrm{P}}=\{z^*\in\mathcal{Z}(\mathcal{X})\colon (z^*-\mathbb{R}^p_{\geqq})\cap\mathcal{Z}(\mathcal{X})=\{z^*\}\},\qquad \mathcal{X}_{\mathrm{E}}:=\{x^*\in\mathcal{X}\colon \exists z^*\in\mathcal{Z}(x^*)\text{ such that }z^*\in\mathcal{Z}_{\mathrm{P}}\}$$

where $\mathbb{R}^p_{\geqq}=\{z\in\mathbb{R}^p\colon z_i\geq 0,\ \forall i=1,2,\dots p\}$.

2 MAIN RESULTS

Under certain regularity conditions, TSSMOLP has a nondominance equivalent reformulation, called the image-set reformulation (P), where the nondominated set in the second-stage is replaced by the image set:

$$\text{minimize}\quad \{\phi(x):=\mathsf{C}x+\mathsf{E}[\mathcal{V}(x,\xi)]\}\quad \text{s.t.}\quad x\in\mathcal{X}, \tag{P}$$

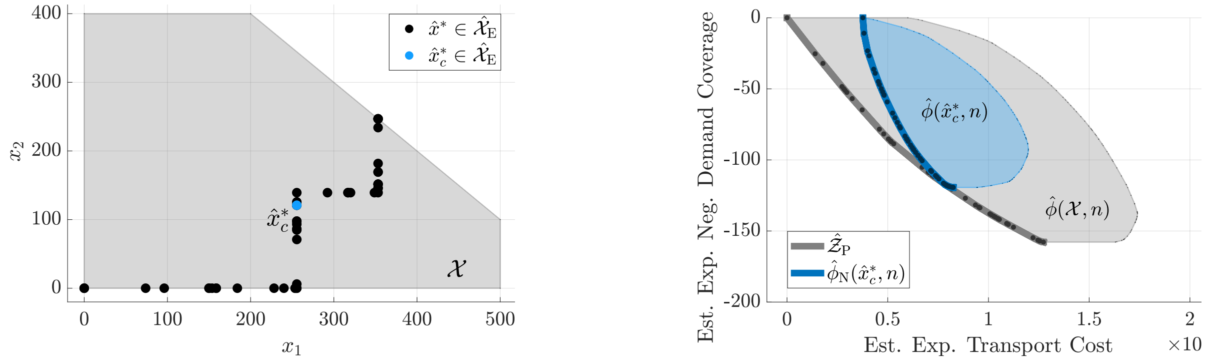

Figure 1: The estimated efficient set $\hat{\mathcal{X}}_{\mathrm{E}}$ (left) and estimated global Pareto set $\hat{\mathcal{Z}}_{\mathrm{P}}$ (right) for solving a two-stage stochastic bi-objective linear program modeling an inventory distribution problem to (i) maximize the expected demand coverage and (ii) minimize the expected transportation cost, for $n = 10$ random scenarios generated with correlation by NORTA. The estimated efficient point $\hat{x}^*_c \in \hat{\mathcal{X}}_{\mathrm{E}}$ is a candidate solution for implementation because its image set, $\hat{\phi}_{\mathrm{N}}(\hat{x}^*_c, n)$, contributes to $\hat{\mathcal{Z}}_{\mathrm{P}}$.

where, under certain regularity conditions, the nondominated set $\phi_{\mathrm{N}}(\mathcal{X}) = \mathcal{Z}_{\mathrm{P}}$ (Gupta and Hunter 2025, Theorem 2). Since the second-stage image sets, $\mathcal{V}(\cdot, \xi)$, are polyhedral for all $x \in \mathcal{X}$, we leverage the properties of selection expectation applied to convex sets, to derive our main result which states that the global Pareto set is compact and cone-convex, i.e., $\mathcal{Z}_{\mathrm{P}} + \mathbb{R}^p$ is convex (Gupta and Hunter 2025, Theorem 4)

To solve a TSSMOLP we adopt a sample average approximation framework for a finite number of samples, n. The formulation of the sample path problem is

$$\text{minimize} \quad \{\hat{\phi}(x,n) := \mathsf{C}x + n^{-1}\textstyle\sum_{i=1}^{n} \mathcal{V}(x, \xi^{(i)})\} \quad \text{s.t.} \quad x \in \mathcal{X}. \tag{P$_n$}$$

Because the empirical distribution has finite support, we show that the estimated nondominated set, $\hat{\mathcal{Z}}_{\mathrm{P}}$, of (P$_n$) can be recovered by solving the large scale MOLP,

$$\text{minimize} \quad \begin{bmatrix}\mathsf{C} & (1/n)\mathsf{D}(\xi^{(1)}) & \ldots & (1/n)\mathsf{D}(\xi^{(n)})\end{bmatrix} w \quad \text{s.t.} \quad w \in \hat{\mathcal{W}}_n, \tag{MOLP$_n$}$$

where $w = (x, y_1, \ldots, y_n) \in \mathbb{R}^{q_1 + nq_2}$ and $\widehat{\mathcal{W}}_n := \{(x, y_1, \ldots, y_n) \colon x \in \mathcal{X},\, y_i \in \mathcal{Y}_i(x, \xi^{(i)})\ \forall i = 1, \ldots, n\}$. We use open-source vector optimization software, bensolve tools (Löhne and Weißing 2017), to solve (MOLP$_n$).

3 CONCLUDING REMARKS

We formally introduce and study the properties of TSSMOLPs for nonatomic probability measures. Leveraging the theoretical results, we solve the sample path problem by reformulating it as a large scale MOLP.

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