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REVISITING AN OPEN QUESTION IN RANKING AND SELECTION UNDER UNKNOWN VARIANCES

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ABSTRACT

Expected improvement (EI) is a common ranking and selection (R&S) method for selecting the optimal system design from a finite set of alternatives. Ryzhov (2016) observed that, under normal sampling distributions with known variances, the limiting budget allocation achieved by EI was closely related to the theoretical optimum. However, when the variances are unknown, the behavior of EI was quite different, giving rise to the question of whether the optimal allocation in this setting was totally distinct from the known-variance case. This research solves that problem with a new analysis that can distinguish between known and unknown variance, unlike previously existing theoretical frameworks. We derive a new optimal budget allocation for this setting, and confirm that the limiting behavior of EI has a similar relationship to this allocation as in the known-variance case.

1 INTRODUCTION

Ranking and selection (R&S) is an important problem in simulation-based optimization. Its goal is to identify the optimal alternative of a complex system from a finite set of alternatives. The mean performance of each alternative is unknown and can be estimated by samples collected from the simulation model of the system under study. Based on these estimates, the optimal alternative can be selected. The quality of the selected alternative depends on the number of samples allocated to each alternative. This has led to the development of numerous R&S methods, which aim to determine the appropriate sample sizes for each alternative to either guarantee or optimize the quality of the selected alternative (Hong et al. 2021; Hunter and Nelson 2017; Xu et al. 2015). Apart from simulation, R&S has also been extensively studied in the machine learning field under the name of best arm identification (BAI) (Jourdan et al. 2023; Yang et al. 2023; Kaufmann et al. 2016).

R&S methods can be classified into two types: fixed-confidence and fixed-budget. Fixed-confidence methods pre-specify a level for the probability of correct selection (PCS) for the optimal alternative and seek to guarantee this probability level using a sample budget as small as possible (Fan et al. 2025; Wang et al. 2024; Zhong and Hong 2022), while fixed-budget methods fix the total sample budget that can be used and aim to efficiently allocate the budget to the alternatives so as to maximize the PCS (Wang and Zhou 2025; Agrawal et al. 2020; Gao and Chen 2016). We consider fixed-budget R&S in this research. Based on the statistical framework adopted, fixed-budget R&S methods can be further divided into two streams: frequentist methods and Bayesian methods (Kim and Nelson 2007; Branke et al. 2007). Frequentist methods use simple statistics such as sample means and sample variances of the alternatives to estimate system performance, guide sample allocation and select the optimal alternative (Chen and Ryzhov 2023; Xiao et al. 2023; Gao and Chen 2017). In contrast, Bayesian methods first utilize historical data or human experiences to construct prior beliefs toward unknown values os the system and update the beliefs dynamically as more simulation samples are collected (Russo 2020; Peng et al. 2018; Chick et al. 2010). The performance estimate for an alternative in Bayesian methods is often its posterior mean.

Expected improvement (EI) is a common Bayesian fixed-budget R&S method (Ryzhov 2016; Qin et al. 2017). In each iteration, EI uses the current best posterior mean as a benchmark, computes the expected improvement of each alternative over the benchmark based on their posterior distributions, and samples the alternative with the largest expected improvement. EI has a straightforward rationale and is widely used not only in R&S but also in other applications, such as continuous black-box optimization (Frazier 2018).

Besides EI, the optimal computing budget allocation (OCBA) is another popular Bayesian fixed-budget R&S method (Chen and Lee 2011). It adopts the Bayesian probability of fasle selection (PFS, equal to 1-PCS) as the quality measure, which quantifies the probability that the selected alternative is not the true optimal one based on the posterior distributions. The method aims to develop a rule for sample allocation that minimizes the Bayesian PFS. Deriving this rule often requires considering the asymptotic case (as the sample budget goes to infinity) to obtain analytical results. Therefore, the OCBA allocation rule is essentially the asymptotic optimal sample allocation. Ryzhov (2016) compared EI and OCBA and observed that the limiting sampling ratios of any two suboptimal alternatives produced by EI are consistent with the asymptotic optimal sample allocation derived by the OCBA method. This is an interesting result because the EI and OCBA methods were developed based on fundamentally different rationales.

However, these two methods (as well as most other fixed-budget R&S methods in the literature) and the aforementioned consistency in their sample allocation were obtained under normal sampling distributions with known variances. Although the normality assumption is standard and reasonable for the samples of simulation models, assuming that their variances are known is unrealistic and does not align with real-world situations. The known-variance assumption underestimates the uncertainty of samples, which will cause the derived sample allocation to be suboptimal.

In view of this issue, Ryzhov (2016) analyzed the EI algorithm when the sampling variances of the alternatives are unknown, and found that the limiting sampling ratios become different from the known-variance case. Since the sampling ratios of EI are closely related to the asymptotic optimal sample allocation under known variances, Ryzhov (2016) inferred that this result likely extends to the unknown-variance case, and further conjectured that the asymptotic optimal sample allocation under unknown variances would differ from the known-variance case as well. It then raised an open question on how to find this potentially different asymptotic optimal sample allocation under unknown variances (possibly using the OCBA method), and whether the sampling ratios of EI under unknown variances match this allocation.

This research provides an answer to this open question by establishing a systematic approach to analyze fixed-budget R&S problems under unknown variances. The new approach is inspired by the basic framework of OCBA and begins by characterizing the convergence rate function of the PFS. Based on this, a sample allocation optimization model is developed, from which the asymptotic optimal sample allocation can be derived. The result confirms that the asymptotic optimal sample allocation under known and unknown variances are indeed different, which validates the conjecture of Ryzhov (2016). In addition, the sampling ratios of EI under unknown variances are shown to align with this optimal allocation, so the consistency in sample allocation between EI and OCBA remains in the unknown-variance case.

We would like to point out that the analysis in this paper is not a simple extension of the OCBA method in the known-variance case (Chen and Lee 2011; Gao et al. 2017). First, the traditional large deviations-based approach for deriving the convergence rate function of PFS (Glynn and Juneja 2004) cannot be applied to the case of normal distributions with unknown variances. This is because the posterior distribution for the mean of each alternative usually does not satisfy the conditions required for a standard large deviations principle. To address this, we developed a new analysis method for derive the rate function of PFS. Second, when sampling variances are unknown, the rate function of PFS no longer has a closed-form expression, the optimization model in defining the rate function is no longer convex, and the corresponding optimal solutions are no longer continuous with respect to the sample allocation. These factors introduce significant challenges in finding the asymptotic optimal sample allocation. In particular, the commonly used solution approach of checking KKT conditions in the known-variance case is not applicable. In this research, we developed new techniques tailored to solve this difficult optimization model.

The remainder of the paper is organized as follows. Section 2 formulates the R&S problem, introduces the EI and OCBA methods and compares their limiting sample allocations. Section 3 shows the convergence rate function of Bayesian PFS under unknown variances. Based on that, Section 4 builds a sample allocation optimization model and derives the asymptotic optimal sample allocation. Section 5 concludes the paper.

2 PROBLEM FORMULATION

In a typical fixed-budget R&S problem, we aim to identify the optimal system design among k alternatives $\{1,\ldots,k\}$ using a total sample budget n. Let Y_{at} denote the t-th simulation sample for alternative a, where Y_{at} follows a normal distribution with mean μ_a and variance σ_a^2, $a=1,\ldots,k$. To find the optimal alternative $a^* \triangleq \arg\max_{a=1,\ldots,k} \mu_a$, we allocate N_a samples to each alternative a with $\sum_{a=1}^k N_a = n$. The mean of each alternative is estimated by samples and the alternative with the highest performance estimate is selected as the optimal one.

When the variances σ_a^2 are unknown, we need to jointly learn (μ_a, σ_a^2) for each alternative a. In a Bayesian framework, (μ_a, σ_a^2) is treated as a random vector, and a prior distribution is specified to represent our initial beliefs about their true values. Once simulation samples for alternative a are observed, the prior distribution can be updated using the Bayes' rule to obtain the posterior distribution. The posterior mean of μ_a serves as its point estimate, and the estimated optimal alternative $\hat{a}^*$ is the one with the highest posterior mean. It is also the selected alternative when the sample budget n is used up. Note that when the variances σ_a^2 are known, only μ_a's need to be learned.

A common measure to quantify the quality of $\hat{a}^*$ is the Bayesian PFS, defined as

$$PFS_n \triangleq \mathbb{P}_n(\cup_{a\neq\hat{a}^*}\{\mu_{\hat{a}^*} \leq \mu_a\}), \tag{1}$$

where $\mathbb{P}_n$ is the posterior probability distribution of $(\mu_1,\mu_2,\ldots,\mu_k)$. The probability PFS_n represents how confident we are about the selected alternative $\hat{a}^*$ not being the true optimal one based on the posterior distribution. Ideally, we hope to allocate the number of samples N_a for each alternative a in a way that minimizes PFS_n. However, since PFS_n rarely has an analytical form, directly optimizing it becomes difficult. In the literature, there have been well-established R&S methods to approximately determine the optimal sample allocations N_a for $a=1,\ldots,k$. In this work, we focus specifically on the EI and OCBA methods.

The EI method sequentially allocates the sample budget. Consider stage $s+1$ where s samples have been allocated to the k alternatives. Let $\hat{\mu}_{s,a}$ denote the posterior mean of alternative a and $\hat{a}_s^* \triangleq \max_{a=1,\ldots,k} \hat{\mu}_{s,a}$ denote the estimated optimal alternative at the end of stage s. The expected improvement of alternative a at stage $s+1$ is defined as

$$\mathscr{E}_{s,a} \triangleq \mathbb{E}_{s,a}(\mu_a - \hat{\mu}_{s,\hat{a}_s^*})^+,$$

where $\mathbb{E}_{s,a}$ is the expectation taken with respect to the posterior distribution of μ_a. $\mathscr{E}_{s,a}$ measures the posterior expected improvement of μ_a over the current best value $\hat{\mu}_{s,\hat{a}_s^*}$, $a=1,\ldots,k$. EI allocates the $(s+1)$st sample to the alternative with the maximum value of $\mathscr{E}_{s,a}$. After the sample is observed, the posterior will be updated and EI will move to stage $s+2$.

The OCBA method formulates the following optimization problem to allocate samples:

$$\min_{N_1,\ldots,N_k} PFS_n \quad s.t. \sum_{a=1}^k N_a = n,\ N_a \geq 0,\ a=1,\ldots,k. \tag{2}$$

Let $p_a \triangleq N_a/n$ denote the proportion of the total sample budget allocated to alternative a. We call $\vec{p} = (p_1,\ldots,p_k)$ a sample allocation. While directly minimizing PFS_n in this formulation is difficult, the convergence rate function of PFS_n is analytically tractable. As a result, an approximate version of the optimization model (2) is often considered, where minimizing PFS_n is replaced by maximizing its rate function. Solving this approximate model yields the asymptotic optimal sample allocation. Based on this allocation, selection algorithms (Chen and Ryzhov 2019; Li and Gao 2023) can be developed.

It is worth noting that the Bayesian PFS is different from the PFS defined in the frequentist framework (Glynn and Juneja 2004; Gao et al. 2017; Chen and Ryzhov 2023). The frequentist PFS is defined as

$$PFS_{freq} \triangleq \mathbb{P}_{freq}(\cup_{a \neq a^*} \{\bar{Y}_{a^*} \leq \bar{Y}_a\})$$

where $\bar{Y}_a$ is the sample mean of alternative a, $a = 1, \ldots, k$, and $\mathbb{P}_{freq}$ is taken with respect to the distribution of sample means $(\bar{Y}_1, \ldots, \bar{Y}_k)$. In the following, we present the existing theoretical results of EI and OCBA under known and unknown variances respectively.

2.1 EI and OCBA under Known Variances

Consider the Bayesian framework where the prior of μ_a is normal for each alternative. When samples of each alternative are normal with known variances, the posterior distribution of μ_a remains normal (Ryzhov 2016). Under this setting, EI is consistent, i.e., it can correctly identify the optimal alternative a^* asymptotically. The number of samples for a^* from EI grows as $n - O(\log n)$, and the limiting sampling ratios of any two suboptimal alternatives satisfy

$$\frac{p_a}{p_{a'}} = \frac{(\mu_{a'} - \mu_{a^*})^2/\sigma_{a'}^2}{(\mu_a - \mu_{a^*})^2/\sigma_a^2}, \ a, a' \neq a^*. \tag{3}$$

As $n \to \infty$, it can be shown that PFS_n converges to zero exponentially (Russo 2020) at the rate of

$$\min_{a \neq a^*} \frac{(\mu_a - \mu_{a^*})^2}{2(\sigma_a^2/p_a + \sigma_{a^*}^2/p_{a^*})}. \tag{4}$$

The OCBA method seeks to maximize this rate function, subject to the constraints $\sum_{a=1}^k p_a = 1$ and $p_a \geq 0$ for $a = 1, \ldots, k$, and the optimal solution (asymptotic optimal sample allocation) satisfies the following conditions (Glynn and Juneja 2004):

$$\frac{(\mu_{a^*} - \mu_a)^2}{2(\sigma_{a^*}^2/p_{a^*} + \sigma_a^2/p_a)} = \frac{(\mu_{a^*} - \mu_{a'})^2}{2(\sigma_{a^*}^2/p_{a^*} + \sigma_{a'}^2/p_{a'})}, \quad a, a' \neq a^*, \tag{5a}$$

$$p_{a^*}^2/\sigma_{a^*}^2 = \sum_{a \neq a^*} p_a^2/\sigma_a^2. \tag{5b}$$

Given the sample proportion p_{a^*} for the optimal alternative a^*, (5a) determine the relative allocation of samples among the suboptimal alternatives. (5b) further establishes a balance between sample sizes allocated to the optimal alternative and the set of suboptimal alternatives. Solving the system in (5) does not yield a closed-form solution and therefore requires numerical methods. However, by assuming that the proportion p_{a^*} of the budget allocated to alternative a^* is significantly greater than that allocated to any suboptimal alternative (i.e., $p_{a^*} \gg p_a$ for all $a \neq a^*$), an approximate solution to (5) can be derived as demonstrated in Chen et al. (2000):

$$\frac{p_a}{p_{a'}} = \frac{(\mu_{a'} - \mu_{a^*})^2/\sigma_{a'}^2}{(\mu_a - \mu_{a^*})^2/\sigma_a^2}, \ a, a' \neq a^*, \tag{6a}$$

$$p_{a^*}^2/\sigma_{a^*}^2 = \sum_{a \neq a^*} p_a^2/\sigma_a^2, \tag{6b}$$

which is a closed-form sample allocation and can be easily computed. Note that the limiting sampling ratios of EI in (3) are exactly the same as the part of the OCBA sample allocation in (6a) that governs the relative allocation among suboptimal alternatives.

2.2 EI under Unknown Variances

The values of variances σ_a^2 (or equivalently, precisions Λ_a ($\triangleq \sigma_a^{-2}$)) are rarely known in practice, $a = 1,\dots,k$. When the precisions Λ_a are treated as unknown, (μ_a, Λ_a) can be learned using the the conjugate Normal-Gamma model. That is, when the prior of (μ_a, Λ_a) is a Normal-Gamma distribution, the posterior is still Normal-Gamma. The relationship of the parameters in the prior and posterior distributions has been reported in the literature (Ryzhov 2016) and is omitted here. The marginal posterior distribution of μ_a follows a Student's t distribution.

Under this model, Ryzhov (2016) showed that the EI algorithm remains consistent, and the optimal alternative a^* still receives $n - O(\log n)$ samples. However, the sampling ratios between any two suboptimal alternatives deviate from the known-variance case. Specifically, the ratios are given by:

$$\frac{p_a}{p_{a'}} = \frac{\log(1+(\mu_{a'}-\mu_{a^*})^2/\sigma_{a'}^2)}{\log(1+(\mu_a-\mu_{a^*})^2/\sigma_a^2)},\ a, a' \neq a^*. \tag{7}$$

Moreover, Ryzhov (2016) showed that the EI under unknown variances demonstrates superior empirical performance to its known-variance version.

3 CONVERGENCE RATE OF BAYESIAN PFS

In this section, we derive the convergence rate function of PFS_n under unknown variances. Some lemmas will be introduced for better understanding of the main conclusion. Suppose the prior of (μ_a, Λ_a) is Normal-Gamma with density denoted by $g_{0,a}(x,\lambda)$, $a = 1,\dots,k$. The priors of different alternatives are independent. For notation simplicity, let $\mathscr{D} = \mathbb{R}^+ \times \mathbb{R}$. The prior density should be non-degenerate such that

$$\iint_{\mathscr{D}} g_{0,a}(x_a,\lambda_a)dx_a d\lambda_a = 1,\ a = 1,\dots,k. \tag{8}$$

After N_a simulation samples (out of the sample budget n) have been collected for alternative a, its posterior distribution is Normal-Gamma with density $g_{n,a}(x_a,\lambda_a)$. The posterior mean of μ_a is denoted by $\hat{\mu}_{n,a}$, $a = 1,\dots,k$, and the selected alternative is $\hat{a}^* = \arg\max_{a=1,\dots,k} \hat{\mu}_{n,a}$.

The event of false selection is fundamentally determined by the outcomes of $k-1$ pairwise comparisons between the selected alternative $\hat{a}^*$ and each of the remaining alternatives. In the following lemma, we reduce the convergence rate of PFS_n to the convergence rates of the Bayesian probabilities of false comparison associated with these $k-1$ pairwise comparisons.

Lemma 1 Suppose $p_a > 0$ for all $a = 1,\dots,k$. The convergence rate function of PFS_n is

$$\lim_{n\to\infty} \frac{1}{n} \log PFS_n = \max_{i\neq i^*} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n(\mu_{a^*} \leq \mu_a).$$

The rationale of Lemma 1 is as follows. Since $p_a > 0$, $N_a \to \infty$ as $n \to \infty$, and the Law of Large Numbers ensures that the posterior mean $\hat{\mu}_{n,a}$ converges to the true mean μ_a as $n \to \infty$ for all $a = 1,\dots,k$. This convergence implies that the selected alternative $\hat{a}_n^* = a^*$ when n is large enough. For such sufficiently large n, it follows directly that:

$$\max_{a\neq a^*} \mathbb{P}_n(\mu_{a^*} \leq \mu_a) \leq PFS_n \leq \sum_{a\neq a^*} \mathbb{P}_n(\mu_{a^*} \leq \mu_a) \leq (k-1) \max_{a\neq a^*} \mathbb{P}_n(\mu_{a^*} \leq \mu_a).$$

Then, using a similar analysis as in Glynn and Juneja (2004), we can establish the proof of Lemma 1.

Let $l(y;x,\lambda)$ denote the density function of a normal distribution with mean x and precision λ. Let $\bar{Y}_a \triangleq N_a^{-1}\sum_{t=1}^{N_a} Y_{at}$ and $\hat{\sigma}_a^2 \triangleq N_a^{-1}\sum_{t=1}^{N_a}(Y_{at}-\bar{Y}_a)^2$ denote the MLE of μ_a and σ_a^2 based on simulation samples

Y_{at}, $t = 1,\dots,N_a$, for alternative a where $N_a = p_a n$. Define a likelihood ratio function as

$$m_{n,a}(x_a,\lambda_a) \triangleq \prod_{t=1}^{N_a} \frac{l(Y_{at};x_a,\lambda_a)}{l(Y_{at};\bar{Y}_a,\hat{\sigma}_a^{-2})}.$$

According to the Bayes' rule, for any suboptimal alternative $a \neq a^*$,

$$\mathbb{P}_n(\mu_{a^*} \le \mu_a) = \iiiint_{\mathscr{D}^2} \mathbb{I}(x_{a^*} \le x_a) g_{n,a}(x_a,\lambda_a) g_{n,a^*}(x_{a^*},\lambda_{a^*}) dx_a d\lambda_a dx_{a^*} d\lambda_{a^*}$$
$$= \frac{\iiiint_{\mathscr{D}^2} \mathbb{I}(x_{a^*} \le x_a) g_{0,a}(x_a,\lambda_a) m_{n,a}(x_a,\lambda_a) g_{0,a^*}(x_{a^*},\lambda_{a^*}) m_{n,a^*}(x_{a^*},\lambda_{a^*}) dx_a d\lambda_a dx_{a^*} d\lambda_{a^*}}{\iiiint_{\mathscr{D}^2} g_{0,a}(x'_a,\lambda'_a) m_{n,a}(x'_a,\lambda'_a) g_{0,a^*}(x'_{a^*},\lambda'_{a^*}) m_{n,a^*}(x'_{a^*},\lambda'_{a^*}) dx'_a d\lambda'_a dx'_{a^*} d\lambda'_{a^*}}, \tag{9}$$

where the posterior density function of $g_{n,a}(x_a,\lambda_a)$ is obtained by

$$g_{n,a}(x_a,\lambda_a) = \frac{g_{0,a}(x_a,\lambda_a)\prod_{t=1}^{N_a} l(Y_{at};x_a,\lambda_a)}{\iint_{\mathscr{D}} g_{0,a}(x'_a,\lambda'_a)\prod_{t=1}^{N_a} l(Y_{at};x'_a,\lambda'_a) dx'_a d\lambda'_a} = \frac{g_{0,a}(x_a,\lambda_a) m_{n,a}(x_a,\lambda_a)}{\iint_{\mathscr{D}} g_{0,a}(x'_a,\lambda'_a) m_{n,a}(x'_a,\lambda'_a) dx'_a d\lambda'_a}.$$

In the formulation above, we used the likelihood ratio function $m_{n,a}(x_a,\lambda_a)$ instead of the likelihood function $\prod_{t=1}^{N_a} l(Y_{at};x_a,\lambda_a)$ to facilitate the discussion on the convergence rate. Let

$$\mathscr{I}_a \triangleq \iiiint_{\mathscr{D}^2} \mathbb{I}(x_{a^*} \le x_a) g_{0,a}(x_a,\lambda_a) m_{n,a}(x_a,\lambda_a) g_{0,a^*}(x_{a^*},\lambda_{a^*}) m_{n,a^*}(x_{a^*},\lambda_{a^*}) dx_a d\lambda_a dx_{a^*} d\lambda_{a^*},$$
$$\mathscr{I}_{a^*} \triangleq \iiiint_{\mathscr{D}^2} g_{0,a}(x_a,\lambda_a) m_{n,a}(x_a,\lambda_a) g_{0,a^*}(x_{a^*},\lambda_{a^*}) m_{n,a^*}(x_{a^*},\lambda_{a^*}) dx_a d\lambda_a dx_{a^*} d\lambda_{a^*}.$$

Then $\mathbb{P}_n(\mu_{a^*} \le \mu_a) = \mathscr{I}_a / \mathscr{I}_{a^*}$ by (9).

We seek to find lower and upper bounds on $\mathscr{I}_a$ and $\mathscr{I}_{a^*}$ so that $\mathbb{P}_n(\mu_{a^*} \le \mu_a)$ can be bounded. Then, using Lemma 1, the convergence rate function of PFS_n can be obtained from the bounds on $\mathscr{I}_a / \mathscr{I}_{a^*}$.

To do it, we consider two maximization problems as follows for alternative $a \neq a^*$:

$$\max_{(x_a,\lambda_a,x_{a^*},\lambda_{a^*}) \in \mathscr{D}^2} m_{n,a}(x_a,\lambda_a) m_{n,a^*}(x_{a^*},\lambda_{a^*}), \tag{10}$$

$$\max_{(x_a,\lambda_a,x_{a^*},\lambda_{a^*}) \in \mathscr{D}^2} m_{n,a}(x_a,\lambda_a) m_{n,a^*}(x_{a^*},\lambda_{a^*}) \quad \text{s.t. } x_{a^*} \le x_a. \tag{11}$$

Let $\vec{v}_{a^*}$ and $\vec{v}_a$ denote the optimal solutions to (10) and (11), respectively. If we can find the optimal values of (10) and (11) and derive lower and upper bounds on their objective values in the neighborhood of the optimal solutions $\vec{v}_{a^*}$ and $\vec{v}_a$, then, bounds on $\mathscr{I}_{a^*}$ and $\mathscr{I}_a$ will be easily obtained by integration over the neighborhood of $\vec{v}_{a^*}$ and $\vec{v}_a$ respectively.

In the case of (10), this process is relatively straightforward because the problem simplifies to the maximum likelihood estimation of the parameters (μ_a,Λ_a) and $(\mu_{a^*},\Lambda_{a^*})$. However, for problem (11), we must address the additional constraint $x_{a^*} \le x_a$, which introduces further complexity to the solution process.

Lemma 2 Let $\bar{\varepsilon} > 0$ denote a small enough constant. For alternative $a \neq a^*$ and ε with $0 < \varepsilon \le \bar{\varepsilon}$, when n is large enough such that $|\bar{Y}_{a'} - \mu_{a'}| \le \varepsilon$ and $|\hat{\sigma}^2_{a'} - \sigma^2_{a'}| \le \varepsilon$, $a' = a, a^*$, the optimal value of (11) equals

$$\exp\Big(-n\min_{x_a}\Big(\frac{p_a}{2}\log\big(1+(\bar{Y}_a - x_a)^2/\hat{\sigma}_a^2\big) + \frac{p_{a^*}}{2}\log\big(1+(\bar{Y}_{a^*} - x_a)^2/\hat{\sigma}_{a^*}^2\big)\Big)\Big).$$

Lemma 2 shows that the optimization over $(x_a,\lambda_a,x_{a^*},\lambda_{a^*})$ under the constraint $x_{a^*} \le x_a$ can be reduced to an one-dimensional optimization over x_a.

For ε with $0 < \varepsilon \le \bar{\varepsilon}$, define the neighborhood around the optimal solutions $\vec{v}_a$ and $\vec{v}_{a^*}$ as

$$B_{n,a} = \{(x_a,\lambda_a,x_{a^*},\lambda_{a^*}) \in \mathscr{D}^2 : \|(x_a,\lambda_a,x_{a^*},\lambda_{a^*}) - \vec{v}_a\|_\infty \le \varepsilon \text{ and } x_{a^*} \le x_a\},$$
$$B_{n,a^*} = \{(x_a,\lambda_a,x_{a^*},\lambda_{a^*}) \in \mathscr{D}^2 : \|(x_a,\lambda_a,x_{a^*},\lambda_{a^*}) - \vec{v}_{a^*}\|_\infty \le \varepsilon\}.$$

Lemma 3 For alternative $a \neq a^*$ and $0 < \varepsilon \leq \bar{\varepsilon}$, when n is large enough such that $|\bar{Y}_{a'} - \mu_{a'}| \leq \varepsilon$ and $|\hat{\sigma}^2_{a'} - \sigma^2_{a'}| \leq \varepsilon$, $a' = a, a^*$, there exists a constant b_R independent of n and ε such that for any $(x_a, \lambda_a, x_{a^*}, \lambda_{a^*}) \in B_{n,a}$, we have $\exp(-n(\mathscr{R}_a(p_a, p_{a^*}) + b_R\varepsilon)) \leq m_{n,a}(x_a, \lambda_a) m_{n,a^*}(x_{a^*}, \lambda_{a^*}) \leq \exp(-n(\mathscr{R}_a(p_a, p_{a^*}) - b_R\varepsilon))$where

$$\mathscr{R}_a(p_a, p_{a^*}) \triangleq \min_{x_a} \left(\frac{p_a}{2} \log\left(1 + (\mu_a - x_a)^2/\sigma_a^2\right) + \frac{p_{a^*}}{2} \log\left(1 + (\mu_{a^*} - x_a)^2/\sigma_{a^*}^2\right) \right). \tag{12}$$

For any $(x_a, \lambda_a, x_{a^*}, \lambda_{a^*}) \in B_{n,a^*}$, we have $\exp(-nb_R\varepsilon) \leq m_{n,a}(x_a, \lambda_a) m_{n,a^*}(x_{a^*}, \lambda_{a^*}) \leq 1$.

Lemma 3 provides lower and upper bounds on the objective value of (11) in the regions $B_{n,a}$ and B_{n,a^*}. It can be proved by investigating the uniform continuity of the objective function of (11) in the logarithmic scale.

Now we can characterize the rate function of $\mathbb{P}_n(\mu_{a^*} \leq \mu_a)$. Let c denote the constant such that $g_{0,a}(x_a, \lambda_a) g_{0,a^*}(x_{a^*}, \lambda_{a^*}) \geq c > 0$ for any $(x_a, \lambda_a, x_{a^*}, \lambda_{a^*}) \in B_{n,a}$ and $a = 1, \ldots, k$. For alternative $a \neq a^*$,

$$\begin{aligned}
\mathscr{I}_a &\geq c \iiiint_{B_{n,a}} m_{n,a}(x_a, \lambda_a) m_{n,a^*}(x_{a^*}, \lambda_{a^*}) dx_a d\lambda_a dx_{a^*} d\lambda_{a^*} \\
&\geq c\text{Size}(B_{n,a}) \min_{(x_a, \lambda_a, x_{a^*}, \lambda_{a^*}) \in B_{n,a}} m_{n,a}(x_a, \lambda_a) m_{n,a^*}(x_{a^*}, \lambda_{a^*}) \geq c\text{Size}(B_{n,a}) \exp(-n(\mathscr{R}_a(p_a, p_{a^*}) + b_R\varepsilon)),
\end{aligned} \tag{13}$$

where $\text{Size}(B_{n,a})$ denotes the volume of $B_{n,a}$ and the last inequality holds by Lemma 3. On the other hand,

$$\begin{aligned}
\mathscr{I}_a &\leq \max_{(x_a, \lambda_a, x_{a^*}, \lambda_{a^*}) \in B_{n,a}} m_{n,a}(x_a, \lambda_a) m_{n,a^*}(x_{a^*}, \lambda_{a^*}) \iiiint_{\mathscr{D}^2} g_{0,a}(x_a, \lambda_a) g_{0,a^*}(x_{a^*}, \lambda_{a^*}) dx_a d\lambda_a dx_{a^*} d\lambda_{a^*} \\
&\leq \exp(-n(\mathscr{R}_a(p_a, p_{a^*}) - b_R\varepsilon))
\end{aligned} \tag{14}$$

where the last inequality holds by Lemma 3 and (8). Similarly, we have

$$c\text{Size}(B_{n,a^*}) \exp(-nb_R\varepsilon) \leq \mathscr{I}_{a^*} \leq 1. \tag{15}$$

Substituting the lower and upper bounds of (13)-(15) into (9),

$$c\text{Size}(B_{n,a}) \exp(-n\mathscr{R}_a(p_a, p_{a^*}) - nb_R\varepsilon) \leq \mathscr{I}_a / \mathscr{I}_{a^*} \leq \exp(-n\mathscr{R}_a(p_a, p_{a^*}) + 2nb_R\varepsilon) / (c\text{Size}(B_{n,a^*})).$$

By letting $n \to \infty$ and $\varepsilon \to 0$, we obtain

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n(\mu_{a^*} \leq \mu_a) = \lim_{n \to \infty} \frac{1}{n} \log \frac{\mathscr{I}_a}{\mathscr{I}_{a^*}} = -\mathscr{R}_a(p_a, p_{a^*}).$$

Using this equation and Lemma 1, we can derive the rate function of PFS_n.

Theorem 1 The rate function of PFS_n is

$$-\lim_{n \to \infty} \frac{1}{n} \log PFS_n = \min_{a \neq a^*} \mathscr{R}_a(p_a, p_{a^*}).$$

Theorem 1 shows that the Bayesian PFS converges exponentially at the rate of $\min_{a \neq a^*} \mathscr{R}_a(p_a, p_{a^*})$. We remark that since $\log(1+x) < x$ for any $x > 0$,

$$\min_{a \neq a^*} \mathscr{R}_a(p_a, p_{a^*}) < \min_{a \neq a^*} \min_{x_a} \left(\frac{p_a}{2} \frac{(\mu_a - x_a)^2}{\sigma_a^2} + \frac{p_{a^*}}{2} \frac{(\mu_{a^*} - x_a)^2}{\sigma_{a^*}^2} \right) = \min_{a \neq a^*} \frac{(\mu_a - \mu_{a^*})^2}{2(\sigma_a^2/p_a + \sigma_{a^*}^2/p_{a^*})},$$

where the right-hand side is the rate function (4) under known variances. This suggests that given the same allocation, the convergence rate under unknown variances is slower than that under known variances.

With Theorem 1, we turn to consider the following optimization problem

$$\max_{\vec{p}} \min_{a \neq a^*} \mathscr{R}_a(p_a, p_{a^*}), \quad s.t. \sum_{a=1}^{k} p_a = 1, \quad 0 < p_a < 1, \ a = 1, \ldots, k, \tag{16}$$

which finds the sample allocation that drives PFS_n to converge to zero at the fastest possible rate. The approach to solving this optimization problem is discussed in detail in the next section.

4 ASYMPTOTIC OPTIMAL SAMPLE ALLOCATION UNDER UNKNOWN VARIANCES

In the known-variance case, maximizing the rate function of PFS_n leads to a well-defined analytical convex optimization problem. However, when the variances are unknown, in the corresponding optimization model (16), the term $\mathscr{R}_a(p_a, p_{a^*})$ does not have a closed form, which significantly increases the complexity of the problem and makes it difficult to solve.

The reason causing this difficulty is that the optimization problem (12) in defining $\mathscr{R}_a(p_a, p_{a^*})$ is non-convex such that the optimal solution to (12) is not analytical. We use an example to illustrate this effect. Consider two alternatives a and a^* and set $\mu_a = 0$, $\mu_{a^*} = 5$, and $\sigma_a^2 = \sigma_{a^*}^2 = 1$. Figure 1 plots the objective function of (12) when (p_a, p_{a^*}) is $(0.19, 0.20)$, $(0.20, 0.20)$ and $(0.21, 0.20)$, respectively. We have three main observations from Figure 1.

1. The objective function is non-convex and has multiple local optimal solutions.
2. When $(p_a, p_{a^*}) = (0.20, 0.20)$, the objective function has two local optimal solutions, approximately 0.209 and 4.791, both of which are also globally optimal.
3. As p_a increases from 0.19 to 0.21 while holding p_{a^*} constant, the global optimal solution to (12) decreases significantly from 4.803 to 0.197.

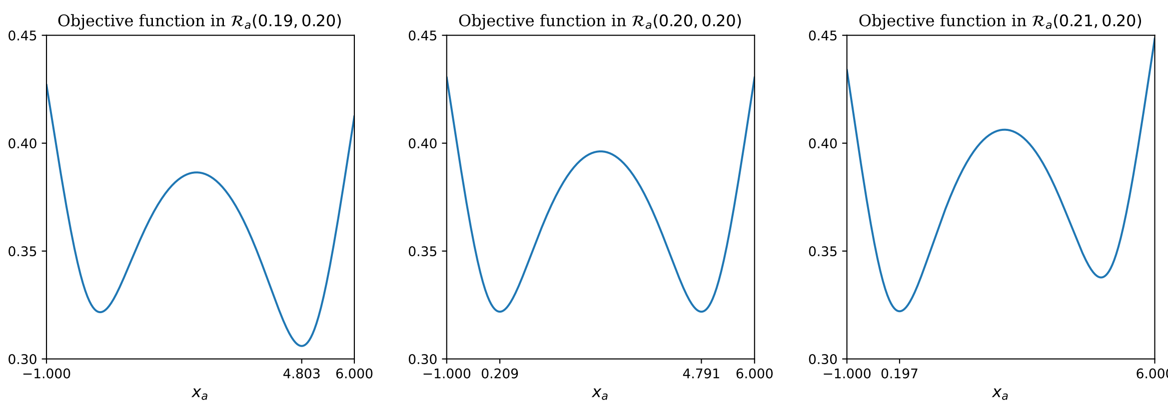

Figure 1: Objective function of (12) under various values of (p_a, p_{a^*}).

For (p_a, p_{a^*}) and (p'_a, p'_{a^*}) with $p_a/p_{a^*} = p'_a/p'_{a^*}$, their corresponding optimal solutions to (12) are the same. We can regard the global optimal solutions to (12) as functions of the sampling ratio p_a/p_{a^*}. Let $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ denote the smallest and largest (global) optimal solution to (12). The two solutions can either coincide (e.g., for $(p_a, p_{a^*}) = (0.19, 0.20)$ and $(p_a, p_{a^*}) = (0.21, 0.20)$ in Figure 1) or differ (e.g., for $(p_a, p_{a^*}) = (0.20, 0.20)$ in Figure 1). In practice, these two solutions can be calculated based on the first-order condition to (12). Lemma 4 shows some basic properties of these two solutions with respect to the sampling ratio p_a/p_{a^*} that are important for the subsequent discussion.

Lemma 4 The optimal solutions $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ to (12) have the following properties.

1. If $p_a/p_{a^*} < p'_a/p'_{a^*}$, then $x_a^1(p_a/p_{a^*}) > x_a^2(p'_a/p'_{a^*})$.
2. Both $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ are in $[\mu_a, \mu_{a^*}]$.
3. As $p_a/p_{a^*} \to 0$, both $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ converge to μ_{a^*}.
4. As $p_a/p_{a^*} \to \infty$, both $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ converge to μ_a.

Property 1 of Lemma 4, combined with the fact that $x_a^2(p_a/p_{a^*}) \geq x_a^1(p_a/p_{a^*})$, establishes that both $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ are strictly decreasing functions of the sampling ratio p_a/p_{a^*}. Properties 2-4 of Lemma 4 demonstrate that these two solutions decrease monotonically from μ_{a^*} to μ_a as p_a/p_{a^*} increases from zero to infinity. However, as illustrated in Figure 1, the decreases in $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ are not continuous with respect to p_a/p_{a^*}. We may find that $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ have a jump at some values of p_a/p_{a^*} where $x_a^1(p_a/p_{a^*}) < x_a^2(p_a/p_{a^*})$, e.g., when $(p_a, p_{a^*}) = (0.20, 0.20)$ in Figure 1. The discontinuity of $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ implies that the partial derivative of these two solutions with respect to p_a or p_{a^*} may not exist. This discontinuity prohibits the use of KKT conditions, a standard solution technique, to finding the asymptotic optimal sample allocation.

Below, we develop a method to solve (16) without the use of the KKT conditions. It consists of two steps.

First, we fix the sample proportion p_{a^*} for the optimal alternative a^* where $0 < p_{a^*} < 1$. Consider the following more restrictive problem than (16):

$$\max_{p_a, a\neq a^*} \min_{a\neq a^*} \mathscr{R}_a(p_a, p_{a^*}), \quad s.t. \sum_{a\neq a^*} p_a = 1 - p_{a^*}, \quad 0 < p_a < 1,\ a \neq a^*. \tag{17}$$

The decision variables of (17) are the sample proportions p_a of the $k-1$ suboptimal alternatives, $a \neq a^*$.

Increasing the sample proportion p_a for the suboptimal alternative a while keeping p_{a^*} fixed results in a larger value of $\mathscr{R}_a(p_a, p_{a^*})$. However, due to the sample budget constraint $\sum_{a\neq a^*} p_a = 1 - p_{a^*}$, if we increase p_a for alternative a, we must decrease $p_{a'}$ for some other suboptimal alternative a', which will lead to a smaller value of $\mathscr{R}_{a'}(p_{a'}, p_{a^*})$. Therefore, problem (17) involves balancing the sample proportions among suboptimal alternatives to maximize the minimum $\mathscr{R}_a(p_a, p_{a^*})$. This problem is investigated in the following theorem.

Theorem 2 The optimal solution (denoted by $p_a^\circ(p_{a^*})$, $a \neq a^*$) to (17) is unique and satisfies

$$\mathscr{R}_a(p_a^\circ(p_{a^*}), p_{a^*}) = \mathscr{R}_{a'}(p_{a'}^\circ(p_{a^*}), p_{a^*}),\ \forall a, a' \neq a^*.$$

As discussed in Section 3, $\mathscr{R}_a(p_a, p_{a^*})$ is the rate function of the posterior probability $\mathbb{P}_n(\mu_{a^*} \leq \mu_a)$ of alternative a^* being inferior to a. Theorem 2 shows that the sample proportions among suboptimal alternatives should be allocated such that the values of the rate functions of $\mathbb{P}_n(\mu_{a^*} \leq \mu_a)$, $a \neq a^*$, are equal. Theorem 2 further implies that the optimal allocation to (16) must be in the set of allocations

$$\mathscr{S} \triangleq \{\vec{p} = (p_1, \ldots, p_k) : 0 < p_{a^*} < 1 \text{ and } p_a = p_a^\circ(p_{a^*}), \forall a \neq a^*\}.$$

Because of the uniqueness of $p_a^\circ(p_{a^*})$, $a \neq a^*$, each possible value of the sample proportion p_{a^*} corresponds to one allocation in $\mathscr{S}$.

Second, we find the value of p_{a^*} whose corresponding allocation in $\mathscr{S}$ leads to the optimal convergence rate for (16). For notation simplicity, for each $a \neq a^*$, define

$$\mathscr{U}_a^1(p_{a^*}) \triangleq \log(1 + (\mu_{a^*} - x_a^1(p_a^\circ(p_{a^*})/p_{a^*}))^2/\sigma_{a^*}^2), \quad \mathscr{V}_a^1(p_{a^*}) \triangleq \log(1 + (\mu_a - x_a^1(p_a^\circ(p_{a^*})/p_{a^*}))^2/\sigma_a^2),$$
$$\mathscr{U}_a^2(p_{a^*}) \triangleq \log(1 + (\mu_{a^*} - x_a^2(p_a^\circ(p_{a^*})/p_{a^*}))^2/\sigma_{a^*}^2), \quad \mathscr{V}_a^2(p_{a^*}) \triangleq \log(1 + (\mu_a - x_a^2(p_a^\circ(p_{a^*})/p_{a^*}))^2/\sigma_a^2).$$

Let $p_{a^*}^*$ be the value of p_{a^*} that satisfies

$$\sum_{a\neq a^*} \mathscr{U}_a^1(p_{a^*}^*)/\mathscr{V}_a^1(p_{a^*}^*) \geq 1, \quad \sum_{a\neq a^*} \mathscr{U}_a^2(p_{a^*}^*)/\mathscr{V}_a^2(p_{a^*}^*) \leq 1. \tag{18}$$

Note that $p_{a^*}^*$ exists and is unique. In the following, we will demonstrate that $p_{a^*}^*$ is the optimal value for p_{a^*}.

Let $\vec{p}^* = (p_1^*, \ldots, p_k^*)$ denote the corresponding allocation in $\mathscr{S}$ given $p_{a^*}^*$. That is, $p_a^* \triangleq p_a^\circ(p_{a^*}^*)$, $a \neq a^*$. Lemma 5 below provides a way to compare the convergence rates associated with an arbitrary allocation $\vec{p}$ in $\mathscr{S}$ and the optimal allocation $\vec{p}^*$. This lemma can be derived based on Lemma 4.

Lemma 5 For any p_{a^*} with $0 < p_{a^*} < 1$ and any suboptimal alternative $a \neq a^*$, we have

$$2\mathscr{R}_a(p_a^\circ(p_{a^*}), p_{a^*}) < 2\mathscr{R}_a(p_a^*, p_{a^*}^*) + (p_{a^*} - p_{a^*}^*)\mathscr{U}_a^1(p_{a^*}^*) + (p_a^\circ(p_{a^*}) - p_a^*)\mathscr{V}_a^1(p_{a^*}^*), \tag{19}$$

$$2\mathscr{R}_a(p_a^\circ(p_{a^*}), p_{a^*}) < 2\mathscr{R}_a(p_a^*, p_{a^*}^*) + (p_{a^*} - p_{a^*}^*)\mathscr{U}_a^2(p_{a^*}^*) + (p_a^\circ(p_{a^*}) - p_a^*)\mathscr{V}_a^2(p_{a^*}^*). \tag{20}$$

An immediate implication from (19) of Lemma 5 is that if there exists p_{a^*} with $0 < p_{a^*} < p_{a^*}^*$ such that $\mathscr{R}_a(p_a^\circ(p_{a^*}), p_{a^*}) \geq \mathscr{R}_a(p_a^*, p_{a^*}^*)$ for some alternative $a \neq a^*$, then the following inequality must hold

$$p_a^\circ(p_{a^*}) - p_a^* > (p_{a^*}^* - p_{a^*})\mathscr{U}_a^1(p_{a^*}^*)/\mathscr{V}_a^1(p_{a^*}^*), \tag{21}$$

because if it does not, we will obtain $\mathscr{R}_a(p_a^\circ(p_{a^*}), p_{a^*}) < \mathscr{R}_a(p_a^*, p_{a^*}^*)$ by (19) of Lemma 5. Taking the summation over $a \neq a^*$ in (21) yields

$$\sum_{a \neq a^*} p_a^\circ(p_{a^*}) - \sum_{a \neq a^*} p_a^* > (p_{a^*}^* - p_{a^*}) \sum_{a \neq a^*} \frac{\mathscr{U}_a^1(p_{a^*}^*)}{\mathscr{V}_a^1(p_{a^*}^*)} \geq p_{a^*}^* - p_{a^*},$$

where the last inequality holds by (18). The above equation leads to the contradiction that $\sum_{a \neq a^*} p_a^\circ(p_{a^*}) + p_{a^*} > \sum_{a \neq a^*} p_a^* + p_{a^*}^* = 1$ because $\sum_{a \neq a^*} p_a^\circ(p_{a^*}) + p_{a^*}$ should equal to one.

Similarly, if there exists p_{a^*} with $p_{a^*}^* < p_{a^*} < 1$ such that $\mathscr{R}_a(p_a^\circ(p_{a^*}), p_{a^*}) \geq \mathscr{R}_a(p_a^*, p_{a^*}^*)$ for some alternative $a \neq a^*$, then the following inequality must hold

$$p_a^* - p_a^\circ(p_{a^*}) < (p_{a^*} - p_{a^*}^*)\mathscr{U}_a^2(p_{a^*}^*)/\mathscr{V}_a^2(p_{a^*}^*). \tag{22}$$

Taking the summation over $a \neq a^*$ in (22) yields $\sum_{a \neq a^*} p_a^* - \sum_{a \neq a^*} p_a^\circ(p_{a^*}) < p_{a^*} - p_{a^*}^*$, which leads to the contradiction that $\sum_{a \neq a^*} p_a^\circ(p_{a^*}) + p_{a^*} > 1$ again. Therefore, for any p_{a^*} with $p_{a^*} \neq p_{a^*}^*$, its corresponding convergence rate of PFS_n is slower than that of $p_{a^*}^*$, making $p_{a^*}^*$ the optimal solution for p_{a^*}. Based on Theorem 2 and the optimality of $p_{a^*}^*$, it can be concluded that that $\vec{p}^*$ is optimal for problem (16), i.e., it is the asymptotic optimal sample allocation for fixed-budget R&S under unknown variances.

Theorem 3 The optimal allocation $\vec{p}^* = (p_1^*, \ldots, p_k^*)$ to (16) is unique and satisfies

$$\mathscr{R}_a(p_a^*, p_{a^*}^*) = \mathscr{R}_{a'}(p_{a'}^*, p_{a^*}^*), \quad \forall a, a' \neq a^*, \tag{23a}$$

$$\sum_{a \neq a^*} \mathscr{U}_a^1(p_{a^*}^*)/\mathscr{V}_a^1(p_{a^*}^*) \geq 1, \quad \sum_{a \neq a^*} \mathscr{U}_a^2(p_{a^*}^*)/\mathscr{V}_a^2(p_{a^*}^*) \leq 1. \tag{23b}$$

It can be seen that conditions (23) that determine the asymptotic optimal sample allocation under unknown variances and those under known variances (5) are indeed different. That is, the conjecture of Ryzhov (2016) is correct. Conditions (23) also consist of two equations, which play similar roles to the two equations in (5). Specifically, (23a) controls the sample proportion allocated to each suboptimal alternative given that of the optimal alternative a^*, while (23b) balances the sample sizes allocated to the optimal alternative a^* and the set of suboptimal alternatives. However, (23) is more complex than (5) because of the lack of analytical forms of the rate function $\mathscr{R}_a(p_a, p_{a^*})$ and the optimal solutions $x_a^1(p_a/p_{a^*})$ and $x_a^2(p_a/p_{a^*})$ to (12).

We provide a brief discussion on how to calculate the optimal sample allocation $\vec{p}^*$ given the mean and variance of each alternative. First, for any $0 < p_{a^*} < 1$, we can balance the sample proportions p_a, $a \neq a^*$, to obtain the optimal solution $p_a^\circ(p_{a^*})$, $a \neq a^*$, in Theorem 2 based on the monotonicity of $\mathscr{R}_a(p_a, p_{a^*})$ in p_a. Second, we can find the optimal value $p_{a^*}^*$ for p_{a^*} based on the fact that both $\sum_{a \neq a^*} \mathscr{U}_a^1(p_{a^*})/\mathscr{V}_a^1(p_{a^*})$ and $\sum_{a \neq a^*} \mathscr{U}_a^2(p_{a^*})/\mathscr{V}_a^2(p_{a^*})$ are monotonically decreasing as p_{a^*} increases. The detailed introduction and analysis of the algorithm to calculate this optimal allocation and the numerical comparison are left to future study.

We can also simplify (23a) in the same way as (6) simplifies (5), by assuming $p^*_{a^*} \gg p^*_a$ for all $a \neq a^*$. Since $x^1_a(p_a/p_{a^*})$ and $x^2_a(p_a/p_{a^*})$ converge to μ_{a^*} as p_a/p_{a^*} decreases to 0 by Lemma 4, we can approximate $x^1_a(p^*_a/p^*_{a^*})$ or $x^2_a(p^*_a/p^*_{a^*})$ by μ_{a^*} when $p^*_a/p^*_{a^*}$ is small enough such that

$$\begin{aligned}\mathscr{R}_a(p^*_a, p^*_{a^*}) &= \min_{x_a}\left(\frac{p^*_a}{2}\log\left(1+(\mu_a - x_a)^2/\sigma_a^2\right) + \frac{p^*_{a^*}}{2}\log\left(1+(\mu_{a^*} - x_a)^2/\sigma_{a^*}^2\right)\right)\\ &\approx \frac{p^*_a}{2}\log(1+(\mu_a-\mu_{a^*})^2/\sigma_a^2).\end{aligned}$$

Then (23a) can be approximated by

$$\frac{p^*_a}{p^*_{a'}} = \frac{\log(1+(\mu_{a'}-\mu_{a^*})^2/\sigma_{a'}^2)}{\log(1+(\mu_a-\mu_{a^*})^2/\sigma_a^2)}, \quad \forall a, a' \neq a^*. \tag{24}$$

Note that the sampling ratios of any two suboptimal alternatives in (24) are exactly the same as the lmiting sampling ratios (7) produced by the EI algorithm under unknown variances. In other words, just like in the known-variance case, the limiting sampling ratios of any two suboptimal alternatives of EI also nearly match the asymptotic optimal sample allocation.

5 CONCLUSION

This paper considers the fixed-budget R&S problem under unknown variances. In real applications, the sampling variances of system alternatives are often unknown, introducing additional uncertainty that most existing fixed-budget R&S methods fail to address. Therefore, it is important to develop effective methods and sample allocation rules tailored to the unknown-variance setting. In this research, we propose a systematic approach to derive the asymptotic optimal sample allocation under normal sampling distributions with unknown variances. We analyze the convergence rate function of PFS, formulate a sample budget optimization model and solve it to identify the optimal sample allocation. Our method overcomes the significant challenges posed by the non-convexity of the underlying optimization problem and the lack of analytical expressions for the rate function of PFS.

The results of our analysis confirm the conjecture made in Ryzhov (2016) that the asymptotic optimal sample allocation under unknown variances is different from that under known variances. In addition, we show that the limiting sampling ratios of any two suboptimal alternatives produced by the EI algorithm nearly match the asymptotic optimal sample allocation in the unknown-variance case.

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