Proceedings of the 2022 Winter Simulation Conference
B. Feng, G. Pedrielli, Y. Peng, S. Shashaani, E. Song, C.G. Corlu, L.H. Lee, E.P. Chew, T. Roeder, and P. Lendermann, eds.

ESTIMATING CONFIDENCE REGIONS FOR DISTORTION RISK MEASURES AND THEIR GRADIENTS

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ABSTRACT

This article constructs confidence regions (CRs) of distortion risk measures and their gradients at different risk levels based on replicate samples obtained from finite-horizon simulations. The CRs are constructed by batching and sectioning methods which partition the sample into nonoverlapping batches. Preliminary numerical results show that the estimated coverage rates of the CRs constructed are close to the nominal values.

1 INTRODUCTION

An important class of risk measures widely used in finance and economics are distortion risk measures (DRMs) that take the form

$$\vartheta(\alpha) = \int_0^\infty \omega(P(Z > z); \alpha) dz, \tag{1}$$

where Z is typically a nonnegative random variable with cumulative distribution function (c.d.f.) $F(z) = P(Z \le z)$, probability density function (p.d.f.) $f(z)$ that is positive and has derivative $\frac{d}{dz}f(z)$ at every $z \in \mathbb{R}^+$ (please refer to Glynn et al. (2021) to see the more-general case allowing negative Z), and the function ω with parameter α is called the distortion function that is nondecreasing and satisfies $\omega(0;\alpha) = 0$ and $\omega(1;\alpha) = 1$ for all α. Suppose the random variable Z is the output of a stochastic model: $Z = g(X;\theta)$ for some function $g : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$, where $X \in \mathbb{R}^m$ is the input random variable and $\theta \in \mathbb{R}^k$ is the decision parameter. Notice that we allow not only the dependence of $Z = g(X;\theta)$ on θ to be affected by θ in the distribution of input random variables, but also the structural dependence on θ in the function g. When $\omega(y;\alpha)$ is left-continuous with respect to y, the expression of the DRM in (1) can also be rewritten as

$$\vartheta(\alpha) = -\int_0^1 F^{-1}(y,\theta) d\tilde{\omega}_\alpha(y), \tag{2}$$

where $\tilde{\omega}_\alpha(y) = \omega(1-y;\alpha)$ (Dhaene et al. 2012, Theorem 6). The well-known risk measures value at risk (VaR), also known as a quantile, and conditional value at risk (CVaR) at level α are special cases of the distortion risk measures, where $\omega(y;\alpha) = 1(y > 1-\alpha)$ ($1(\cdot)$ denotes the indicator function) and $\omega(y;\alpha) = \min\{y/(1-\alpha), 1\}$, respectively.

Previous studies discussed the estimation of $\vartheta(\alpha)$ as well as gradients of $\vartheta(\alpha)$ with respect to risk level α (Gourieroux and Liu 2006) or parameter θ in the distribution function of Z (Glynn et al. 2021), i.e., the estimation of $\partial\vartheta/\partial\alpha$ or $\partial\vartheta/\partial\theta$. A confidence interval (CI) can be constructed of DRMs and the gradients of distortion risk measures (GDRMs) at one risk level α. However, comprehensive risk analysis requires the joint consideration for risk measures at various levels, so we may be interested in estimating not only the DRM at one risk level but also simultaneously estimating a vector of DRMs or GDRMs at risk measures $\alpha_1,\ldots,\alpha_d$. In addition, we may also wish to simultaneously estimate the GDRMs with respect to different parameters $\theta_1,\ldots,\theta_k$. Simultaneous estimation of distortion risk measures or their gradients corresponding to either different risk levels $\alpha_1,\ldots,\alpha_d$ or with respect to different parameters $\theta_1,\ldots,\theta_k$ requires constructing a confidence region (CR) to measure the accuracy and precision of the vector of estimates. Specifically, when we estimate a DRM at different risk levels, we wish to construct a region $\mathscr{R}(n,\beta) \subset \mathbb{R}^d$ based on a dataset consisting of n independent and identically distributed (i.i.d.) simulation responses $Z_u : u = 1,\ldots,n$ such that $\lim_{n\to\infty} \Pr[(\vartheta(\alpha_1),\ldots,\vartheta(\alpha_d)) \in \mathscr{R}(n,\beta)] = 1-\beta$; and when we estimate the respective GDRM with respect to θ_ℓ at different risk levels, we wish to construct a region $\mathscr{R}(n,\beta) \subset \mathbb{R}^d$ such that $\lim_{n\to\infty} \Pr[(\nabla_\ell\vartheta(\alpha_1),\ldots,\nabla_\ell\vartheta(\alpha_d)) \in \mathscr{R}(n,\beta)] = 1-\beta$. Then when we estimate the GDRM with respect to different parameters $\theta_1,\ldots,\theta_k$, we aim at constructing a region $\mathscr{R}(n,\beta) \subset \mathbb{R}^k$ such that $\lim_{n\to\infty} \Pr[\nabla\vartheta(\alpha) \in \mathscr{R}(n,\beta)] = 1-\beta$ for given confidence coefficient $\beta \in (0,1)$, where $\nabla_\ell\vartheta(\alpha) \doteq \partial\vartheta(\alpha)/\partial\theta_\ell$ and $\nabla\vartheta(\alpha) \doteq \big(\nabla_1\vartheta(\alpha),\ldots,\nabla_k\vartheta(\alpha)\big)$.

In this paper, we discuss CR construction for distortion risk measures and their gradients with respect to different parameters $\theta_1,\ldots,\theta_k$ at different risk levels $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_d < 1$, in an unified framework via batching and sectioning methods. GDRMs with respect to a given θ_ℓ at different risk levels $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_d < 1$, and GDRMs with respect to different $\theta_1,\ldots,\theta_k$ at a fixed risk level α_h are special cases with $k = 1$ and $d = 1$, respectively. Batching and sectioning methods have been studied in the literature to construct the CIs for quantiles (e.g., Nakayama 2014, Dong and Nakayama 2017), and CRs for a vector of quantiles (e.g., Lei et al. 2020, Glynn et al. 2021). The batching method divides the n outputs into $b \geq 2$ batches, each of size $m = n/b$, and computes the DRM or GDRM estimator from each batch. Then the sample mean and sample variance of the estimates across the batches are used to construct a CR for the DRM and GDRM. Similar to batching, the sectioning method also divides the outputs into $b \geq 2$ batches, but it replaces the average batched estimator with the estimator using all samples (Asmussen and Glynn 2007, Dong and Nakayama 2017).

The construction of CRs for DRMs or GDRMs is related to two streams of literature: gradient estimation and CR construction. Gradient estimation or sensitivity analysis is an important area in stochastic optimization. The information provided by the gradient of distortion risk measures is useful for selecting an appropriate risk management strategy (Gourieroux and Liu 2006). Sensitivity estimation for VaRs and CVaRs has been widely discussed in the literature (Hong 2009, Liu and Hong 2009, Fu et al. 2009, Lei et al. 2018). Peng et al. (2017) developed central limit theorems for the quantile sensitivity estimators in Liu and Hong (2009), Fu et al. (2009) and Lei et al. (2018). Further, sensitivity estimation for distortion risk measures more general than VaR and CVaR have been studied in Gourieroux and Liu (2006), Cao and Wan (2017) and Glynn et al. (2021). Gourieroux and Liu (2006) studied the gradient estimation of distortion risk measures with respect to the parameters in the distortion function and established a central limit theorem for the estimators. Cao and Wan (2017) developed a gradient estimator for distortion risk measures, but they did not study the asymptotic properties of the proposed estimator. Glynn et al. (2021) provided a gradient estimator for distortion risk measures with respect to the parameters in the underlying stochastic models and established a central limit theorem.

This study is also closely related to the literature on CR construction for distortion risk measures or their gradients. Lei et al. (2020) studied the construction of CRs for the estimated quantiles at different risk levels $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_d < 1$ by generalized likelihood ratio (GLR) methods as well as methods based on batching and sectioning methods in finite-horizon simulation. Lei et al. (2022) further extend the CRs for quantiles to steady-state simulation. Numerical results in Lei et al. (2020) and Lei et al. (2022) show the validity of GLR methods based on batching and sectioning for CRs construction for quantiles in both finite-horizon and steady-state simulations. In this regard, this study can be viewed as a generalization of Lei et al. (2020). Gourieroux and Liu (2006) derived the functional asymptotic properties of the functional distortion risk measures, but not CR construction for distortion risk measures at different risk levels $\alpha_i, i = 1, \cdots, d$. In this regard, this study proposes the batching and sectioning methods to build CRs of DRMs in Gourieroux and Liu (2006). Moreover, this study also considers the estimators of GDRMs with respect to parameters in the stochastic models and discusses CRs for GDRMs at different risk levels α_i; this extends Gourieroux and Liu (2006). Glynn et al. (2021) developed estimators of gradients of distortion risk measures with respect to parameters in the underlying stochastic models, i.e., θ in the function g. They also discussed the construction of CRs for the proposed sensitivities of distortion risk measures, along with sectioning-based method. However, Glynn et al. (2021) did not implement or provide any numerical experiments related to the CRs for distortion risk measures. In this regard, our paper fills a gap by numerically analyzing the performance of the sectioning method discussed in Glynn et al. (2021).

The rest of the paper is organized as follows. Section 2 describes the DRM and GDRM estimation problem and constructs CRs for DRMs and GDRMs based on batching and sectioning methods. The GLR method is used in this section towards the estimation of GDRMs. Section 3 evaluates the efficiency of the proposed methodology using a single example. Finally, Section 4 offers some concluding remarks.

2 ESTIMATORS AND CONFIDENCE REGIONS

Let $q(y) = F^{-1}(y;\theta) \doteq \inf\{q : F(q;\theta) \geq y\}$ be the quantile at probability level y for the random variable Z. If one can obtain a quantile estimator $\hat{q}_n(y)$, then the distortion risk measure can be estimated via

$$\tilde{\zeta}_n(\alpha) = -\int_{[0,1]} \hat{q}_n(y) d\tilde{\omega}_\alpha(y).$$

In the finite-horizon case where the data Z_i are obtained from n independent replications, $q(y)$ can be estimated by $\hat{q}_n(y) \doteq Z_{(\lceil ny \rceil, n)}$, where $Z_{(1,n)} \leq \cdots \leq Z_{(n,n)}$ are the respective order statistics and $\lceil \cdot \rceil$ denotes the ceiling function. As a result, Gourieroux and Liu (2006) derived the following estimator of the distortion risk measure,

$$\zeta_n(\alpha) = -\sum_{i=1}^{n} Z_{(i,n)} \cdot \left[\tilde{\omega}_\alpha\left(\frac{i-1}{n}\right) - \tilde{\omega}_\alpha\left(\frac{i}{n}\right) \right],$$

which is a linear combination of the order statistics $Z_{(i,n)}$.

Based on the expression of distortion risk measure in (2), and assuming the gradient and integral can be interchanged, Glynn et al. (2021) derived the expression of $\nabla\vartheta(\theta)$, i.e., the gradient of the distortion risk measure, as:

$$\nabla\vartheta(\alpha) = -\int_{[0,1]} \nabla_\theta F^{-1}(y;\theta)\tilde{\omega}_\alpha(dy). \tag{3}$$

The interchange of gradient and integration is typically justified by the condition for applying the dominated convergence theorem. If $F(q(y);\theta)$ is continuous at each $q(y) \in \mathbb{R}$, then $F(q(y);\theta) = y$ for each $y \in (0,1)$. Denote $\nabla_\ell F^{-1}(y;\theta) \doteq \partial F^{-1}(y;\theta)/\partial\theta_\ell, 1 \leq \ell \leq k$, as the gradient of the quantile with respect to a specific θ_ℓ, and $\nabla_\theta F^{-1}(y;\theta) \doteq \left(\frac{\partial F^{-1}(y;\theta)}{\partial\theta_1}, \ldots, \frac{\partial F^{-1}(y;\theta)}{\partial\theta_k}\right)$ as the vector of the gradients of the quantile. The estimation of the gradient of quantiles has been widely discussed in the literature. For non-batched estimators, Fu et al. (2009) proposed a Conditional Monte Carlo (CMC) estimator, Liu and Hong (2009) developed a kernel-based estimator, and Lei et al. (2018) derived an estimator based on GLR method. Peng et al. (2017) discussed the asymptotic properties of these estimators (CMC, kernel-based, and GLR estimators), and presented these three (non-batched) quantile gradient estimators of $\nabla_\ell F^{-1}(y;\theta), 1 \le \ell \le k$, in the form

$$\hat{D}_n^\ell(y) = \frac{\sum_{j=1}^n \Phi_1^\ell(X_j, Z_{(\lceil yn\rceil,n)})}{\sum_{j=1}^n \Phi_2(X_j, Z_{(\lceil yn\rceil,n)})},$$

where Φ_1^ℓ and Φ_2 are measurable functions that satisfy different requirements for different quantile sensitivity estimators outlined in Peng et al. (2017), and the gradient of distortion risk measure can be estimated via

$$\delta_n^\ell(\alpha) = -\int_{[0,1]} \hat{D}_n^\ell(y)\tilde{\omega}_\alpha(dy).$$

Let $\delta_n(\alpha) \doteq \big(\delta_n^1(\alpha),\ldots,\delta_n^k(\alpha)\big)$ be the vector of estimators. Denote $\hat{D}_n(y) \doteq \big(\hat{D}_n^1(y),\ldots,\hat{D}_n^k(y)\big)$ as the vector of quantile sensitivity estimators, $\hat{D}_{(i,n)}^\ell \doteq \frac{\sum_{j=1}^n \Phi_1^\ell(X_j,Z_{(i,n)})}{\sum_{j=1}^n \Phi_2(X_j,Z_{(i,n)})}$ as the gradients of quantiles with respect to θ_ℓ at y for $y \in (\frac{i-1}{n},\frac{i}{n}]$, and $\hat{D}_{(i,n)} \doteq \big(\hat{D}_{(i,n)}^1,\ldots,\hat{D}_{(i,n)}^k\big)$. Then the GDRM can be estimated by

$$\hat{\delta}_n^\ell(\alpha) = -\sum_{i=1}^n \hat{D}_{(i,n)}^\ell \cdot \left[\tilde{\omega}_\alpha\big(\frac{i-1}{n}\big) - \tilde{\omega}_\alpha\big(\frac{i}{n}\big)\right].$$

Similar to DRM, the estimator of GDRM is a linear combination of the estimators of quantile sensitivity at different levels.

In the remainder of this paper, we focus on GLR quantile sensitivity estimators; see Peng et al. 2018 or Glynn et al. 2021. We assume the following technical conditions hold:

- **Regularity Condition (A.1)** The inverse function $g^{-1}(\cdot,x_{-1};\theta)$ of g with respect to the first argument exists for all $x_{-1} = (x_2,\ldots,x_m)$.
- **Regularity Condition (A.2)** There exists $\varepsilon > 0$ such that $|\big(\partial g(x;\theta)/\partial x_1\big)^{-1}| > \varepsilon$ for every $x \in \mathbb{R}^m$ and for every $\theta \in \mathbb{R}^k$.
- **Regularity Condition (A.3)** The partial derivatives $\frac{\partial g(x;\theta)}{\partial\theta_\ell}$, $\frac{\partial^2 g(x;\theta)}{\partial x_1^2}$, and $\frac{\partial^2 g(x;\theta)}{\partial\theta_\ell\partial x_1}$ exist for every $x \in \mathbb{R}^m$; every $\ell \in \{1,\ldots,k\}$; and for every $\theta \in \mathbb{R}^k$. Moreover, $E\left[\left|\frac{\partial^2 g}{\partial x_1^2}(x;\theta)\right|\right] < \infty$ for every $\theta \in \mathbb{R}^k$.
- **Regularity Condition (A.4)** Let h be the joint density of X, and h_1 be the marginal density of the first coordinate in X. Then

  $$\lim_{x_1\to\pm\infty} h_1(x_1;\theta) = 0 \quad \text{and} \quad 0 < h(x;\theta) < \infty$$

  for every $x \in \mathbb{R}^m$ and for every $\theta \in \mathbb{R}^k$.
- **Regularity Condition (A.5)** The first-order partial derivative $\frac{\partial h(x;\theta)}{\partial\theta_\ell}$ exist for every $x \in \mathbb{R}^m$; every $\ell \in \{1,\ldots,k\}$; and for every $\theta \in \mathbb{R}^k$.

Conditions (A.1) and (A.2) can justify that $g^{-1}(\cdot,x_{-1};\theta)$ is globally Lipchitz continuous with respect to the first argument. When g is a linear function of x, $\partial g(x;\theta)/\partial x_1$ is a constant and $\partial^2 g(x;\theta)/\partial x_1^2$ is zero, so condition (A.2) and (A.3) hold. Condition (A.4) holds for most distributions supported on the whole space, i.e., the normal distributions. Condition (A.5) also satisfies for most distributions.

Let $\Phi(x,q) \doteq \big(\Phi_1^1(x,q),\ldots,\Phi_1^k(x,q),\Phi_2(x,q)\big)$. From Glynn et al. (2021), we know that under regularity conditions (A.1)–(A.4), we can write $\Phi(x,q) = \Psi(x)\cdot 1\{g(x;\theta) \le q\}$, where $\Psi(x) \doteq \big(\Psi_1^1(x),\ldots,\Psi_1^k(x),\Psi_2(x)\big)$,

$$\Psi_1^\ell(x) = \frac{\partial \ln h(x;\theta)}{\partial\theta_\ell} - \left(\frac{\partial g(x;\theta)}{\partial x_1}\right)^{-1} \times \left[\frac{\partial g(x;\theta)}{\partial\theta_\ell} \times \left(\frac{\partial \ln h(x;\theta)}{\partial x_1} - \frac{\partial^2 g(x;\theta)}{\partial x_1^2} \times \left(\frac{\partial g(x;\theta)}{\partial x_1}\right)^{-1}\right) + \frac{\partial^2 g(x;\theta)}{\partial\theta_\ell\partial x_1}\right], \tag{4}$$

and

$$\Psi_2(x) = \left(\frac{\partial g(x;\theta)}{\partial x_1}\right)^{-1} \times \left[\frac{\partial \ln h(x;\theta)}{\partial x_1} - \frac{\partial^2 g(x;\theta)}{\partial x_1^2}\left(\frac{\partial g(x;\theta)}{\partial x_1}\right)^{-1}\right]. \quad (5)$$

Gourieroux and Liu (2006) proved that for i.i.d. random samples from Z with probability density function (p.d.f.) f,

$$\sqrt{n}[\tilde{\zeta}_n(\alpha) - \vartheta(\alpha)] \Rightarrow \int_0^1 \frac{B(y)}{f(q(y))}\tilde{\omega}_\alpha(dy), \quad (6)$$

where $\Rightarrow$ denotes weak convergence and $B(\cdot)$ is a standard Brownian bridge.

In order to establish the asymptotic properties of $\delta_n(\alpha)$, we let

$$\overline{\Phi}_n(\cdot) = \frac{1}{n}\sum_{j=1}^{n}\big(\Phi_1^1(X_j,\cdot),\ldots,\Phi_1^k(X_j,\cdot),\Phi_2(X_j,\cdot)\big),$$

and set $\gamma_n = \varphi(\overline{\Phi}_n(\hat{q}_n))$ for some deterministic function $\varphi : \mathbb{R}^{k+1} \to \mathbb{R}^{kd}$. Clearly, $\delta_n(\alpha)$ is a special case of γ_n with $\varphi = (\varphi_1,\ldots,\varphi_d)$, where $\varphi_i = \int_{[0,1]} \phi(\overline{\Phi}_n(\hat{q}_n(y)))\tilde{\omega}_{\alpha_i}(dy)$, and $\phi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ with $\phi(x_1,\ldots,x_{k+1}) = \big(x_1/x_{k+1},\ldots,x_k/x_{k+1}\big)$ is continuously differentiable with $k\times(k+1)$ Jacobian matrix $J\phi(\cdot)$. Denote

$$P\Phi(\cdot) \doteq \Big(\int \Phi_1^1(x,\cdot)P(dx),\ldots,\int \Phi_1^k(x,\cdot)P(dx),\int \Phi_2(x,\cdot)P(dx)\Big),$$

and assume that $P\Phi(\cdot)$ is continuously differentiable on $\mathbb{R}$ with derivative $(P\Phi)'$. Theorem 2 in Glynn et al. (2021) implies the following result.

Theorem 1 Assume there exists an $\mathbb{R}^{k+2}$-valued continuous-path Gaussian process $G' = (G_1'(z), G_2'(z))$ such that

$$n^{1/2}\big(\overline{\Phi}_n(\cdot) - P\Phi(\cdot), \hat{q}_n(\cdot) - q(\cdot)\big) \Rightarrow (\tilde{G}_1(\cdot), \tilde{G}_2(\cdot)), \text{ as } n\to\infty,$$

where $\Rightarrow$ denotes weak convergence. Then

$$n^{1/2}\bigg(\gamma_n - \varphi(P\Phi(q))\bigg) \Rightarrow (G_1,\ldots,G_k), \text{ as } n\to\infty, \quad (7)$$

where $G_i = \int_{[0,1]}(J\phi)((P\Phi(q(y))))\Big[\tilde{G}_1(q(y)) + (P\Phi)'(q(y))\cdot\tilde{G}_2(q(y))\Big]\omega_{\alpha_i}(dy),\ i = 1,\ldots,k$ are $\mathbb{R}^d$-valued continuous-path Gaussian processes.

A class $\mathscr{F}$ of measurable functions is called P-Donsker if the sequence of processes $\mathbb{G}_n\psi : \psi \in \mathscr{F}$ converges in distribution to a tight limit process in the space $\ell^\infty(\mathscr{F})$ where $\mathbb{G}_n\psi \doteq \frac{\sum_{i=1}^n\big(\psi(X_i)-P\psi\big)}{\sqrt{n}}$, $\ell^\infty(\mathscr{F})$ is a collection of bounded real-valued functionals on $\mathscr{F}$, equipped with the uniform (sup) norm $\|\cdot\|_{\mathscr{F}}$ (p. 269 in Van der Vaart 2000). From Peng et al. (2017), it is known that the function classes $\{\Phi_1^\ell(x,q),\Phi_2(x,q) : q\in\mathbb{R}\}$ is P-Donsker if

$$E[(\Psi_1^\ell(X))^2] < \infty\ ,\ell = 1,\ldots,k, \quad \text{and} \quad E[(\Psi_2(X))^2] < \infty. \quad (8)$$

Therefore, there exists an $\mathbb{R}^{k+1}$-valued continuous-path Gaussian process $\tilde{G}_1(z)$ such that $n^{1/2}(\bar{\Phi}(\cdot) - P\Phi(\cdot)) \Rightarrow \tilde{G}_1(z)$ if (8) is satisfied. In addition, if the p.d.f $f(\cdot)$ of Z is differentiable and $f(q(y)) > 0$ for $y\in(0,1)$, then there exists an $\mathbb{R}$-valued continuous-path Gaussian process $\tilde{G}_2(z)$ such that $n^{1/2}\big(\hat{q}_n(\cdot) - q(\cdot)\big) \Rightarrow \tilde{G}_2(\cdot)$ (Serfling 1980 §2.3.3, Theorem B).

However, the asymptotic variance and covariance of the estimated DRMs and GDRMs is complicated to derive and estimate. Instead, we use batching- and sectioning-based methods to build CRs for DRMs and GDRMs. For given $b \geq 2$, one forms b nonoverlapping batches of simulation responses, each of size m $(n = bm)$. In addition to the estimators ζ_n or δ_n^ℓ, we also compute the batched estimators

$$\zeta_{j,m}(\alpha_h) = -\sum_{i=(j-1)m+1}^{jm} Z_{(i,m)} \cdot \left[\tilde{\omega}\big(\frac{i-1}{m}, \alpha_h\big) - \tilde{\omega}\big(\frac{i}{m}, \alpha_h\big)\right],$$

and

$$\delta_{j,m}^\ell(\alpha_h) = -\sum_{i=(j-1)m+1}^{jm} \hat{D}_{(i,m)}^\ell \cdot \left[\tilde{\omega}\big(\frac{i-1}{m}, \alpha_h\big) - \tilde{\omega}\big(\frac{i}{m}, \alpha_h\big)\right],$$

for $j = 1, ..., b$, $h = 1, ..., d$, and $\ell = 1, ..., k$.

Denote

$$\zeta_{j,m}(\alpha) \doteq \big(\zeta_{j,m}(\alpha_1), ..., \zeta_{j,m}(\alpha_d)\big),\ \delta_{j,m}^\ell(\alpha) \doteq \big(\delta_{j,m}^\ell(\alpha_1), ..., \delta_{j,m}^\ell(\alpha_d)\big), \text{ and } \delta_{j,m}(\alpha) \doteq \big(\delta_{j,m}^1(\alpha), ..., \delta_{j,m}^k(\alpha)\big).$$

Note that $\delta_{j,m}^\ell(\alpha)$ is a vector on $\mathbb{R}^d$ and $\delta_{j,m}(\alpha)$ is a vector on $\mathbb{R}^{kd}$. Then we compute the sample covariance matrices for $\zeta_{j,m}(\alpha)$ and $\delta_{j,m}(\alpha)$, namely,

$$S_\zeta(n) = \frac{1}{b-1} \sum_{i=1}^{b} \left[\zeta_{j,m}(\alpha) - \overline{\zeta}_{b,m}(\alpha)\right] \cdot \left[\zeta_{j,m}(\alpha) - \overline{\zeta}_{b,m}(\alpha)\right]^{\mathrm{T}}, \tag{9}$$

and

$$S_\delta(n) = \frac{1}{b-1} \sum_{i=1}^{b} \left[\delta_{j,m}(\alpha) - \overline{\delta}_{b,m}(\alpha)\right] \cdot \left[\delta_{j,m}(\alpha) - \overline{\delta}_{b,m}(\alpha)\right]^{\mathrm{T}}, \tag{10}$$

where

$$\overline{\zeta}_{b,m}(\alpha) = \frac{1}{b} \sum_{j=1}^{b} \zeta_{j,m}(\alpha) \quad \text{and} \quad \overline{\delta}_{b,m}(\alpha) = \frac{1}{b} \sum_{j=1}^{b} \delta_{j,m}(\alpha)$$

are the batched estimators of $\vartheta(\alpha)$ and $\nabla\vartheta(\alpha)$, respectively. The next proposition establishes the asymptotic validity of the CRs for $\vartheta(\alpha)$ and $\nabla\vartheta(\alpha)$ based on batching.

Proposition 1 Fix $b \geq 2$,
(i) If $b > d$ and the covariance matrix of $\left\{\int_0^1 \frac{B(y)}{f(q(y))} \tilde{\omega}_{\alpha_i}(dy), i = 1, ..., d\right\}$ in (6) is invertible almost surely (a.s.), then

$$b\big[\overline{\zeta}_{b,m}(\alpha) - \vartheta(\alpha)\big] S_\zeta^{-1}(n) \big[\overline{\zeta}_{b,m}(\alpha) - \vartheta(\alpha)\big]^{\mathrm{T}} \Rightarrow \frac{d(b-1)}{(b-d)} F_{d,b-d} \text{ as } m \to \infty,$$

where F_{v_1,v_2} is denotes a r.v. from Snedecor's F distribution with v_1 and v_2 degrees of freedom.
(ii) If $b > kd$ and the covariance matrix of $\big(G_1, ..., G_d\big)$ in (7) is invertible a.s., then

$$b\big[\overline{\delta}_{b,m}(\alpha) - \nabla\vartheta(\alpha)\big] S_\delta^{-1}(n) \big[\overline{\delta}_{b,m}(\alpha) - \nabla\vartheta(\alpha)\big]^{\mathrm{T}} \Rightarrow \frac{kd(b-1)}{(b-kd)} F_{(kd,b-kd)} \text{ as } m \to \infty.$$

Proof. For $j = 1, \ldots, b$, let $Q_j \doteq m^{1/2}\big[\zeta_{j,m} - \vartheta\big]$ so that the $\{Q_j\}$ are i.i.d. For each j, (6) ensures that $Q_j \Rightarrow Y_j \sim N_d(0, \Sigma)$ as $m \to \infty$. Let $\overline{Q}_b = b^{-1} \sum_{j=1}^{b} Q_j$ and $S_Q = (b-1)^{-1} \sum_{j=1}^{b} \big(Q_j - \overline{Q}_b\big)^{\mathrm{T}} \big(Q_j - \overline{Q}_b\big)$, respectively, denote the sample mean and sample covariance matrix of the $\{Q_j\}$. Similarly, we define $\overline{Y}_b = b^{-1} \sum_{j=1}^{b} Y_j$, and $S_Y = (b-1)^{-1} \sum_{j=1}^{b} \big(Y_j - \overline{Y}_b\big)^{\mathrm{T}} \big(Y_j - \overline{Y}_b\big)$. We note that the mapping $Q \mapsto b\overline{Q}_b S_Q^{-1} \overline{Q}_b^{\mathrm{T}}$ is continuous at each point $Q \in \mathbb{R}^b$ such that $\det(S_Q) > 0$. Since Σ is invertible a.s., i.e., $\det(\Sigma) > 0$,

$\det(S_Q) > 0$ with probability 1 (Dykstra 1970). Because $Q \Rightarrow Y$ as $m \to \infty$, the continuous-mapping theorem (Whitt 2002, Theorem 3.4.3) implies

$$b\big[\overline{\zeta}_{b,m}(\alpha) - \vartheta(\alpha)\big] S_\zeta^{-1}(n) \big[\overline{\zeta}_{b,m}(\alpha) - \vartheta(\alpha)\big]^{\mathrm{T}} \Rightarrow b\overline{Y}_b S_Y^{-1} \overline{Y}_b^{\mathrm{T}} \text{ as } m \to \infty.$$

Finally by Corollary 5.2.1 in Anderson (2003), we have

$$b\overline{Y}_b S_Y^{-1} \overline{Y}_b^{\mathrm{T}} \sim \frac{d(b-1)}{(b-d)} F_{d,b-d}.$$

Similarly, we can prove that

$$b\big[\overline{\delta}_{b,m}(\alpha) - \nabla\vartheta(\alpha)\big] S_\delta^{-1}(n) \big[\overline{\delta}_{b,m}(\alpha) - \nabla\vartheta(\alpha)\big]^{\mathrm{T}} \Rightarrow \frac{kd(b-1)}{(b-kd)} F_{kd,b-kd} \text{ as } m \to \infty.$$

□

Therefore, as $m \to \infty$, an asymptotically valid $100(1-\beta)\%$ CR for ζ_n and δ_n based on the batching method is given by

$$\mathscr{R}_1(n,\beta) = \left\{ \vartheta(\alpha) \in \mathbb{R}^d : b\big[\overline{\zeta}_{b,m}(\alpha) - \vartheta(\alpha)\big] S_\zeta^{-1}(n) \big[\overline{\zeta}_{b,m}(\alpha) - \vartheta(\alpha)\big]^{\mathrm{T}} \leq \frac{d(b-1)}{(b-d)} F_{d,b-d,\beta} \right\}, \tag{11}$$

and

$$\mathscr{R}_2(n,\beta) = \left\{ \nabla\vartheta(\alpha) \in \mathbb{R}^{kd} : b\big[\overline{\delta}_{b,m}(\alpha) - \nabla\vartheta(\alpha)\big] S_\delta^{-1}(n) \big[\overline{\delta}_{b,m}(\alpha) - \nabla\vartheta(\alpha)\big]^{\mathrm{T}} \leq \frac{kd(b-1)}{(b-kd)} F_{kd,b-kd,\beta} \right\}, \tag{12}$$

where $F_{\nu_1,\nu_2,\gamma}$ is the γ-quantile of F_{ν_1,ν_2} distribution. For the sectioning method, the samples are also split into b batches with $b > d$ for DRM and $b > kd$ for GDRM, respectively, but the estimators $\zeta_n(\alpha)$ and $\delta_n(\alpha)$ of $\vartheta(\alpha)$ and $\nabla\vartheta(\alpha)$, respectively are obtained from the full sample. The sample covariance matrices are the same as those used in the batching method; see equations (9) and (10).

Proposition 2 Fix $b \geq 2$.
(i) If $b > d$ and the covariance matrix of $\left\{ \int_0^1 \frac{B(y)}{f(q(y))} \tilde{\omega}_{\alpha_i}(dy), i = 1,\ldots,d \right\}$ in (6) is invertible a.s., then

$$b\big(\zeta_n(\alpha) - \vartheta(\alpha)\big) S_\zeta^{-1}(n) \big(\zeta_n(\alpha) - \vartheta(\alpha)\big)^{\mathrm{T}} \Rightarrow \frac{d(b-1)}{(b-d)} F_{d,b-d} \text{ as } m \to \infty.$$

(ii) If $b > kd$ and the covariance matrix of $\big(G_1,\ldots,G_d\big)$ in (7) is invertible a.s., then

$$b\big[\delta_n(\alpha) - \nabla\vartheta(\alpha)\big] S_\delta^{-1}(n) \big[\delta_n(\alpha) - \nabla\vartheta(\alpha)\big]^{\mathrm{T}} \Rightarrow \frac{kd(b-1)}{(b-kd)} F_{(kd,b-kd)} \text{ as } m \to \infty.$$

The proof of Proposition 2 will be presented in a forthcoming full paper. Briefly, in order to prove part (i), we plan to show that $\mathbb{Q}_{BM} - \mathbb{Q}_{SBM} \Rightarrow 0$, as $m \to \infty$, where

$$\mathbb{Q}_{BM} \doteq b\big[\overline{\zeta}_{b,m}(\alpha) - \vartheta(\alpha)\big] S_\zeta^{-1}(n) \big[\overline{\zeta}_{b,m}(\alpha) - \vartheta(\alpha)\big]^{\mathrm{T}} \text{ and } \mathbb{Q}_{SBM} = b\big[\zeta_n(\alpha) - \vartheta(\alpha)\big] S_\zeta^{-1}(n) \big[\zeta_n(\alpha) - \vartheta(\alpha)\big]^{\mathrm{T}},$$

and then proceed based on Proposition 1. The concepts apply to the proof of part (ii). The remainder of the proof of Proposition 2 will exploit the delta method (Theorem 3.1 in Van der Vaart (2000)) and Proposition 2 in Lei et al. (2022).

Therefore, as $m \to \infty$, an asymptotically valid $100(1-\beta)\%$ CR for ζ_n and δ_n based on the sectioning method is given by

$$\mathscr{R}_3(n,\beta) = \left\{ \vartheta(\alpha) \in \mathbb{R}^d : b\big[\zeta_n(\alpha) - \vartheta(\alpha)\big] S_\zeta^{-1}(n) \big[\zeta_n(\alpha) - \vartheta(\alpha)\big]^{\mathrm{T}} \le \frac{d(b-1)}{(b-d)} F_{d,b-d,\beta} \right\}, \tag{13}$$

and

$$\mathscr{R}_4(n,\beta) = \left\{ \nabla\vartheta(\alpha) \in \mathbb{R}^{kd} : b\big[\delta_n(\alpha) - \nabla\vartheta(\alpha)\big] S_\delta^{-1}(n) \big[\delta_n(\alpha) - \nabla\vartheta(\alpha)\big]^{\mathrm{T}} \le \frac{kd(b-1)}{(b-kd)} F_{kd,b-kd,\beta} \right\}. \tag{14}$$

Propositions 1 and 2 justify the asymptotic validity of CRs for DRMs and GDRMs with respect to different parameters i.e., $\theta_\ell, \ell = 1,...,k$ at different risk levels $\alpha_1, ...\alpha_d$ via batching and sectioning based methods, respectively. If one is concerned about the gradients of distortion risk measures with respect to a specific θ_ℓ, $1 \le \ell \le k$ at different risk levels $\alpha_1, ..., \alpha_d$, then one can set $k = 1$ in $\mathscr{R}_2(n,\beta)$ or $\mathscr{R}_4(n,\beta)$ to obtain the CRs of GDRMs, whilst if one wishes to estimate the gradients of distortion risk measures with respect to different $\theta_\ell, \ell = 1,...,k$ at a fixed risk level α_h, $1 \le h \le d$, then one can set $d = 1$ in $\mathscr{R}_2(n,\beta)$ or $\mathscr{R}_4(n,\beta)$ to obtain the CRs for the GDRMs.

3 NUMERICAL RESULTS

In this section, we use three examples to test the performance of the batching method (BM) and sectioning method (SM) with regard to the CR construction for DRMs and GDRMs at different risk levels (Example 1 and Example 2) and with respect to different parameters (Examples 3). We estimate the coverage rate of $100(1-\beta)\%$ CRs by the proportion of the constructed CRs that contain the true vector of DRMs or GDRMs from 10^4 independent trials. In each example we divide the 10^4 trials into 100 replications with 100 trials in each replication and report the point estimates and standard errors of the coverage rates.

Example 1 Consider the case where $X_i \sim N(0,\sigma_i^2), i = 1,...,9$ are independent, and $Y = \sum_{i=1}^9 \theta_i X_i$. We are interested in estimating CVaR at different risk level α, and constructing CRs for CVaRs via BM and SM. We know that $Y \sim N(0, \sum_{i=1}^9 \theta_i^2\sigma_i^2)$, so the true value of the CVaR at risk level α is $\mathrm{CVaR}_\alpha = \sigma_Y^2 \cdot \phi(q(\alpha);0,\sigma_Y)/(1-p)$ where σ_Y is the standard deviation of Y, $\phi(x;a,b)$ is the p.d.f of normal distribution with mean a and variance b, and $q(\alpha)$ is the quantile of Y at level α; i.e., $q(\alpha) = \sigma_Y z_\alpha$, where z_α is the quantile of standard normal distribution at level α. The results are obtained at $\boldsymbol{\theta} = [1,2,4,1,2,1,2,2,4]$ and $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_9) = [1,2,4,8,8,9,10,11,12]$.

The probabilities α_i are spaced uniformly: $\alpha_i = i/(d+1)$, $i = 1,\dots,d$. Asmussen and Glynn (2007) suggest choosing $b \le 30$ for both BM and SM in CI construction, and we have to choose b such that $b > d$ for CR construction. The results in Table 1 show cases where $d = 4$ and $d = 9$. For large n, the estimated coverages of all the CRs are close to the target coverage rate for both BM and SM, demonstrating their asymptotic validity. Clearly, for fixed relatively small sample sizes, the gap between the estimated coverage rates and nominal rates for BM- and SM-based CRs widens as d increases. The experimental results displayed in Table 1 also indicate significant advantages for the SM-based method with regard to CR construction for CVaR compared to BM, especially when the sample size is small and the dimension of the probability vector is relatively large.

Example 2 This example has the same setting as Example 1, but we are interested in the experimental evaluation of the CRs for the vector of GLR estimators of the gradients of quantiles with respect to θ_1 obtained by the two methods (BM and SM). Therefore, this example corresponds to the case where $k = 1$ in CR construction of GDRMs. The derivatives in the estimators (4) and (5) are given by $\frac{\partial g(x)}{\partial x_1} = \theta_1$, $\frac{\partial^2 g(x)}{\partial x_1^2} = 0$, $\frac{\partial \ln h(X;\theta)}{\partial x_1} = -\frac{x_1}{\sigma_1^2}$, $\frac{\partial \ln h(X;\theta)}{\partial \theta_1} = 0$, and $\frac{\partial^2 g(X;\theta)}{\partial \theta_1 \partial x_1} = 1$. The probabilities α_i are spaced uniformly as in Example 1 and the true values of the gradients of quantiles at level $\alpha_i, i = 1,...,d$ are $\frac{\partial \sigma_Y}{\partial \theta} \cdot z_{\alpha_i} = \frac{\theta_1\sigma_1}{\sigma_Y} \cdot z_{\alpha_i}$.

Table 1: Coverage rates for confidence regions (means ± standard errors) based on 100 independent runs with 100 independent experiments in each run for the BM and SM based on Example 1.

n	$b=16$		$b=32$		$b=64$	
	BM	SM	BM	SM	BM	SM
$d=4,\ 1-\beta=0.9$						
2^{12}	0.829 ± 0.036	0.891 ±0.028	0.842 ± 0.033	0.902 ± 0.028	0.524 ± 0.051	0.888 ± 0.031
2^{14}	0.899 ± 0.029	0.904 ± 0.027	0.854 ± 0.035	0.898 ± 0.028	0.527 ± 0.049	0.869 ± 0.034
2^{16}	0.896 ± 0.029	0.900 ± 0.028	0.900 ± 0.026	0.903 ± 0.026	0.878 ± 0.029	0.902 ± 0.026
2^{18}	0.897 ± 0.034	0.898 ± 0.034	0.897 ± 0.032	0.898 ± 0.031	0.878 ± 0.031	0.897 ± 0.029
$d=9,\ 1-\beta=0.9$						
2^{12}	0.204 ± 0.038	0.859 ± 0.035	0.581 ± 0.050	0.92 ± 0.027	0 ± 0	0.656 ± 0.046
2^{14}	0.665 ± 0.042	0.886 ± 0.031	0.590 ± 0.047	0.891 ± 0.030	0 ± 0	0.745 ± 0.044
2^{16}	0.855 ± 0.036	0.894 ± 0.032	0.884 ± 0.034	0.897 ± 0.030	0.006 ± 0.033	0.817 ± 0.037
2^{18}	0.889 ± 0.035	0.901 ± 0.031	0.880 ± 0.032	0.897 ± 0.029	0.514 ± 0.049	0.871 ± 0.031

The experimental results are displayed in Table 2. For large n, the estimated coverages of all CRs are close to the true value, demonstrating their asymptotic validity. However, the experimental results displayed in Table 2 do not show significant advantages for the SM method compared to BM as in Example 1.

Table 2: Coverage rates for confidence regions (means ± standard errors) based on 100 independent runs with 100 independent experiments in each run for the BM and SM based on Example 2.

n	$b=16$		$b=32$		$b=64$	
	BM	SM	BM	SM	BM	SM
$d=4,\ 1-\beta=0.9$						
2^{12}	0.959 ± 0.019	0.480 ± 0.050	0.955 ± 0.021	0.474 ± 0.054	0.957 ± 0.017	0.478 ± 0.052
2^{14}	0.958 ± 0.018	0.603 ± 0.046	0.959 ± 0.018	0.601 ± 0.051	0.955 ± 0.021	0.598 ± 0.045
2^{16}	0.958 ± 0.020	0.839 ± 0.042	0.955 ± 0.021	0.830 ± 0.037	0.952 ± 0.020	0.833 ± 0.037
2^{18}	0.956 ± 0.019	0.978 ± 0.014	0.959 ± 0.017	0.983 ± 0.012	0.952 ± 0.020	0.984 ± 0.012
$d=9,\ 1-\beta=0.9$						
2^{12}	0.960 ± 0.019	0.307 ± 0.046	0.968 ± 0.018	0.317 ± 0.048	0.962 ± 0.021	0.332 ± 0.047
2^{14}	0.964 ± 0.018	0.438 ± 0.047	0.962 ± 0.019	0.452 ± 0.052	0.963 ± 0.020	0.467 ± 0.049
2^{16}	0.964 ± 0.019	0.691 ± 0.044	0.964 ± 0.016	0.716 ± 0.044	0.962 ± 0.018	0.724 ± 0.045
2^{18}	0.955 ± 0.021	0.901 ± 0.031	0.969 ± 0.014	0.921 ± 0.026	0.964 ± 0.018	0.925 ± 0.030

Example 3 This example has the same setting as Example 1, but we are interested in the experimental evaluation of the CRs for the vector of GLR estimators of the gradients of quantiles with respect to $\theta_\ell, \ell = 1, \ldots, 9$ at a fixed risk level obtained by the two methods (BM and SM). Therefore, this example corresponds to the case where $k = 9$ and $d = 1$ in CR construction of GDRMs. The derivatives in the estimators (4) and (5) are given by $\frac{\partial g(x)}{\partial x_1} = \theta_1$, $\frac{\partial^2 g(x)}{\partial x_1^2} = 0$, $\frac{\partial \ln h(X;\theta)}{\partial x_1} = -\frac{x_1}{\sigma_1^2}$, $\frac{\partial \ln h(X;\theta)}{\partial \theta_\ell} = 0$ for $\ell = 1, \ldots, 9$, $\frac{\partial^2 g(X;\theta)}{\partial \theta_1 \partial x_1} = 1$ and $\frac{\partial^2 g(X;\theta)}{\partial \theta_\ell \partial x_1} = 0$ for $\ell = 2, \ldots, 9$. The true values of the gradients of quantiles with respect to $\theta_i, i = 1, \ldots, k$ are $\frac{\partial \sigma_Y}{\partial \theta_i} \cdot z_\alpha = \frac{\theta_i \sigma_i}{\sigma_Y} \cdot z_\alpha$.

The experimental results are listed in Table 3. For large n, the estimated coverages of all CRs are close to the true value, demonstrating their asymptotic validity. When n is large enough, (e.g., $n = 2^{18}$), SM performs better when b is smaller. However, if k is large, b has to be significantly larger to achieve a high accuracy for the target coverage rates of CRs.

Table 3: Coverage rates for confidence regions (means $\pm$ standard errors) based on 100 independent runs with 100 independent experiments in each run for the BM and SM based on Example 3.

n	$b = 16$		$b = 32$		$b = 64$	
	BM	SM	BM	SM	BM	SM
$\alpha = 0.85,\ 1-\beta = 0.9$						
2^{12}	0.869 $\pm$ 0.033	0.931 $\pm$ 0.026	0.842 $\pm$ 0.037	0.911 $\pm$ 0.025	0.831 $\pm$ 0.032	0.921 $\pm$ 0.025
2^{14}	0.867 $\pm$ 0.033	0.919 $\pm$ 0.027	0.840 $\pm$ 0.035	0.914 $\pm$ 0.032	0.831 $\pm$ 0.037	0.843 $\pm$ 0.036
2^{16}	0.867 $\pm$ 0.032	0.913 $\pm$ 0.031	0.843 $\pm$ 0.041	0.894 $\pm$ 0.030	0.836 $\pm$ 0.038	0.823 $\pm$ 0.034
2^{18}	0.869 $\pm$ 0.038	0.904 $\pm$ 0.028	0.847 $\pm$ 0.039	0.899 $\pm$ 0.032	0.836 $\pm$ 0.041	0.857 $\pm$ 0.036
$\alpha = 0.95,\ 1-\beta = 0.9$						
2^{12}	0.923 $\pm$ 0.029	0.923 $\pm$ 0.028	0.912 $\pm$ 0.029	0.914 $\pm$ 0.029	0.883 $\pm$ 0.029	0.884 $\pm$ 0.032
2^{14}	0.925 $\pm$ 0.026	0.922 $\pm$ 0.028	0.921 $\pm$ 0.024	0.920 $\pm$ 0.027	0.919 $\pm$ 0.026	0.921 $\pm$ 0.029
2^{16}	0.920 $\pm$ 0.027	0.922 $\pm$ 0.025	0.914 $\pm$ 0.028	0.911 $\pm$ 0.028	0.853 $\pm$ 0.034	0.836 $\pm$ 0.037
2^{18}	0.905 $\pm$ 0.026	0.905 $\pm$ 0.029	0.888 $\pm$ 0.030	0.887 $\pm$ 0.032	0.829 $\pm$ 0.033	0.831 $\pm$ 0.034

4 CONCLUSION

In this article, we have proposed methods based on batching and sectioning for constructing CRs of distortion risk measures and gradients of risk measures at different risk levels, as well as for gradients of risk measures with respect to different parameters based on independent samples from finite-horizon simulations. The validity of the CRs based on batching method has been established under a set of sufficient conditions. Numerical experiments buttress the effectiveness of the proposed methods. Future work will focus on more complex settings and extensions to stationary output processes.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation of China (NSFC) under Grants 71901003, 71720107003, 71690232, 91846301, 71790615, and 71972019.

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