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HIGHER-ORDER COVERAGE ERROR ANALYSIS FOR BATCHING AND SECTIONING

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ABSTRACT

While batching and sectioning have been widely used in simulation, it is open regarding their higher-order coverage behaviors and whether one is better than the other in this regard. We develop techniques to obtain higher-order coverage errors for sectioning and batching via an Edgeworth-type expansion on t-statistics. Based on our expansion, we give insights on the effect of the number of batches on the coverage error. Moreover, we theoretically argue that none of batching or sectioning is uniformly better than the other in terms of coverage, but sectioning usually has a smaller coverage error when the number of batches is large. We also support our theoretical findings via numerical experiments.

1 INTRODUCTION

Sectioning and batching are widely used methods in simulation analysis. The basic idea of these methods is to divide the data into batches and quantify the variability of point estimates by suitably combining the batch estimates. They are especially useful tools to construct confidence intervals (CI) when the variance of the output is hard to compute, such as quantile (Nakayama 2014) whose variance estimation involves density estimation, and in serially dependent problems and steady-state estimation (Asmussen and Glynn 2007; Nakayama 2007).

While widely used, the detailed coverage behaviors of sectioning and batching are not well understood. To understand the statistical performances of these methods and to conduct comparisons, however, this question seems imminent. To put things in perspective, note that a good CI should have a small half width and coverage error. By construction, with the same choice of batch size, the CI half widths of sectioning and batching are equal. Therefore, their difference lies in the coverage errors. Nonetheless, under regularity conditions, both sectioning and batching lead to asymptotically exact CIs. Thus, both methods only have higher-order coverage errors, and it is these errors that can differ from each other.

There are very few studies on the higher-order coverage probabilities for sectioning and batching. The challenge is that the statistics used in sectioning and batching have an asymptotic t-distribution rather than a normal distribution, so conventional Edgeworth expansion cannot be directly applied. The most relevant result is the heuristic argument given in Nakayama (2014), which argued that since the estimator based on the whole empirical distribution has smaller bias, sectioning appears to lead to better coverage. Nakayama (2014) supported this claim with numerical results.

In this paper, we develop tools to study the higher-order expansion for the coverage probabilities of sectioning and batching. Under regularity conditions, we show that the coverage errors of sectioning and batching can be expanded as series of $n^{-1/2}$ where n is the data size in each batch. For a symmetric CI, we show that both methods have coverage errors of order $O(n^{-1})$. The coefficients in the expansion involve some integration which can not be explicitly calculated in general, but we provide examples where explicit calculation is possible and sufficient to draw some conclusions. In terms of methodology, our analysis utilizes Edgeworth expansion and Taylor expansion techniques combined with oddness and evenness arguments for functions. To the best of our knowledge, we are the first to study the higher-order coverage of sectioning and batching using this type of analysis.

Based on the expansion, we compare the coverage errors of sectioning and batching, and draw insights on the effect of the number of batches. From our analyses, we conclude that whether sectioning or batching has smaller higher-order coverage error depends on the problem parameters, so none of them is uniformly better. But for a fixed problem, when the number of batches is large, batching suffers from a significant bias and sectioning has better coverage.

Here are some additional results in our full paper which are not included in the WSC paper due to page limit: 1) We derive the coverage error expansion for sectioned jackknife which is another sectioning-based scheme that is known to be bias-corrected. 2) We provide a simulation scheme to estimate the coefficient of the n^{-1} error term in the expansion of coverage probability that works for any smooth function models. 3) We compare sectioning, batching, and sectioned jackknife using more general models and analyse the effect of the number of batches. Our results suggest that sectioned jackknife may not be the best when the number of batches is small, but when the number of batches is large, sectioned jackknife has the smallest coverage error, sectioning has over-coverage issues, while batching has under-coverage issues. Moreover, we observe that for each method, when the total sample size is fixed, the coverage error increases as the number of batches increases.

We briefly review the literature on sectioning and batching techniques. Pope (1995) analyzes the coverage error of sectioning using Edgeworth expansion, but it focuses on the case when the number of batches goes to infinity so that the problem statistic can be approximated by normal. This is different from our analysis for the t distribution approximation which is our key novelty and faced challenge. For the CI half width, Schmeiser (1982) shows that if the data size is large enough so that the non-normality of the batch estimators is negligible, then the expected half width would decrease as the number of batches increases, but the rate of decrease would become much slower when the number of batches is large.

2 SUMMARY OF THE MAIN RESULTS

Here is a summary of the main results:

1. Let K be the number of sections. Suppose that $K \geq r+2$ for some positive integer r. Then, under regularity conditions, the coverage error of sectioning, batching, and sectioned jackknife can be expanded as a series of $n^{-1/2}$ with residual $o(n^{-r/2})$. Moreover, for a symmetric CI (i.e., an interval whose center is the point estimation), the coverage error is of order $O(n^{-1})$.
2. The coefficients in the expansion are intricate analytically, but amenable to simulation. We provide an algorithm to estimate the coefficient of the n^{-1} error term. We prove that the algorithm gives an unbiased estimator for the coefficient.
3. When $K = 2, r = 2$, the expansion mentioned in Point 1 does not exist for some examples.
4. When the number of data in each section is fixed and $K \to \infty$, the coverage probability of batching goes to 0, the coverage probability of sectioning goes to a limit that is different from the nominal level, while the coverage probability of sectioned jackknife converges to the nominal level.

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