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ON THE CONVERGENCE OF OPTIMAL COMPUTING BUDGET ALLOCATION ALGORITHMS

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ABSTRACT

This paper considers a well-known ranking and selection (R&S) framework, called optimal computing budget allocation (OCBA). This framework includes a set of equations that optimally determine the number of samples allocated to each design in a finite design set. Sample allocations that satisfy these equations have been shown to be the asymptotic optimizer of the probability of correct selection (PCS) for the best design and the expected opportunity cost (EOC) if false selection occurs. In this paper, we analyze two popular OCBA algorithms and study their convergence rates, assuming known variances for samples of each design. It fills the gap of convergence analysis for algorithms that are developed based on the OCBA optimality equations. In addition, we propose modifications of the OCBA algorithms for cumulative regret, an objective commonly studied in machine learning, and derive their convergence rates. Last, the convergence behaviors of these algorithms are demonstrated using numerical examples.

1 INTRODUCTION

Discrete-event system (DES) simulation plays a prominent role in complex system analysis and decision making, e.g., for traffic control systems, manufacturing engineering, queueing models, supply chain management, communication network reliability, etc. In these applications, analytical models are rarely applicable due to their complex operation logic and dynamics; in contrast, DES can faithfully model the mechanism of the systems in detail, and thus becomes a powerful tool for evaluating the performance of these systems. Despite the rapid improvement of computing capacity, deep concern for simulation efficiency still remains. The reason is that given an insufficient simulation budget, the simulation costs are frequently higher than expected; in the meantime, an accurate DES performance estimator generally demands a large number of simulation replications due to the typically slow convergence of the estimator (Lee et al. 2010).

To this end, R&S techniques have been developed as an effective tool for smart budget allocation and efficient utilization of limited simulation resources. After conducting simulation, the budget allocation algorithm recommends an estimated best design based on the sample mean of each design, which is expected to be identical to the true best one, i.e., the design with the maximal (or minimal) mean performance. The key concern in the R&S problems is called the "exploration and exploitation" trade-off, that is, the budget allocation algorithm ought to provide more computing resources to designs that are likely to be the true best one, i.e., to exploit; in the meantime, the algorithms should also focus on the designs that we do not know much, i.e., to explore. These two goals must be well balanced for the best design to be correctly selected. For a broad review of the field R&S, see Fu (2014), Hong and Nelson (2009).

There are three popular streams of approaches in R&S. The indifference-zone (IZ) method aims to ensure the probability of correct selection (PCS) for the best design exceeding a pre-specified level P_*. It assumes that the difference of mean performance between the best design and each non-best one is at least δ_*, where δ_* denotes the minimum difference deserving discrimination (Kim and Nelson 2001; Nelson et al. 2001). Furthermore, Frazier (2014), Toscano-Palmerin and Frazier (2015) presented some IZ procedures whose tight lower bounds on PCS are exactly P_*'s in certain cases. It improves sampling efficiency and allows IZ to select the best with fewer simulation resources. The second approach is the expected value of information procedure (VIP). It applies the Bayesian framework to characterize the event of correct selection. It is intended to allocate the simulation budget in order to maximize the expected value of information obtained during the simulation process (Chick and Inoue 2001a; Chick and Inoue 2001b). Last is the optimal computing budget allocation (OCBA) method, which is a famous sequential budget allocation strategy to efficiently assign a limited simulation budget for the purpose of achieving a maximal PCS. It was first developed under contexts where simulation outputs of the designs follow independent normal distributions with known variance (Chen et al. 2000). Diverse simulation tests demonstrate the high efficiency of OCBA under not only normal underlying distributions but also many other non-normal circumstances (Branke et al. 2007; Gao and Gao 2016). Furthermore, Glynn and Juneja (2004), Gao et al. (2017) applied the large deviation technique to study the single best selection problem without the normal dependence and seek to maximize the convergence rates of the probability of false selection (PFS, equals 1-PCS) and expected opportunity cost (EOC, which is another commonly used performance criterion in R&S). After that, Chen et al. (2008), Lee et al. (2012), Chen et al. (2014), Gao and Chen (2015), Gao and Chen (2016), Gao et al. (2017), Xiao and Gao (2018) extended OCBA framework to optimal subset selection and many other variant problems.

The OCBA approach attracts considerable attention benefitting from both its simulation efficiency and nice closed-form of allocation rules. It provides a set of equations for sample allocation that can serve as the optimality conditions to achieve effective optimization for performance criteria such as PCS and EOC. Based on the OCBA optimality conditions, efficient budget allocation algorithms can be developed. Wu and Zhou (2018) did some exploratory research along this direction. In this paper, we focus on two different OCBA algorithms designed based on the optimality conditions from Chen et al. (2000) and Glynn and Juneja (2004), under the setting of normal underlying distributions with known variances. While the two OCBA algorithms are constructed based on the optimality conditions, theoretical analysis for the sample allocation convergence and convergence rates of the two algorithms is still lacking, which can be significant concerns when they are applied in practice.

In addition to the commonly studied PCS and EOC, we also consider a third performance criterion called cumulative regret (CR). CR is the sum of the difference in means between each sampled design and the true best one over all the iterations of the algorithm. This is a reasonable measure if samples are directly collected from the real systems, because consequences need to be considered for each time the real system is operated. Thus, it can act as a criterion to describe the algorithm's performance with respect to a different "exploration and exploitation" trade-off. In the literature, CR is frequently investigated in problem of multi-armed bandits (Bubeck and Cesa-Bianchi 2012) but is rarely considered in R&S. Since original OCBA algorithms are developed from OCBA optimality conditions for optimizing PCS and EOC, they should achieve a sub-optimal convergence rate with respect to CR. In this paper, we slightly modify the OCBA algorithms for the objective of CR, and show that the modified algorithms can achieve the optimal convergence rate.

The contributions of this paper are summarized as follows:

- We analyze the asymptotic sample allocation and convergence rates of PFS, EOC, and CR for the OCBA algorithms. It provides strong theoretical support for the efficiency of the OCBA approach.
- We propose two slightly modified OCBA algorithms which can achieve the optimal convergence rate under CR. It provides insights into potentially broader applications of the OCBA algorithms.

- Numerical experiments are conducted to further understand the convergence property of these OCBA algorithms.

The rest of the paper is organized as below. Section 2 introduces the two OCBA algorithms under consideration and three performance criteria used to evaluate the algorithms. The analysis and modifications of the OCBA algorithms are provided in Section 3. Section 4 presents the numerical results and Section 5 summarizes the paper.

2 PROBLEM STATEMENT

This section first briefly reviews the development of optimality conditions. After that, we show two OCBA algorithms based on the optimality conditions. Three performance criteria are subsequently introduced for algorithm evaluation.

2.1 Optimality Conditions

Throughout the paper, the best design means the design with the largest mean performance (the case with the smallest mean performance can be similarly analyzed). The goal of R&S is to find the best design from finite alternative designs. The simulation outputs follow normal distributions. They are independent from replication to replication, as well as across different designs. The means are unknown and have no ties among the designs; meanwhile, the variances are known for the normal distributions. For expression simplicity, we introduce the following notations:

n	total number of simulation replications (budget);
k	total number of designs;
$L_{i,j}$	simulation output of the jth simulation replication for design i, $i \in \{1,2,\ldots,k\}$, $j \in \mathbb{N}_+$;
θ_i	mean of $L_{i,j}$ i.e., $\theta_i = \mathbb{E}[L_{i,j}]$;
λ_i^2	variance of $L_{i,j}$ i.e., $\lambda_i^2 = \mathrm{Var}[L_{i,j}]$;
b	the best design, i.e., $b = \arg\max_{i\in\{1,\ldots,k\}} \theta_i$ and $\theta_b > \theta_i$, $\forall i \neq b$;
n_i	number of simulation replications to design i;
α_i	proportion of simulation replications to design i, i.e., $n_i = \alpha_i n$;
$\bar{\theta}_i$	sample mean of $L_{i,j}$, i.e., $\bar{\theta}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} L_{i,j}$;
$\bar{b}$	the estimated best design (the design with the largest sample mean), i.e., $\bar{b} = \arg\max_{i\in\{1,\ldots,k\}} \bar{\theta}_i$.

Define the probability of correct selection as the probability that the estimated best design $\bar{b}$ equals the best design b, i.e., $\mathrm{PCS} = \mathbb{P}\left(\bar{b} = b\right)$. Expected opportunity cost is defined as the expectation of the opportunity cost, where opportunity cost means the difference of means between the true best design b and the estimated best design $\bar{b}$. That is, $\mathrm{EOC} = \mathbb{E}[\mu_b - \mu_{\bar{b}}]$. PCS and EOC can be formulated as a function of n_i, $i = 1,\ldots,k$. Then, the best selection problem in R&S can be formulated as the following optimization problems:

$$\max_{n_1,\ldots,n_k} \mathrm{PCS} \quad \text{s.t.} \sum_{i=1}^{k} n_i = n \text{ and } n_i \geq 0, i = 1,\ldots,k. \tag{1}$$

$$\min_{n_1,\ldots,n_k} \mathrm{EOC} \quad \text{s.t.} \sum_{i=1}^{k} n_i = n \text{ and } n_i \geq 0, i = 1,\ldots,k. \tag{2}$$

For simplicity of the analysis, we will treat n_i's as real numbers instead of integers.

Both the PCS in (1) and EOC in (2) do not have analytical expressions and are time-consuming to evaluate using the Monte-Carlo estimates. Chen et al. (2000) replaced PCS by an analytical approximation using the Bonferroni inequality which is convenient to compute. They utilized the Karush–Kuhn–Tucker

(KKT) conditions to find the optimal allocation proportion $\alpha_1^*, \ldots, \alpha_k^*$ that satisfies the following equations:

$$\left(\frac{\alpha_b^*}{\lambda_b}\right)^2 - \sum_{i \neq b}\left(\frac{\alpha_i^*}{\lambda_i}\right)^2 = 0; \quad \frac{\alpha_{i_1}^*}{\alpha_{i_2}^*} = \frac{\lambda_{i_1}^2 \left(\theta_b - \theta_{i_2}\right)^2}{\lambda_{i_2}^2 \left(\theta_b - \theta_{i_1}\right)^2}, i_1, i_2 \neq b, \tag{3}$$

with the assumption that $\alpha_b^* \gg \alpha_i^*$ for $\forall i \neq b$. On the other hand, Glynn and Juneja (2004) introduced the large deviation (LD) techniques to characterize the convergence rate of PFS. Specifically, denote $\mathscr{M}_i(\gamma) = \mathbb{E}\left[\exp\left(\gamma L_{i,j}\right)\right]$ as the moment generating function of $L_{i,j}$, $\mathscr{F}_i(x) = \sup_{\gamma \in \mathbb{R}}\left(\gamma x - \log\left(\mathscr{M}_i(\gamma)\right)\right)$ by the Fenchel-Legendre transform of $\mathscr{M}_i(\gamma)$. Let $\mathscr{R}_i\left(\alpha_i, \alpha_b\right) = \inf_x\left(\alpha_i \log\left(\mathscr{M}_i(\gamma)\right) + \alpha_b \log\left(\mathscr{M}_b(\gamma)\right)\right)$, $i \neq b$. It can be derived that $\lim_{n\to\infty} \frac{1}{n}\log\left(\text{PFS}\right) = -\min_{i \neq b} \mathscr{R}_i\left(\alpha_i, \alpha_b\right)$. Then, the optimization problem (1) can be re-expressed as:

$$\max_{\alpha_1,\ldots,\alpha_k}\left[\min_{i \neq b} \mathscr{R}_i\left(\alpha_i, \alpha_b\right)\right] \qquad \text{s.t.} \sum_{i=1}^{k} \alpha_i = 1 \text{ and } \alpha_i \geq 0, i = 1, \ldots, k. \tag{4}$$

By making use of the KKT conditions, the optimal allocation propotion $\alpha_1^{**}, \ldots, \alpha_k^{**}$ satisfies the following equations:

$$\left(\frac{\alpha_b^{**}}{\lambda_b}\right)^2 - \sum_{i \neq b}\left(\frac{\alpha_i^{**}}{\lambda_i}\right)^2 = 0; \quad \frac{\left(\theta_b - \theta_{i_1}\right)^2}{\frac{\lambda_{i_1}^2}{\alpha_{i_1}^{**}} + \frac{\lambda_b^2}{\alpha_b^{**}}} = \frac{\left(\theta_b - \theta_{i_2}\right)^2}{\frac{\lambda_{i_2}^2}{\alpha_{i_2}^{**}} + \frac{\lambda_b^2}{\alpha_b^{**}}}, i_1, i_2 \neq b, \tag{5}$$

under the condition of underlying normally distributed simulation outputs. Gao et al. (2017) applied LD to study optimization problem (2) and showed that the same allocation proportion $\alpha_1^{**}, \ldots, \alpha_k^{**}$ can also asymptotically minimize EOC in normal contexts. According to Glynn and Juneja (2004), the optimality conditions (3) and (5) are identical when we assume $\alpha_b^{**} \gg \alpha_i^{**}$ for $\forall i \neq b$.

2.2 Algorithm Description

In this section, we present two OCBA algorithms respectively based on optimality conditions (3) and (5). We call the algorithm based on (3) OCBA-1 algorithm and call the algorithm based on (5) OCBA-2 algorithm.

We first introduce additional notations for the description of the two algorithms.

I_t	the design sampled in the iteration t, $t = 1, \ldots, n$;
$L_{I_t,j}$	simulation output of the jth simulation replication for design I_t, $j \in \mathbb{N}_+$;
$N_{i,t}$	number of simulation replications to design i until iteration t;
$\alpha_{i,t}$	proportion of simulation replications to design i until iteration t, i.e., $N_{i,t} = \alpha_{i,t} t$;
$\bar{\theta}_{i,t}$	sample mean of $L_{i,j}$ until iteration t, i.e., $\bar{\theta}_i = \frac{1}{N_{i,t}} \sum_{j=1}^{N_{i,t}} L_{i,j}$;
$\bar{b}_t$	the estimated best design (the design with the largest sample mean) at iteration t.

Chen and Lee (2011) developed the OCBA-1 algorithm based on the optimality conditions (3). Meanwhile, we can follow a similar algorithm structure in Gao et al. (2017) and present the OCBA-2 algorithm on the basis of optimality conditions (5). The two OCBA algorithms are shown in the next page.

In the subsequent sections, we will analyze the asymptotic behaviors of the two OCBA algorithms, including their sample allocation and the convergence rate under different performance criteria.

2.3 Performance Criteria

In this section, we present the three performance criteria that will be used to evaluate the OCBA algorithms.

The first is the PFS, which is defined as the probability that the estimated best design does not equal the true best one. Denote the value of PFS at iteration t as

$$\mathrm{PFS}_t = \mathbb{P}\left(\bar{b}_t \neq b\right).$$

EOC is the difference of means between the true best design and the estimated best one. Denote the value of EOC in the iteration t by

$$\mathrm{EOC}_t = \mathbb{E}\left[\theta_b - \theta_{\bar{b}_t,t}\right].$$

The third one is the cumulative regret (CR), which is the sum of difference in means between the true best design and each sampled design (Bubeck and Cesa-Bianchi 2012). We can express it at iteration t as

$$\mathrm{CR}_t = t\theta_b - \sum_{s=1}^{t} \mathbb{E}\left[\theta_{I_s}\right].$$

The first two criteria (PFS and EOC) seek to evaluate the mean performance of the final recommendation $\bar{b}$; meanwhile, CR is introduced to assess the algorithm's overall performance during the experiment. We evaluate the two OCBA algorithms from these two aspects and provide some further modifications based on the analytical results.

OCBA-1 algorithm

- Initialize k, n, n_0.
- $t \leftarrow 0$, $N_{1,t} = \cdots = N_{k,t} = n_0$.
- Conduct n_0 simulation replications to each design i, $i = 1,\ldots,k$.
- **While** $\sum_{i=1}^{k} N_{i,t} < n$ **Do**
  - Update $\bar{\theta}_{i,t}$ using the new simulation outputs, $i = 1,\ldots,k$.
  - $\bar{b}_t = \arg\max_i \bar{\theta}_{i,t}$.
  - Compute $\hat{\alpha}_{1,t},\ldots,\hat{\alpha}_{k,t}$ by (3), with $\theta_{i,t}$ and b replaced by $\bar{\theta}_{i,t}$ and $\bar{b}_t$.
  - $I_{t+1} = \arg\max_i \left(\hat{\alpha}_{i,t}\left(1+\sum_{i=1}^{k} N_{i,t}\right) - N_{i,t}\right)$.
  - Conduct one simulation replication to design I_{t+1}.
  - $N_{I_{t+1},t+1} = N_{I_{t+1},t} + 1$. $N_{i,t+1} = N_{i,t}$ for $i \neq I_{t+1}$.
  - $t \leftarrow t+1$.

  End While

OCBA-2 algorithm

- Initialize k, n, n_0.
- $t \leftarrow 0$, $N_{1,t} = \cdots = N_{k,t} = n_0$.
- Conduct n_0 simulation replications to each design i, $i = 1,\ldots,k$.
- **While** $\sum_{i=1}^{k} N_{i,t} < n$ **Do**
  - Update $\bar{\theta}_{i,t}$ using the new simulation outputs, $i = 1,\ldots,k$.
  - $\bar{b}_t = \arg\max_i \bar{\theta}_{i,t}$.
  - **If** $\left(\frac{N_{\bar{b}_t,t}}{\lambda_{\bar{b}_t}}\right)^2 - \sum_{i\neq\bar{b}_t}\left(\frac{N_{i,t}}{\lambda_i}\right)^2 < 0$

    $I_{t+1} = \bar{b}_t$.
  - **Else**

    $I_{t+1} = \arg\max_{i\neq\bar{b}_t}\left(\frac{\left(\bar{\theta}_{\bar{b}_t,t}-\bar{\theta}_{i,t}\right)^2}{\frac{\lambda_i^2}{N_{i,t}}+\frac{\lambda_{\bar{b}_t}^2}{N_{\bar{b}_t,t}}}\right)$.

    End If
  - Conduct one simulation replication to design I_{t+1}.
  - $N_{I_{t+1},t+1} = N_{I_{t+1},t} + 1$. $N_{i,t+1} = N_{i,t}$ for $i \neq I_{t+1}$.
  - $t \leftarrow t+1$.

  End While

3 MAIN RESULTS

In this section, we first provide some definitions regarding the asymptotic performance of the OCBA algorithms. Then, we study the convergence property of the algorithms. All the proofs are omitted due to the limitation of space. They are provided with details in Li, Y., and S. Gao. (2021).

Definition 1 defines some asymptotic relationships for sequences of real numbers. Definition 2 offers two criteria for evaluating the convergence rate of CR.

Definition 1 Denote $\{p_t | t \in \mathbb{N}_+\}$ and $\{q_t | t \in \mathbb{N}_+\}$ as two positive real-value sequences. The following statements hold for the two sequences:

- p_t is asymptotically equivalent to q_t (denoted by $p_t \sim q_t$) $\Leftrightarrow \lim_{t\to\infty} \frac{p_t}{q_t} = 1$.
- p_t is logrithmically equivalent to q_t (denoted as $p_t \doteq q_t$) $\Leftrightarrow \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{p_t}{q_t}\right) = 0$.
- p_t is asymptotically dominated by q_t (symbolized by $p_t = o(q_t)$) $\Leftrightarrow \lim_{t\to\infty} \frac{p_t}{q_t} = 0$.

In Definition 1, asymptotic equivalence is a sufficient condition of logarithmatic equivalence, but not a necessary condition, i.e., $p_t \sim q_t \Rightarrow p_t \doteq q_t$ but $p_t \doteq q_t \not\Rightarrow p_t \sim q_t$. Besides, if $p_t \sim q_t$, then $p_t - q_t = o(p_t)$ and $p_t - q_t = o(q_t)$; if $p_t \doteq q_t$, then $\frac{p_t}{q_t} = o(e^t)$.

Definition 2 (Burnetas and Katehakis 1997) A selection procedure $\mathscr{P}$ is uniformly maximum convergent (UM) $\Leftrightarrow \mathrm{CR}_t = \sum_{i\neq b} \frac{\theta_b - \theta_i}{\mathscr{K}_{i,b}} \log t + o(\log t)$, where $\mathscr{K}_{i,b} = \frac{(\theta_b-\theta_i)^2+\lambda_i^2}{2\lambda_b^2} + \log\left(\frac{\lambda_b}{\lambda_i}\right) - \frac{1}{2}$, $i \neq b$.

In Theorems 1 and 2 below, we show that the estimated best designs from the OCBA-1 and OCBA-2 algorithms converge to the true best one and the sample allocations from the two algorithms converge to the allocations that satisfy optimality conditions (3) and (5). We also show the convergence rates of PFS, EOC and CR from the two algorithms.

Theorem 1 For the OCBA-1 algorithm, the following statements hold ("a.s." means "almost surely"):

- $\lim_{t\to\infty} \bar{b}_t = b$ a.s.
- $\lim_{t\to\infty} \alpha_{i,t} = \alpha_i^*$ a.s., where α_i^* satisfies (3), $i = 1, \ldots, k$.
- $\mathrm{PFS}_t \doteq \exp\left\{-\frac{\beta^*}{2} t\right\}$ a.s., where $\beta^* = \min_{i\neq b} \left(\frac{(\theta_b-\theta_i)^2}{\frac{\lambda_i^2}{\alpha_i^*} + \frac{\lambda_b^2}{\alpha_b^*}}\right)$.
- $\mathrm{EOC}_t \doteq \exp\left\{-\frac{\beta^*}{2} t\right\}$ a.s.
- $\mathrm{CR}_t \sim \sum_{i\neq b} (\theta_b - \theta_i)\, \alpha_i^* t$ a.s.

Theorem 2 For the OCBA-2 algorithm, the following statements hold:

- $\lim_{t\to\infty} \bar{b}_t = b$ a.s.
- $\lim_{t\to\infty} \alpha_{i,t} = \alpha_i^{**}$ a.s., where α_i^{**} satisfies (5), $i = 1, \ldots, k$.
- $\mathrm{PFS}_t \doteq \exp\left\{-\frac{\beta^{**}}{2} t\right\}$ a.s., where $\beta^{**} = \min_{i\neq b} \left(\frac{(\theta_b-\theta_i)^2}{\frac{\lambda_i^2}{\alpha_i^{**}} + \frac{\lambda_b^2}{\alpha_b^{**}}}\right)$.
- $\mathrm{EOC}_t \doteq \exp\left\{-\frac{\beta^{**}}{2} t\right\}$ a.s.
- $\mathrm{CR}_t \sim \sum_{i\neq b} (\theta_b - \theta_i)\, \alpha_i^{**} t$ a.s.

Theorems 1 and 2 show that the sample allocations of OCBA-1 and OCBA-2 asymptotically satisfy the optimality conditions, and the PFS and EOC decrease exponentially with rate $\frac{\beta^*}{2}$ for OCBA-1 and rate $\frac{\beta^{**}}{2}$ for OCBA-2. According to Section 2.1, optimality conditions (3) and (5) are the asymptotic minimizer of PFS and EOC. It indicates that the two OCBA algorithms can recover this optimal allocation asymptotically.

Meanwhile, Theorems 1 and 2 show that CR increases polynomially with respect to t for OCBA-1 and OCBA-2. This result aligns with the conclusion in Bubeck et al. (2011) that for any algorithm with CR increasing linearly with iteration number t, its EOC can achieve an exponential convergence rate at best. According to Definition 2, the two OCBA algorithms do not perform well with respect to CR. This result coincides with the previous discussion about two different types of "exploration and exploitation" trade-offs in Section 1.

Next, we conduct minor modifications for the OCBA-1 and OCBA-2 algorithms such that their CR can achieve the convergence rates in Definition 2. We modify the two algorithms to possess UM property. We call them OCBA-1-UM and OCBA-2-UM.

The OCBA-1-UM and OCBA-2-UM algorithms follow a simple mechanism. We first introduce $h_t = \frac{\sum_{i\neq \bar{b}_t} \frac{\bar{\theta}_{\bar{b}_t,t}-\bar{\theta}_{i,t}}{\mathscr{K}_{i,\bar{b}_t}}}{\sum_{i\neq \bar{b}_t}\left(\bar{\theta}_{\bar{b}_t,t}-\bar{\theta}_{i,t}\right)\tilde{\alpha}_{i,t}}$, where $\mathscr{K}_{i,\bar{b}_t} = \frac{\left(\bar{\theta}_{\bar{b}_t,t}-\bar{\theta}_{i,t}\right)^2+\lambda_i^2}{\lambda_{\bar{b}_t}^2} + \log\left(\frac{\lambda_{\bar{b}_t}}{\lambda_i}\right) - \frac{1}{2}$, $\tilde{\alpha}_{i,t} = \frac{\tilde{N}_{i,t}}{\sum_{j=1}^k \tilde{N}_{j,t}}$ for $i \neq \bar{b}_t$. Then, in the iteration t, we determine the next sampled design by the guidance of optimality conditions with probability $\min\left\{\frac{h_t}{t},1\right\}$; otherwise, we directly sample the estimated best design $\bar{b}_t$. In this way, the average number of sampling non-best designs would logarithmically increase with respect to t. This modification on the OCBA algorithms can effectively reduce the number of samples allocated to the non-best designs.

The OCBA-1-UM and OCBA-2-UM algorithms are described as follows.

OCBA-1-UM algorithm

- Initialize k, n, n_0.
- $t \leftarrow 0$, $N_{1,t} = \cdots = N_{k,t} = n_0$.
- Conduct n_0 simulation replications to each design i, $i = 1,\ldots,k$.
- $\tilde{N}_{1,t} = \cdots = \tilde{N}_{k,t} = n_0$.
- **While** $\sum_{i=1}^k N_{i,t} < n$ **Do**
  - Update $\bar{\theta}_{i,t}$ using the new simulation outputs, $i = 1,\ldots,k$.
  - $\bar{b}_t = \arg\max_i \bar{\theta}_{i,t}$.
  - Uniformly choose $u \in [0,1]$.
  - **If** $u \leq \frac{h_t}{t}$

    Compute $\hat{\alpha}_{1,t},\ldots,\hat{\alpha}_{k,t}$ by (3).

    $I_{t+1} = \arg\max_i \left(\hat{\alpha}_{i,t}\left(1+\sum_{i=1}^k \tilde{N}_{i,t}\right) - \tilde{N}_{i,t}\right)$.

    $\tilde{N}_{I_{t+1},t+1} = \tilde{N}_{I_{t+1},t}+1$.

    $\tilde{N}_{i,t+1} = \tilde{N}_{i,t}$ for $i \neq I_{t+1}$.
  - **Else**

    $I_{t+1} = \bar{b}_t$;

    $\tilde{N}_{i,t+1} = \tilde{N}_{i,t}$ for $i = 1,\ldots,k$.

    End If
  - Conduct one simulation replication to design I_{t+1}.
  - $N_{I_{t+1},t+1} = N_{I_{t+1},t}+1$.

    $N_{i,t+1} = N_{i,t}$ for $i \neq I_{t+1}$. $t \leftarrow t+1$.

  End While

OCBA-2-UM algorithm

- Initialize k, n, n_0.
- $t \leftarrow 0$, $N_{1,t} = \cdots = N_{k,t} = n_0$.
- Conduct n_0 simulation replications to each design i, $i = 1,\ldots,k$.
- $\tilde{N}_{1,t} = \cdots = \tilde{N}_{k,t} = n_0$.
- **While** $\sum_{i=1}^k N_{i,t} < n$ **Do**
  - Update $\bar{\theta}_{i,t}$ using the new simulation outputs, $i = 1,\ldots,k$.
  - $\bar{b}_t = \arg\max_i \bar{\theta}_{i,t}$.
  - Uniformly choose $u \in [0,1]$.
  - **If** $u \leq \frac{h_t}{t}$

    If $\left(\frac{\tilde{N}_{\bar{b}_t,t}}{\lambda_{\bar{b}_t}}\right)^2 - \sum_{i\neq \bar{b}_t}\left(\frac{\tilde{N}_{i,t}}{\lambda_i}\right)^2 < 0$

    $I_{t+1} = \bar{b}_t$.

    Else

    $I_{t+1} = \arg\max_{i\neq \bar{b}_t}\left(\frac{\left(\bar{\theta}_{\bar{b}_t,t}-\bar{\theta}_{i,t}\right)^2}{\frac{\lambda_i^2}{\tilde{N}_{i,t}}+\frac{\lambda_{\bar{b}_t}^2}{\tilde{N}_{\bar{b}_t,t}}}\right)$.

    End If

    $\tilde{N}_{I_{t+1},t+1} = \tilde{N}_{I_{t+1},t}+1$.

    $\tilde{N}_{i,t+1} = \tilde{N}_{i,t}$ for $i \neq I_{t+1}$.
  - **Else**

    $I_{t+1} = \bar{b}_t$;

    $\tilde{N}_{i,t+1} = \tilde{N}_{i,t}$ for $i = 1,\ldots,k$.

    End If
  - Conduct one simulation replication to design I_{t+1}.
  - $N_{I_{t+1},t+1} = N_{I_{t+1},t}+1$.

    $N_{i,t+1} = N_{i,t}$ for $i \neq I_{t+1}$. $t \leftarrow t+1$.

  End While

In Theorems 3 and 4 below, we show the convergence rates of the OCBA-1-UM and OCBA-2-UM algorithms with respect to PFS, EOC and CR.

Theorem 3 For the OCBA-1-UM algorithm, the following statements hold:

- $\mathrm{PFS}_t \doteq t^{-\frac{h^*}{2\tau^*}}$ a.s., where $h^* = \frac{\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}}{\sum_{i\neq b}(\theta_b-\theta_i)\alpha_i^*}$, $\mathscr{K}_{i,b} = \frac{(\theta_b-\theta_i)^2+\lambda_i^2}{2\lambda_b^2} + \log\left(\frac{\lambda_b}{\lambda_i}\right) - \frac{1}{2}$, $i \neq b$.
- $\mathrm{EOC}_t \doteq t^{-\frac{h^*}{2\tau^*}}$ a.s.
- $\mathrm{CR}_t \sim \sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log t$ a.s.

Theorem 4 For the OCBA-2-UM algorithm, the following statements hold:

- $\mathrm{PFS}_t \doteq t^{-\frac{h^{**}}{2\tau^{**}}}$ a.s., where $h^{**} = \frac{\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}}{\sum_{i\neq b}(\theta_b-\theta_i)\alpha_i^{**}}$.
- $\mathrm{EOC}_t \doteq t^{-\frac{h^{**}}{2\tau^{**}}}$ a.s.
- $\mathrm{CR}_t \sim \sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log t$ a.s.

According to Theorems 3 and 4, the OCBA-1-UM and OCBA-2-UM algorithms possess the UM property. It can also be observed that the convergence rates of PFS and EOC of the OCBA-1-UM and OCBA-2-UM algorithms become slower compared to those of the OCBA-1 and OCBA-2 algorithms, from exponential rates to polynomial rates. It is not surprising. CR is associated with comparison result in each iteration of the algorithm, while PFS and EOC focus only on the comparison result in the last iteration. An algorithm that works well for CR typically works badly for PFS and EOC, and vice versa. This observation is in line with the analysis conducted for the best arm identification problem in Audibert et al. (2010), Bubeck et al. (2011).

4 NUMERICAL EXPERIMENTS

In this section, we perform numerical experiments for the four algorithms studied in this paper, i.e., the OCBA-1, OCBA-2, OCBA-1-UM and OCBA-2-UM algorithms. In addition, we compare the performance of the OCBA-1-UM and OCBA-2-UM algorithms with that of Thompson Sampling (Thompson 1933) under noninformative priors and the condition of normally distributed simulation outputs with known variances.

We consider a frequently used numerical example called slippage configuration. There are five designs whose observations are i.i.d. normal with unknown means and known variances. We want to select the best (the one with the largest mean) from the alternative five designs. For design 1, we set the mean $\theta_1 = 2$; for designs $i = 2,\ldots,5$, we set the mean $\theta_i = 1$. The variances $\lambda_i^2 = 1$ for designs $i = 1,\ldots,5$. We can see that the true best design $b = 1$.

We set $n_0 = 2$. To reduce the influence of randomness in algorithm evaluation, each algorithm is repeated for one thousand times to obtain its estimates of PFS, EOC and CR. Figure 1 shows the performance of the OCBA-1 and OCBA-2 algorithms. Specifically, $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{\exp\left(-\frac{\beta^*}{2}n\right)}\right)$ in Figure 1(a) and $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{\exp\left(-\frac{\beta^{**}}{2}n\right)}\right)$ in Figure 1(d) tend to zero as the simulation budget n increases. It suggests that $\mathrm{PFS}_n \doteq \exp\left(-\frac{\beta^*}{2}n\right)$ for the OCBA-1 algorithm and $\mathrm{PFS}_n \doteq \exp\left(-\frac{\beta^{**}}{2}n\right)$ for the OCBA-2 algorithm. In other words, the PFS of the OCBA-1 and OCBA-2 algorithms converges exponentially fast with rates $\frac{\beta^*}{2}$ and $\frac{\beta^{**}}{2}$ respectively. We can also find that $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{\exp\left(-\frac{\beta^*}{2}n\right)}\right)$ in Figure 1(b) and $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{\exp\left(-\frac{\beta^{**}}{2}n\right)}\right)$ in Figure 1(e) tend to zero as n increases. It indicates that the EOC of the two OCBA algorithms converges exponentially fast with rates $\frac{\beta^*}{2}$ and $\frac{\beta^{**}}{2}$. Comparing the performance of the algorithms under PFS (Figure 1(a), Figure 1(d)) and

EOC (Figure 1(b), Figure 1(e)), we can see that the convergence patterns are basically the same. It implies the asymptotic similarity between PFS and EOC, which was also discussed in Gao et al. (2017). Last, for CR, $\frac{\mathrm{CR}_n}{\sum_{i\neq b}(\theta_b-\theta_i)\alpha_i^* n}$ in Figure 1(c) and $\frac{\mathrm{CR}_n}{\sum_{i\neq b}(\theta_b-\theta_i)\alpha_i^{**} n}$ in Figure 1(f) tend to one as the simulation budget n increases. It implies that the CR of the OCBA-1 and OCBA-2 algorithms converges polynomially fast with rates $\sum_{i\neq b}(\theta_b-\theta_i)\alpha_i^*$ and $\sum_{i\neq b}(\theta_b-\theta_i)\alpha_i^{**}$.

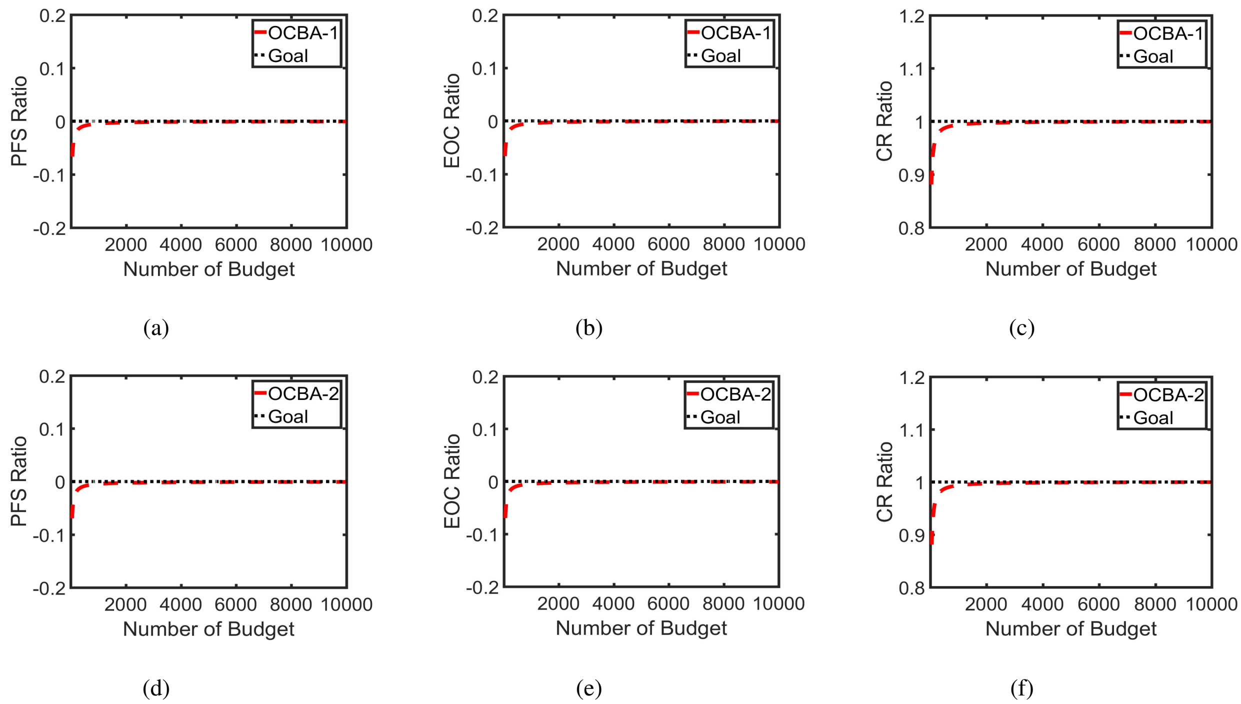

Figure 1: Asymptotic performance of the OCBA-1 and OCBA-2 algorithms with respect to PFS, EOC and CR. In Figure 1(a)-Figure 1(c), PFS ratio, EOC ratio and CR ratio respectively denote $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{\exp\left(-\frac{\beta^*}{2}n\right)}\right)$, $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{\exp\left(-\frac{\beta^*}{2}n\right)}\right)$ and $\frac{\mathrm{CR}_n}{\sum_{i\neq b}(\theta_b-\theta_i)\alpha_i^*\log n}$. In Figure 1(d)-Figure 1(f), PFS ratio, EOC ratio and CR ratio respectively denote $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{\exp\left(-\frac{\beta^{**}}{2}n\right)}\right)$, $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{\exp\left(-\frac{\beta^{**}}{2}n\right)}\right)$ and $\frac{\mathrm{CR}_n}{\sum_{i\neq b}(\theta_b-\theta_i)\alpha_i^{**}\log n}$.

Figure 2 presents the performance of the OCBA-1-UM and OCBA-2-UM algorithms. As n increases, $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{n^{-\frac{h^*}{2\tau^*}}}\right)\to 0$ in Figure 2(a), $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{n^{-\frac{h^*}{2\tau^*}}}\right)\to 0$ in Figure 2(b), $\frac{\mathrm{CR}_n}{\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log n}\to 1$ in Figure 2(c). It means that the PFS and EOC of the OCBA-1-UM algorithm converges polynomially fast, and the OCBA-1-UM algorithm possesses UM property. Similar convergence patterns for OCBA-2-UM can be observed in Figure 2(d)-Figure 2(f), i.e., $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{n^{-\frac{h^{**}}{2\tau^{**}}}}\right)\to 0$, $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{n^{-\frac{h^{**}}{2\tau^{**}}}}\right)\to 0$, and $\frac{\mathrm{CR}_n}{\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log n}\to 1$ as n increases. It means that the PFS and EOC of the OCBA-2-UM algorithm converge polynomially fast, and the OCBA-2-UM algorithm possesses UM property.

Figure 2 also shows the performance comparison of the OCBA-1-UM and OCBA-2-UM algorithms and Thompson Sampling. From Figure 2(a) and Figure 2(b), $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{n^{-\frac{h^*}{2\tau^*}}}\right)$ and $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{n^{-\frac{h^*}{2\tau^*}}}\right)$ of Thompson

Sampling is slightly larger than those of the OCBA-1-UM algorithm, then the gap vanishes as n increases. It indicates that the PFS and EOC of Thompson Sampling converge polynomially fast, and the PFS and EOC of the OCBA-1-UM algorithm converges to zero slightly faster than those of Thompson Sampling converging to zero. For CR in Figure 2(c), $\frac{\mathrm{CR}_n}{\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log n}$ of Thompson Sampling is smaller than that of the OCBA-1-UM algorithm. It indicates that the CR of Thompson Sampling is smaller than that of the OCBA-1-UM algorithm but the CR of the OCBA-1-UM algorithm converges to $\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log n$ faster than that of Thompson Sampling converging to $\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log n$ if Thompson Sampling possesses UM property under the condition of normally distributed simulation outputs instead of Bernoulli sampling (Kaufmann et al. 2012). The comparison results of the OCBA-2-UM algorithm and Thompson Sampling in Figure 2(d)-Figure 2(f) are similar to those of the OCBA-1-UM algorithm and Thompson Sampling in Figure 2(a)-Figure 2(c). It indicates that the sample allocations from (3) and (5) are nearly identical in this numerical experiment, which leads to similar performance of the OCBA-1-UM and OCBA-2-UM algorithms and similar results as shown in Figure 2(a)-Figure 2(c) and Figure 2(d)-Figure 2(f).

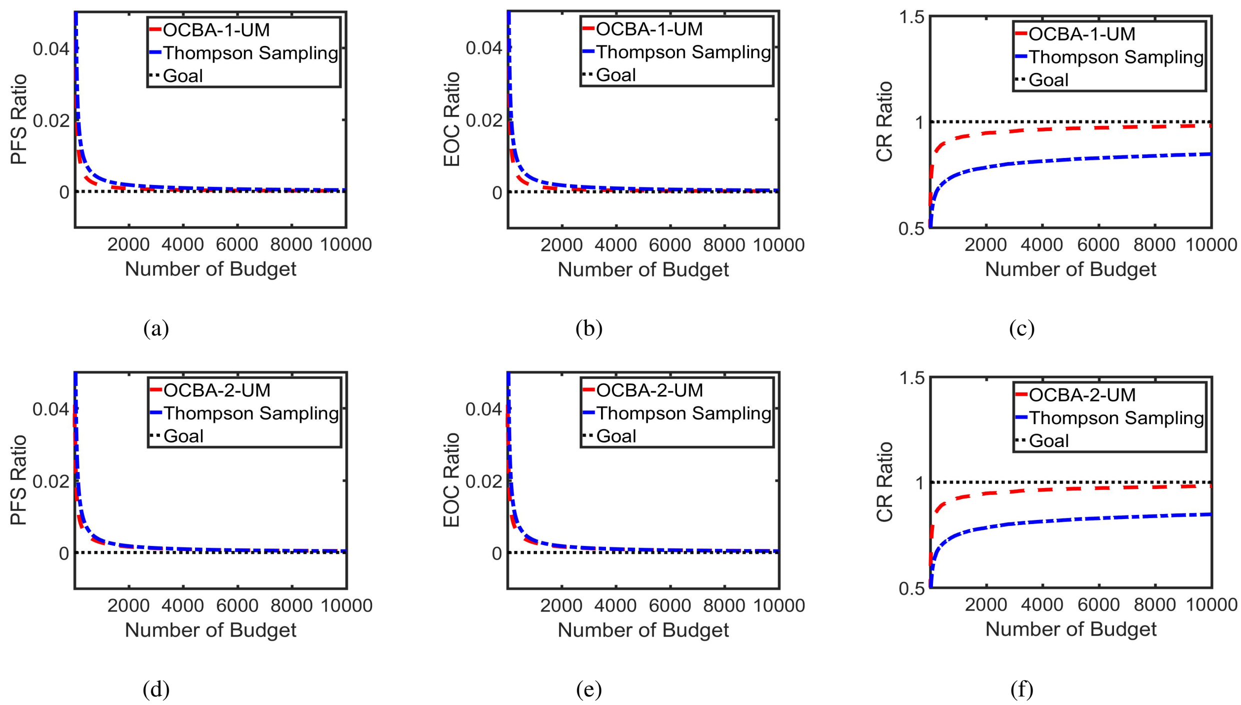

Figure 2: Asymptotic performance of the OCBA-1-UM and OCBA-2-UM algorithms and Thompson Sampling with respect to PFS, EOC and CR. In Figure 2(a)-Figure 2(c), PFS ratio, EOC ratio and CR ratio respectively denote $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{n^{-\frac{h^*}{2\tau^*}}}\right)$, $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{n^{-\frac{h^*}{2\tau^*}}}\right)$ and $\frac{\mathrm{CR}_n}{\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log n}$. In Figure 2(d)-Figure 2(f), PFS ratio, EOC ratio and CR ratio respectively denote $\frac{1}{n}\log\left(\frac{\mathrm{PFS}_n}{n^{-\frac{h^{**}}{2\tau^{**}}}}\right)$, $\frac{1}{n}\log\left(\frac{\mathrm{EOC}_n}{n^{-\frac{h^{**}}{2\tau^{**}}}}\right)$ and $\frac{\mathrm{CR}_n}{\sum_{i\neq b}\frac{\theta_b-\theta_i}{\mathscr{K}_{i,b}}\log n}$.

5 CONCLUSIONS

This paper focuses on a popular R&S approach called OCBA and analyzes the convergence rates of two OCBA algorithms with respect to three performance criteria: PFS, EOC, and CR. We first show that the OCBA procedures possess optimal convergence rates with respect to PFS and EOC. Then, we conduct minor modifications on the OCBA algorithms to achieve the optimal convergence rate with respect to CR. Numerical tests are conducted to demonstrate the performance of the algorithms.

The OCBA algorithms discussed in this paper are assumed to possess known variances for samples of the designs. However, the variances are unknown in a lot of applications and are typically estimated by sample variances. It is an important future research direction to study the convergence rates of the OCBA algorithms in this case.

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