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ESTIMATING A CONDITIONAL EXPECTATION WITH THE GENERALIZED LIKELIHOOD RATIO METHOD

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ABSTRACT

In this paper, we consider the problem of efficiently estimating a conditional expectation. By formulating the conditional expectation as a ratio of two derivatives, we can apply the generalized likelihood ratio method to express the conditional expectation using ordinary expectations with indicator functions, which generalizes the conditional density method. Based on an empirical distribution estimated from simulation, we provide guidance on selecting the appropriate formulation of the derivatives to reduce the variance of the estimator.

1 INTRODUCTION

Many real-world problems, such as energy storage (Carmona and Ludkovski 2010) and option pricing (Merton 1973), can be modeled using stochastic processes, with parameters estimated from real-world data. When the underlying stochastic process evolves, new observations are collected, which usually enables us to make forecasts about the process in the future. One can interpret the forecasting problem as estimating the expected performance of the random system in the future, conditional on information collected up to the current period. This often involves a complicated integration. One approach to estimate the conditional expectation is to use Monte Carlo methods, i.e., simulate a number of sample paths starting from a given information set and calculate the sample average of the performance function as the estimator. However, this approach would require additional simulations, which may be time-consuming. Such an approach would be poorly suited for applications that require quick responses, such as portfolio risk management in a rapid-changing market (Jiang et al. 2020). In this paper, we consider estimating a conditional expectation under the setting where a collection of pre-simulated sample paths is given to the user and no additional simulations should be performed. Many approaches have been proposed for this purpose. One line of research employs least squares Monte Carlo (LSM) to estimate the conditional expectation from cross-sectional information (Longstaff and Schwartz 2001). Besides LSM, Fournié et al. (2001) applied the density method (DM) and Malliavin calculus to estimate the Greeks of European options. Daveloose et al. (2019) further extends their results by considering a jump-diffusion setting with both the conditional density method (CDM) and Malliavin calculus. Our focus in this paper is to generalize the CDM method by noticing that by definition, the conditional expectation can be represented as a ratio of two derivatives that involve indicator functions. With this formulation, we are able to employ existing stochastic gradient estimation (SGE) techniques.

There are many well-known SGE techniques, among which the finite difference (FD) method may be the most straightforward and easiest to implement; however, FD estimators are biased. Two widely-used approaches for deriving an unbiased gradient estimator are infinitesimal perturbation analysis (IPA) (Glasserman and Ho 1991) and the likelihood ratio (LR) method (Ho and Cao 1983), which is also

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known as the Score Function (SF) method (Rubinstein and Shapiro 1993). However, IPA cannot handle a discontinuous sample performance, while LR fails if the parameters of interest appear explicitly (structural parameters) in the sample performance. In our formulation, the performance functions in the corresponding SGE problems are discontinuous with structural parameters. Therefore, IPA and LR are not valid for our purpose. To overcome this shortcoming, a push-out LR technique (Rubinstein 1992) that pushes the structural parameter into the probability measure, can be applied for some problems under appropriate conditions. Recently, Peng et al. (2018) developed a generalized likelihood ratio (GLR) method that extends IPA and LR, allowing discontinuities in the sample performance with the presence of structural parameters. As shown in Peng et al. (2018), GLR is a generalization of the push-out LR method.

Our work shows that CDM is essentially equivalent to push-out LR. By applying GLR, we generalize CDM under some assumptions, one of which requires the output function whose value is conditioned on to be globally invertible. We address this limitation by locally isolating the stationary points of the output function. Since our derived ratio estimator of the conditional expectation involves an indicator function, we then propose a simple way to reduce the variance of the estimator, which is particularly effective for rare events

The rest of this paper is organized as follows. In Section 2, we review the problem of estimating a conditional expectation with the conditional density method under some assumptions, and introduce the stochastic gradient estimation formulation. In Section 3, we derive a ratio representation with GLR for two cases. In Section 4, we discuss the quality of the derived estimator. We then present some numerical experiments in Section 5 and conclude in Section 6.

2 ESTIMATING A CONDITIONAL EXPECTATION

In this section, we review the problem of estimating the conditional expectation with the condition density method. We then show that the conditional expectation can be expressed as a ratio of two derivatives, which motivates the application of the generalized likelihood ratio method.

2.1 Problem Formulation

Consider the problem of estimating the conditional expectation:

$$\mathbb{E}\left[g_1(X,Y)|g_2(U,V)=\alpha\right], \tag{1}$$

where X, Y, U, V are random variables, $g_i:\mathbb{R}^2\to\mathbb{R}$, $i=1,2$, are Borel-measurable functions, and α is a fixed real number. Our goal is to derive an expression for (1) using ordinary expectations.

This problem has been studied by Daveloose et al. (2019) using conditional density methods (CDM). Here, we state their major assumptions and results, and we will show later that our approach extends their result for more complicated settings.

Assumption 1

(i) (X,U) is independent of (Y,V);
(ii) (X,U) has joint density $f_{X,U}(x,u)$ with unbounded support;
(iii) $\ln f_{X,U}(x,\cdot)$ is differentiable for all $x\in\mathbb{R}$;
(iv) $\mathbb{E}\left[(\pi_{X,U}(X,U))^2\right]<\infty$, where $\pi_{X,U}(x,u)=-\partial_u \ln f_{X,U}(x,u)$.

Assumption 2 There exist a Borel-measurable function g^* and a strictly increasing differentiable function h such that $g_2(u,v)=h^{-1}(u+g^*(v))$.

By first rewriting (1) as

$$\frac{\mathbb{E}\left[g_1(X,Y)\delta(g_2(U,V)-\alpha)\right]}{\mathbb{E}\left[\delta(g_2(U,V)-\alpha)\right]},$$

where δ is the Dirac delta function, and then applying the co-area formula (Kanwal 2012) and integration by parts, Daveloose et al. (2019) derived the following representation of the conditional expectation.

Theorem 1 (Daveloose et al. 2019) Under Assumptions 1 and 2, suppose $\mathbb{E}\left[|g_1(X,Y)|^2\right] < \infty$, then for any $\alpha \in Dom(h)$,

$$\mathbb{E}\left[g_1(X,Y)|g_2(U,V)=\alpha\right] = \frac{\mathbb{E}\left[g_1(X,Y)\mathbf{1}\left\{g_2(U,V)\geq\alpha\right\}\pi_{X,U}(X,U)\right]}{\mathbb{E}\left[\mathbf{1}\left\{g_2(U,V)\geq\alpha\right\}\pi_{X,U}(X,U)\right]},$$

where $\mathbf{1}\{\cdot\}$ is the indicator function.

From Theorem 1, the conditional expectation (1) can be estimated by a ratio of two sample averages. However, the existence of Assumption 2 limits the application of the above result. In the next section, we propose an approach to overcome this limitation by applying the generalized likelihood ratio (GLR) method (Peng et al. 2018).

2.2 Stochastic Gradient Estimation

First, we show that (1) can be expressed as a ratio of two derivatives. By the definition of conditional expectation, under mild regularity conditions, we have

$$\begin{aligned}
\mathbb{E}\left[g_1(X,Y)|g_2(U,V)=\alpha\right] &= \lim_{\varepsilon\to 0}\mathbb{E}\left[g_1(X,Y)|g_2(U,V)\in[\alpha-\varepsilon,\alpha-\varepsilon]\right] \\
&= \lim_{\varepsilon\to 0}\frac{\mathbb{E}\left[g_1(X,Y)\mathbf{1}\left\{g_2(U,V)\in[\alpha-\varepsilon,\alpha+\varepsilon]\right\}\right]}{\mathbb{E}\left[\mathbf{1}\left\{g_2(U,V)\in[\alpha-\varepsilon,\alpha+\varepsilon]\right\}\right]} \\
&= \frac{\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\mathbb{E}\left[g_1(X,Y)\mathbf{1}\left\{g_2(U,V)\in[\alpha-\varepsilon,\alpha+\varepsilon]\right\}\right]}{\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\mathbb{E}\left[\mathbf{1}\left\{g_2(U,V)\in[\alpha-\varepsilon,\alpha+\varepsilon]\right\}\right]} \\
&= \frac{\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\mathbb{E}\left[g_1(X,Y)\left(\mathbf{1}\left\{g_2(U,V)\leq\alpha+\varepsilon\right\}-\mathbf{1}\left\{g_2(U,V)\leq\alpha-\varepsilon\right\}\right)\right]}{\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\mathbb{E}\left[\mathbf{1}\left\{g_2(U,V)\leq\alpha+\varepsilon\right\}-\mathbf{1}\left\{g_2(U,V)\leq\alpha-\varepsilon\right\}\right]} \\
&= \frac{\frac{d}{d\theta}\mathbb{E}\left[g_1(X,Y)\mathbf{1}\left\{g_2(U,V)\leq\theta\right\}\right]\big|_{\theta=\alpha}}{\frac{d}{d\theta}\mathbb{E}\left[\mathbf{1}\left\{g_2(U,V)\leq\theta\right\}\right]\big|_{\theta=\alpha}}. \qquad (2)
\end{aligned}$$

Using equation (2), we can employ SGE techniques to derive estimators of both the numerator and the denominator. The finite difference (FD) method is easy to implement, but often results in large variances. Since the performance functions in both parts are discontinuous with respect to θ, and θ appears explicitly in the indicator functions, infinitesimal perturbation analysis (IPA) and the likelihood ratio (LR) method are not valid. However, for a particular g_2, a push-out LR technique can be applied to push θ into the density function by a change of variable. For example, if g_2 satisfies Assumption 2, $\mathbf{1}\{g_2(U,V)\leq\alpha\}$ is equivalent to $\mathbf{1}\{U\leq h(\alpha)-g^*(V)\}$. Letting $Z_\theta=\frac{U}{\theta}$, under appropriate regularity conditions, we can first derive a single-run unbiased estimator of $\frac{d}{d\theta}\mathbb{E}\left[\mathbf{1}\{Z_\theta\leq 1\}\right]\big|_{\theta=h(\alpha)-g^*(V)}$ by conditioning on V using LR and then taking expectation w.r.t to V. As discussed before, such methodology only works when g_2 enjoys certain properties. To deal with more complicated g_2, we apply the idea of generalized likelihood ratio (GLR) method developed by Peng et al. (2018), which is shown to be a generalization of the push-out LR when the latter can be applied.

3 MAIN RESULTS

Before we introduce the main results, we clarify some notation. When many random variables are involved in an expectation, we use a subscript to indicate with respect to which variables the expectation is taken. If a subscript is not present, the expectation is taken with respect to all present random variables. In addition, we sometimes use $\chi(\cdot)$ in place of the indicator function, i.e., $\chi(z)=\mathbf{1}\{z\leq 0\}$.

3.1 Globally Invertible g_2

The main idea of GLR involves three ingredients (Peng et al. 2018): approximating $\chi(\cdot)$ by a smooth sequence, applying integration by parts, and taking limits. These steps can be justified under some regularity conditions, as presented in Assumption 3.

To summarize, (i) guarantees that the existence of a certain smooth sequence for approximation, (ii) justifies the application of integration by parts and the resulted surface integral vanishes from (iii), and the convergence of the approximated sequences can be shown from (iv).

With these assumptions, a generalized result of Theorem 1 can be obtained, as shown in Theorem 2, whose proof follows Peng et al. (2018) by additionally conditioning on the σ-algebra generated by random variables Y and V.

Assumption 3 Let $\Theta(\alpha) = [\alpha-\tau, \alpha+\tau]$ for some $\tau > 0$ be the interval around α.

(i) There exists a sequence of smooth functions $\{\chi_\varepsilon\}$ such that for some $p \in [1,\infty]$,

$$\lim_{\varepsilon\to 0}\sup_{\theta\in\Theta(\alpha)}\left(\int_{\mathbb{R}}|\chi(g_2(u,V)-\theta))-\chi_\varepsilon(g_2(u,V)-\theta))|^p\,du\right)^{1/p} = 0 \quad a.s.,$$

and for q satisfying $\frac{1}{p}+\frac{1}{q}=1$,

$$\mathbb{E}_{X,U}\left[|g_1(X,Y)w(X,U,V)|^q\right] < \infty, \quad \mathbb{E}_{X,U}\left[|w(X,U,V)|^q\right] < \infty \quad a.s.$$

(ii) The function $g_2(u,V(\omega))$ is twice continuously differentiable with respect to u a.s. and $\partial_u g_2(u,V(\omega))^{-1}$ is differentiable w.r.t $u\in\mathbb{R}$ a.s.

(iii) $\lim_{u\to\pm\infty} g_1(x,Y(\omega)))\partial_u g_2(u,V(\omega))^{-1} f_{X,U}(x,u) = 0$ for all $x\in\mathbb{R}$ and $\theta\in\Theta(\alpha)$, $\lim_{u\to\pm\infty}\partial_u g_2(u,V(\omega))^{-1} f_{X,U}(x,u) = 0$ for all $x\in\mathbb{R}$ and $\theta\in\Theta(\alpha)$.

(iv) $\mathbb{E}_{X,U}\left[\sup_{\theta\in\Theta(\alpha)}|g_1(X,Y)\chi(g_2(u,V)-\theta)w(X,U,V)|\right] < \infty$ a.s., $\mathbb{E}_{X,U}\left[\sup_{\theta\in\Theta(\alpha)}|\chi(g_2(u,V)-\theta)w(X,U,V)|\right] < \infty$ a.s.

Theorem 2 Under Assumption 1 (i)-(iii) and Assumption 3, the conditional expectation (1) can be expressed as

$$\mathbb{E}[g_1(X,Y)|g_2(U,V)=\alpha] = \frac{\mathbb{E}[g_1(X,Y)\chi(g_2(U,V)-\alpha)w(X,U,V)]}{\mathbb{E}[\chi(g_2(U,V)-\alpha)w(X,U,V)]},$$

where

$$w(x,u,v) = -\frac{\partial_u^2 g_2(u,v)}{(\partial_u g_2(u,v))^2} + \frac{\partial_u \ln f_{X,U}(x,u)}{\partial_u g_2(u,v)}. \tag{3}$$

Proof. Under Assumption 1 (ii), by the definition of the conditional expectation, (1) can be formulated as (2). Now we first consider the numerator of (2).

Let $\sigma(Y,V)$ be the σ-algebra generated by random variables Y and V and define

$$\tilde{g}_2(u,v;\theta) = g_2(u,v)-\theta.$$

By Assumption 1 (ii), we have

$$\begin{aligned}
&\frac{d}{d\theta}\mathbb{E}[\mathbb{E}[g_1(X,Y)\chi(g_2(U,V)-\theta))|\sigma(Y,V)]]\\
=&\frac{d}{d\theta}\mathbb{E}\left[\int_{\mathbb{R}^2} g_1(x,Y)\chi(\tilde{g}_2(u,V;\theta))f_{X,U}(x,u)dxdu\right]\\
=&\mathbb{E}\left[\frac{d}{d\theta}\int_{\mathbb{R}^2} g_1(x,Y)\chi(\tilde{g}_2(u,V;\theta))f_{X,U}(x,u)dxdu\right],
\end{aligned} \tag{4}$$

The justification of the second equality will be given later in this proof. Consider the argument inside the expectation operator in (4):

$$\begin{aligned}
&\frac{d}{d\theta}\int_{\mathbb{R}^2} g_1(x,Y)\chi_\varepsilon(\tilde{g}_2(u,V;\theta))f_{X,U}(x,u)dxdu \\
=&\int_{\mathbb{R}^2} g_1(x,Y)\frac{d}{d\theta}\chi_\varepsilon(\tilde{g}_2(u,V;\theta))f_{X,U}(x,u)dxdu \qquad (5)\\
=&-\int_{\mathbb{R}^2} g_1(x,Y)\partial_u\tilde{g}_2(u,V;\theta)^{-1}\frac{d}{du}\chi_\varepsilon(\tilde{g}_2(u,V;\theta))f_{X,U}(x,u)dxdu \\
=&\int_{\mathbb{R}}\Big(-g_1(x,Y)\partial_u\tilde{g}_2(u,V;\theta)^{-1}\chi_\varepsilon(\tilde{g}_2(u,V;\theta))f_{X,U}(x,u)|_{-\infty}^{\infty} \\
&+\int_{\mathbb{R}} g_1(x,Y)\chi_\varepsilon(\tilde{g}_2(u,V;\theta))\partial_u\left(\partial_u\tilde{g}_2(u,V;\theta)^{-1}f_{X,U}(x,u)\right)du\Big)dx \qquad (6)\\
=&\int_{\mathbb{R}^2} g_1(x,Y)\chi_\varepsilon(\tilde{g}_2(u,V;\theta))w(x,u,V)f_{X,U}(x,u)dudx \\
=&\mathbb{E}\left[g_1(X,Y)\chi_\varepsilon(\tilde{g}_2(U,V;\theta))w(X,U,V)\right].
\end{aligned}$$

The interchange of integration and differentiation will also be justified later in this proof. The second equality is obtained by $\frac{d}{du}\chi_\varepsilon(\tilde{g}_2(u,V;\theta)) = \frac{d}{dz}\chi_\varepsilon(z)|_{z=\tilde{g}_2(u,V;\theta)}\cdot\partial_u\tilde{g}_2(u,V;\theta)$ and $\frac{d}{d\theta}\chi_\varepsilon(\tilde{g}_2(u,V;\theta)) = \frac{d}{dz}\chi_\varepsilon(z)|_{z=\tilde{g}_2(u,V;\theta)}\cdot\partial_\theta\tilde{g}_2(u,V;\theta) = -\frac{d}{dz}\chi_\varepsilon(z)|_{z=\tilde{g}_2(u,V;\theta)}$. The third equality is obtained by integration by parts under Assumption 3 (ii). The last equality holds since the first term in (6) vanishes by Assumption 3 (iii). Then we prove the following uniform convergence result:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_{\theta\in\Theta(\alpha)}\left|\mathbb{E}_{X,U}\left[g_1(X,Y)\chi(\tilde{g}_2(U,V;\theta))w(X,U,V)\right]-\mathbb{E}_{X,U}\left[g_1(X,Y)\chi_\varepsilon(\tilde{g}_2(U,V;\theta))w(X,U,V)\right]\right| \\
\le&\lim_{\varepsilon\to 0}\sup_{\theta\in\Theta(\alpha)}\mathbb{E}_{X,U}\left[|g_1(X,Y)w(X,U,V)|\,|\chi(\tilde{g}_2(U,V;\theta))-\chi_\varepsilon(\tilde{g}_2(U,V;\theta))|\right] \\
\le&\lim_{\varepsilon\to 0}\sup_{\theta\in\Theta(\alpha)}\left(\mathbb{E}_{X,U}\left[|g_1(X,Y)w(X,U,V)|^q\right]\right)^{1/q}\left(\int_{\mathbb{R}}|\chi(\tilde{g}_2(u,V;\theta))-\chi_\varepsilon(\tilde{g}_2(u,V;\theta))|^p du\right)^{1/p} \\
\le&C_1\lim_{\varepsilon\to 0}\sup_{\theta\in\Theta(\alpha)}\left(\int_{\mathbb{R}}|\chi(\tilde{g}_2(u,V;\theta))-\chi_\varepsilon(\tilde{g}_2(u,V;\theta))|^p du\right)^{1/p}=0,
\end{aligned}$$

for some constant $C_1 \ge 0$. The second inequality is obtained by Holder's inequality. The third inequality and the last equality are obtained by Assumption 3 (i).

Then by Assumption 3 (iv), we have

$$\mathbb{E}_{X,U}\left[\sup_{\theta\in\Theta(\alpha)}|g_1(X,Y)\chi_\varepsilon(\tilde{g}_2(U,V;\theta))w(X,U,V)|\right]<\infty,$$

which can be used together with the mean value theorem to justify the interchange of integration and differentiation to obtain (5).

Similarly, we have

$$\lim_{\varepsilon\to 0}\sup_{\theta\in\Theta(\alpha)}\left|\mathbb{E}_{X,U}\left[g_1(X,Y)\chi(\tilde{g}_2(U,V;\theta))\right]-\mathbb{E}_{X,U}\left[g_1(X,Y)\chi_\varepsilon(\tilde{g}_2(U,V;\theta))\right]\right|=0.$$

Therefore, by the uniform convergence of $\frac{d}{d\theta}\mathbb{E}_{X,U}\left[g_1(X,Y)\chi_\varepsilon(\tilde{g}_2(U,V;\theta))\right]$ and the convergence of the expectation, we have

$$\frac{d}{d\theta}\mathbb{E}_{X,U}\left[g_1(X,Y)\chi(\tilde{g}_2(U,V;\theta))\right]=\mathbb{E}_{X,U}\left[g_1(X,Y)\chi(\tilde{g}_2(U,V;\theta))w(X,U,V)\right].$$

By Assumption 3 (iv), the interchange of integration and differentiation to obtain (4) can be justified, then

$$\frac{d}{d\theta}\mathbb{E}\left[g_1(X,Y)\chi(\tilde{g}_2(U,V;\theta))\right] = \mathbb{E}\left[g_1(X,Y)\chi(\tilde{g}_2(U,V;\theta))w(X,U,V)\right].$$

Repeating the above steps for the denominator in (2), we have

$$\frac{d}{d\theta}\mathbb{E}\left[\chi(\tilde{g}_2(U,V;\theta))\right] = \mathbb{E}\left[\chi(\tilde{g}_2(U,V;\theta))w(X,U,V)\right].$$

The proof is completed by letting $\theta = \alpha$. □

The verification of Assumption 3 is usually not very straightforward, especially condition (i). One possible choice of the approximated sequence of smooth functions $\{\chi_\varepsilon\}$, as suggested in Peng et al. (2018), can be given by $\chi_\varepsilon(z) = \chi_\varepsilon * \phi_\varepsilon(z)$, where $*$ stands for the convolution operator, ϕ_ε is the density function of a normal distribution $\mathscr{N}(0,\varepsilon^2)$, and χ_ε is defined by

$$\chi_\varepsilon(z) = \begin{cases} 1 & z < -\varepsilon \\ -\frac{1}{2\varepsilon}z + \frac{1}{2} & -\varepsilon \le z \le \varepsilon \\ 0 & z > \varepsilon \end{cases}.$$

Then condition (i) can be checked by taking the function g_2 into account. In the following, we provide some simplified conditions that can imply Assumption 3 (i), (iv), but are easier to check for some cases.

Corollary 3 Under Assumption 1 (i)-(iii) and Assumption 3 (ii)-(iii), suppose that $\mathbb{E}\left[|g_1(X,Y)|^2\right] < \infty$ and $\mathbb{E}\left[|w(X,U,V)|^2\right] < \infty$, the representation of the conditional expectation (1) given in Theorem 2 holds.

Proof. By Holder's inequality, we have

$$\mathbb{E}\left[|g_1(X,Y)w(X,U,V)|\right] \le \left(\mathbb{E}\left[|g_1(X,Y)|^2\right]\right)^{1/2}\left(\mathbb{E}\left[|w(X,U,V)|^2\right]\right)^{1/2} < \infty.$$

By Jensen's inequality, we have $\mathbb{E}\left[|w(X,U,V)|\right] \le \left(\mathbb{E}\left[|w(X,U,V)|^2\right]\right)^{1/2} < \infty$. Taking $p = \infty$ and $q = 1$, we can see Assumption 3 (i) holds. Assumption 3 (iv) can also be verified by noticing that $\chi(\cdot)$ is bounded. □

Comparing Corollary 3 and Theorem 1, we can clearly see that our approach is a generalization of CDM for estimating conditional expectations since Assumption 2 is no longer required. However, to apply our results, some integrability conditions have to be verified.

3.2 g_2 with Stationary Points

One limitation of Theorem 2 is that Assumption 3 (ii) requires that $(\partial_u g_2(u,V))^{-1}$ to be differentiable with respect to u a.s. Generally speaking, this implies that for a fixed sample path, g_2 does not have a stationary point and is a strictly monotonic function of u, which is however weaker than Assumption 2. For instance, $g_2(u,v) := e^{uv} + v$, $v > 0$, satisfies Assumption 3 (ii), but not Assumption 2.

Next, we consider a simplified setting where random variables Y and V are not included and g_2 has stationary points, i.e., $g_2'(u) = 0$ has real roots. The major problem in this setting is that the function $w(x,u) = \partial_u\left(g_2'(u)^{-1}\right) + g_2'(u)^{-1}\partial_u \ln f_{X,U}(x,u)$ defined similar to (3) may not be integrable w.r.t $f_{X,U}(x,u)$. The idea to handle this problem is to consider isolated intervals around the stationary points.

Suppose that $g_2'(u) = 0$ has a finite number of stationary points p_i, $i = 1,\dots,n$, and $p_1 < p_2 < \cdots < p_n$, with $p_0 = -\infty$ and $p_{n+1} = \infty$. Then $g_2(u) = \alpha$ can have at most $n+1$ roots, since it is continuous. We

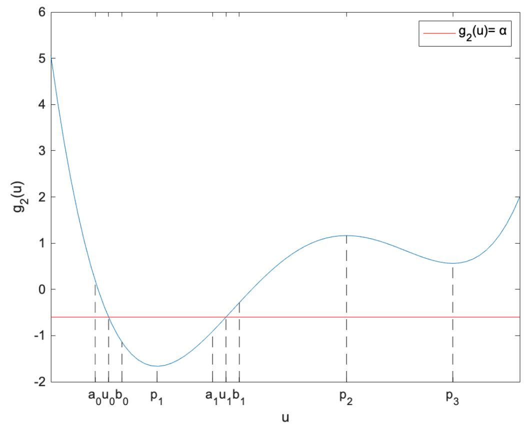

Figure 1: Plot of $g_2(u)$ with stationary points.

denote the roots by u_i, $i = 0, \ldots, k-1$, where $u_0 < u_1 < \cdots < u_{k-1}$ and k is the number of roots. We further assume for any u_i, $g_2(u)$ is locally invertible in a neighborhood around it. Then, there exist points a_i and b_i, such that $p_j < a_i \leq u_i \leq b_i < p_{j+1}$. A simple example is depicted in Figure 1.

Since the conditional expectation can be formulated as a ratio of two SGE problems, as seen in (2), and the integration around the stationary points p_i does not affect the SGE results around $\theta = \alpha$, it is sufficient to consider the SGE problems for $U \in \bigcup_{i=0}^{k-1}[a_i, b_i]$.

Next, we consider the SGE problems around a particular u_i where g_2 is locally invertible. Without loss of generality, we assume that $g_2(a_i) > \alpha$ and $g_2(b_i) \leq \alpha$. Here, we omit the technical details, since they are similar to Theorem 2. By the three ingredients of GLR, under appropriate conditions, for the numerator of the SGE formulation, we have

$$
\begin{aligned}
&\frac{d}{d\theta}\Big(\int g_1(x)\int_{a_i}^{b_i}\mathbf{1}\{g_2(u)\leq\theta\}f_{X,U}(x,u)dudx\Big)|_{\theta=\alpha}\\
&=-\frac{1}{g_2'(b_i)}\int g_1(x)f_{X,U}(x,b_i)dx+\int\int_{a_i}^{b_i}g_1(x)\mathbf{1}\{g_2(u)\leq\alpha\}w(x,u)f_{X,U}(x,u)dudx\\
&=-\frac{1}{g_2'(b_i)}\mathbb{E}[g_1(X)\delta(U-b_i)]]+\mathbb{E}[g_1(X)\mathbf{1}\{g_2(U)\leq\alpha\}w(X,U)\mathbf{1}\{a_i\leq U\leq b_i\}]\\
&=-\frac{1}{g_2'(b_i)}\mathbb{E}[g_1(X)\mathbf{1}\{U\leq b_i\}\partial_u\ln f_{X,U}(X,U)]]+\mathbb{E}[g_1(X)\mathbf{1}\{g_2(U)\leq\alpha\}w(X,U)\mathbf{1}\{a_i\leq U\leq b_i\}],
\end{aligned}
$$

where the first equality is obtained by noticing that $\mathbf{1}\{g_2(a_i)\leq\alpha\}=0$ and $\mathbf{1}\{g_2(b_i)\leq\alpha\}=1$, and the last equality is obtained by GLR again. Thus, by considering piecewise SGE problems, we can represent the conditional expectation by ordinary expectations when g_2 has stationary points.

In most cases, the application of the above results requires a rough knowledge of locations of p_i, $i = 0, 1, \ldots, n+1$, and u_i, $i = 0, \ldots, k-1$. For the special case where $g_2(u)$ has a unique stationary point, a simplified representation can be obtained, an example of which is given in the numerical experiment section (Experiment 3, setting (i)).

3.3 Confidence Interval of the Ratio Estimator

Theorem 2 indicates that we can construct a ratio of two sample averages as an estimator of the conditional expectation from N independent sample paths. Suppose the samples of the numerator and denominator are denoted by A_i and B_i, $i = 1, \ldots, n$, respectively. The sample means $\hat{\mu}_A$, $\hat{\mu}_B$, the sample variances S_A^2, S_B^2, and the sample covariance S_{AB}^2 can be computed correspondingly. Then, the mean and the variance of the ratio estimator

$$R = \frac{\frac{1}{N}\sum_{i=1}^{N} A_i}{\frac{1}{N}\sum_{i=1}^{N} B_i},$$

can be estimated by the delta method (Beyene and Moineddin 2005) as

$$\mathbb{E}[R] \approx \hat{r} = \frac{\hat{\mu}_A}{\hat{\mu}_B},$$
$$\mathbb{V}[R] \approx \hat{\sigma}_R^2 = \frac{1}{N\hat{\mu}_B^2}\left(S_A^2 - 2S_{AB}^2\hat{r} + \hat{r}^2 S_B^2\right).$$

For a large N, a $(1-a)\%$ confidence interval for the ratio estimator is given by

$$\left[\hat{r} - z_{\frac{a}{2}}\hat{\sigma}_R,\ \hat{r} + z_{\frac{a}{2}}\hat{\sigma}_R\right],$$

where $z_{\frac{a}{2}}$ is the $(1-a/2)\%$ quantile of the standard normal distribution.

3.4 Variance Reduction

When formulating the associated stochastic gradient estimation problems, we can use either $\mathbf{1}\{g_2(U,V) \geq \theta\}$ or $\mathbf{1}\{g_2(U,V) \leq \theta\}$. Theoretically, they are both correct, but the sign inside the indicator function can have a significant impact on the variance of the estimator. Suppose that $P(g_2(U,V) \leq \alpha)$ is very small; then, the event $\mathbf{1}\{(g_2(U,V) \leq \alpha)\}$ is a rare event. Thus, if the number of independent replications is relatively limited, it is highly possible that the variances are large. Moreover, for the finite difference method, the denominator may be zero, which results in an invalid ratio estimator. Therefore, a preliminary estimate of the distribution of $g_2(U,V)$ around the given α can be used to guide the appropriate formulation.

4 SIMULATION EXPERIMENTS

In this section, we test our approach on three estimation problems. In the first experiment, we consider a financial option problem and demonstrate the usefulness of our simple variance reduction technique. In the second experiment, we verify our approach by applying it to a toy problem of estimating a conditional expectation where g_2 is globally invertible. In the last experiment, we consider the case where g_2 is locally invertible. We show that our approach can yield better estimations and lower variances, and can outperform the finite difference method.

4.1 Experiment 1: Financial Option

We consider the European option pricing problem for which Daveloose et al. (2019) applied the conditional density method to estimate the option price at maturity T given the stock price at time t, $0 < t < T < \infty$:

$$\mathbb{E}\left[\psi(S_T)|S_t = \alpha\right], \tag{7}$$

where S_t is the stock price at time t, $t \geq 0$. S_t is assumed to follow the Merton model (Merton 1973), i.e.,

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t + \tilde{N}_t\right),$$

where $\{W_t\}$ is the standard Brownian motion, $r > 0$ is the risk-free interest rate and σ is the volatility, $\tilde{N}_t = \sum_{i=1}^{N(t)} Z_i$, $\{N(t)\}$ is a Poisson process with arrival rate λ, and Z_i are i.i.d. random variables following $\mathcal{N}\left(-\frac{\rho^2}{2}, \frac{\rho^2}{2}\right)$ with a constant ρ. Thus, $\{\tilde{N}_t\}$ is a compound Poisson process. For a put option, $\psi(z) =$

$e^{-r(T-t)}(K-z)^{+}$, where K is the strike price, and the true value of (7) can be computed using an analytic formula which is in the form of an infinite sum (Merton 1973).

Let $X=\beta W_T$, $U=\beta W_t$, $Y=\mu T+\tilde{N}_T$, $V=\mu t+\tilde{N}_t$, $g_1(x,y)=\psi(S_0e^{x+y})$, $g_2(u,v)=u+v$. The joint density $f_{X,U}(x,u)$ has an analytic formula. Since the exponential function is monotonically increasing, (7) can be rewritten as

$$\mathbb{E}\left[g_1(X,Y)|g_2(U,V)=\ln(\alpha/S_0)\right].$$

For this problem, Assumption 2 holds, therefore, the CDM is applicable. Since the GLR method is a generalization of CDM, both methods yield the same result:

$$\mathbb{E}\left[\psi(S_T)|S_t=\alpha\right]=\frac{\mathbb{E}\left[\psi(S_T)\mathbf{1}\{S_t\le\alpha\}\frac{TW_t-tW_T}{\sigma t(T-t)}\right]}{\mathbb{E}\left[\mathbf{1}\{S_t\le\alpha\}\frac{TW_t-tW_T}{\sigma t(T-t)}\right]}. \tag{8}$$

Daveloose et al. (2019) considered variance reduction techniques such as localization and control variates. Here, we reduce the estimation variances by appropriately choosing the inequality sign inside the indicator function in (8) based on the value of α, as discussed in Section 3.4.

We use the same parameter values as in Daveloose et al. (2019): $S_0=40$, $K=45$, $r=0.08$, $\sigma^2=\rho^2=0.05$, $\lambda=5$, $T=1$. For CDM, we apply the localization techniques for variance reduction. For GLR, we first estimate $P(S_t\le\alpha)$ using existing sample paths. If the probability exceeds 0.5, we use $\mathbf{1}\{S_t\le\alpha\}$; otherwise, we use $\mathbf{1}\{S_t\ge\alpha\}$. We run the experiments for $\alpha=10$ using $N=5\times10^6$ independent runs. The numerical results are shown in Table 1, where the true values are approximated by truncating the infinite series after 25 terms. From the results, we can see that an appropriate choice of the inequality sign inside the indicator function can have a great impact on the estimation quality. For example, since $P(S_{0.1}\le\alpha)$ is very small, if we use $\mathbf{1}\{S_{0.1}\le\alpha\}$ in the estimator even with the localization technique, the estimator can have a large relative error, whereas using $\mathbf{1}\{S_{0.1}\ge\alpha\}$ as in the GLR estimators leads to much better results.

Table 1: Put option price estimates in Experiment 1 ($\hat{\sigma}_R$ in parentheses, ε is FD perturbation).

t	CDM	GLR	FD ($\varepsilon=0.05$)	TRUE
0.1	28.48(2.01)	32.14(0.98)	27.92(1.49)	31.89
0.2	31.49(0.96)	32.13(0.39)	31.47(1.30)	32.22
0.3	32.54(0.59)	32.84(0.19)	33.06(0.53)	32.56
0.4	32.45(0.37)	32.86(0.13)	32.66(0.43)	32.90
0.5	33.07(0.28)	33.44(0.08)	33.04(0.26)	33.24
0.6	33.08(0.21)	33.65(0.07)	33.49(0.19)	33.58
0.7	33.55(0.19)	33.96(0.05)	33.91(0.14)	33.93
0.8	34.06(0.17)	34.31(0.05)	34.20(0.09)	34.29
0.9	34.53(0.18)	34.66(0.05)	34.68(0.05)	34.64

4.2 Experiment 2: g_2 without Explicit Inverse

Consider estimating

$$\mathbb{E}\left[g_1(X)|g_2(U,V)=\alpha\right], \tag{9}$$

where U is a random variable following a Student's t-distribution with degrees of freedom $\nu=4$, the conditional distribution of $X|U$ is $\mathscr{N}(U,1)$, V follows a Bernoulli distribution, which takes value 1 with

Table 2: Estimations in Experiment 2 ($\hat{\sigma}_R$ in parentheses, ε is FD perturbation).

	$N=10^3$		$N=10^6$		
method	GLR	FD($\varepsilon=0.1$)	GLR	FD($\varepsilon=0.01$)	TRUE
estimation	2.45(0.42)	2.91(0.72)	2.48(0.01)	2.55(0.03)	2.50

probability 3/4 and value 2 with probability 1/4, and $g_1(x)=e^x$, $g_2(u,v)=e^{uv}+u$. For a given $\alpha=0$, the conditional expectation (9) can be computed exactly by applying the properties of the Dirac function.

Since g_2 does not satisfy Assumption 2, we cannot directly apply the conditional density method. Before applying our approach, we need to check the conditions. The joint density function of (X,U) is given by

$$f_{X,U}(x,u)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-u)^2}{2}}\cdot\frac{3}{8\left(1+\frac{u^2}{4}\right)^{5/2}},\quad -\infty<x,u<\infty.$$

It is easy to see that Assumption 1 and Assumption 3 (iii) hold. Since $\chi(\cdot)$ is bounded and the decay rate of joint density $f_{X,U}(x,u)$ when x and u go to $\pm\infty$ suppresses $g_1(x)$ and $\partial_u g_2(u,V)^{-1}$ for any x a.s., Assumption 3 (ii) holds. The other conditions in Corollary 3 can also be verified. Therefore, (9) can be represented by

$$\frac{\mathbb{E}[g_1(X)\mathbf{1}\{g_2(U,V)\le\alpha\}w(X,U,V)]}{\mathbb{E}[\mathbf{1}\{g_2(U,V)\le\alpha\}w(X,U,V)]},$$

where

$$w(x,u,v)=-\frac{v^2e^{uv}}{(ve^{uv}+1)^2}+\frac{-u+x-\frac{5u}{u^2+4}}{ve^{uv}+1}.$$

Given $\alpha=2$, the results are shown in Table 2. When the number of independent runs is small, e.g., $N=1000$, the samples of the denominator may be all zero if using the FD method with a small perturbation ($\varepsilon=0.01$). Thus, for a small number of sample paths, FD requires a larger perturbation ($\varepsilon=0.1$). The results show that GLR yields more accurate estimations, with lower variances than FD.

4.3 Experiment 3: g_2 with Stationary Points

Here, we consider the problem of estimating $\mathbb{E}[g_1(X)|g_2(U)=0]$ where $g_2(u)$ has stationary points for two settings: (i) $g_1(x)=x^2e^{-x}$ and $g_2(u)=1-u^2$, (ii) $g_1(x)=x$ and $g_2(u)=(u-1)(u-2)(u-3)$. In both settings, we assume that $U\sim\mathcal{N}(2,1)$ and $X|U\sim\mathcal{N}(U,1)$, and the true values of the conditional expectation can be computed analytically. The joint density function is given by

$$f_{X,U}(x,u)=\frac{1}{2\pi}e^{-\frac{(x-u)^2}{2}-\frac{(u-2)^2}{2}},\quad -\infty<x,u<\infty.$$

Setting (i). The associated function $w(x,u)$ is given by $w(x,u)=\frac{1}{2u^2}+\frac{2u-x-2}{2u}$. Since $\mathbf{1}\{g_2(u)\le 0\}=0$ around a small neighborhood of the unique stationary point $p_1=0$ of $g_2(u)$, $\mathbb{E}[g_1(X)\mathbf{1}\{g_2(U)\le 0\}w(X,U)]<\infty$ and $\mathbb{E}[\mathbf{1}\{g_2(U)\le 0\}w(X,U)]<\infty$. Therefore, we have

$$\mathbb{E}[g_1(X)|g_2(U)=0]=\frac{\mathbb{E}[g_1(X)\mathbf{1}\{g_2(U)\le 0\}w(X,U)]}{\mathbb{E}[\mathbf{1}\{g_2(U)\le 0\}w(X,U)]}.$$

Setting (ii). The associated function $w(x,u)$ is given by $w(x,u)=-\frac{6u-12}{(3u^2-12u+11)^2}-\frac{2u-x}{3u^2-12u+11}$. $g_2(u)$ has two stationary points $p_1=1.423$ and $p_2=2.577$ and the roots of $g_2(u)=0$ are $u_0=1$, $u_1=2$ and $u_3=3$.

Applying the method introduced in section 3.2, we obtain

$$\begin{aligned}&\mathbb{E}\left[g_1(X)|g_2(U)=0\right]\\ =&\frac{\mathbb{E}\left[g_1(X)\left((\mathbf{1}\{g_2(U)\le 0\}+\mathbf{1}\{U\le b\}-1)\,w(X,U)-\frac{\mathbf{1}\{U\le b\}\partial_u \ln f_{X,U}(X,U)}{g_2'(b)}\right)\right]}{\mathbb{E}\left[(\mathbf{1}\{g_2(U)\le 0\}+\mathbf{1}\{U\le b\}-1)\,w(X,U)-\frac{\mathbf{1}\{U\le b\}\partial_u \ln f_{X,U}(X,U)}{g_2'(b)}\right]},\end{aligned}$$

where b can be any value in $(2,2.577)$. The numerical results for the two settings are given in Table 3, showing that the GLR estimator is more accurate and has a smaller variance compared to the FD estimator.

Table 3: Estimation results in Experiment 3 with 10^6 independent runs ($\hat{\sigma}_R$ in parentheses, ε is FD perturbation, $b=2.3$ for Setting (ii)).

	GLR	FD ($\varepsilon=0.01$)	TRUE
Setting (i)	1.02(0.02)	0.82(0.12)	1.00
Setting (ii)	2.00(0.004)	1.98(0.01)	2.00

5 CONCLUSION

Expressing the conditional expectation as a ratio of two derivatives, we apply the generalized likelihood ratio method to both SGE problems (numerator and denominator). Our method generalizes the conditional density method and thus can handle more complicated measurement functions. For the case where the measurement function has stationary points, we consider the simplified setting without the random variable V. However, our methodology can still work if the random variable V does not affect the locations of the stationary points of $g_2(u,V)$ and the roots of $g_2(u,V)=\alpha$. Future work is needed for the more general setting when this is not the case. We also provide a simple way to reduce the variance of the estimator, which may be incorporated with jackknifing to reduce the ratio bias and importance sampling to achieve further variance reduction.

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