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# VARIANCE REDUCTION FOR GENERALIZED LIKELIHOOD RATIO METHOD IN QUANTILE SENSITIVITY ESTIMATION

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## ABSTRACT

We apply the generalized likelihood ratio (GLR) methods in Peng et al. (2018) and Peng et al. (2021) to estimate quantile sensitivities. Conditional Monte Carlo and randomized quasi-Monte Carlo methods are used to reduce the variance of the GLR estimators. The proposed methods are applied to a toy example and a stochastic activity network example. Numerical results show that the variance reduction is significant.

## 1 INTRODUCTION

Quantile, also known as value-at-risk (VaR), is an important risk measure. Quantile sensitivity estimation has been actively studied in simulation due to its centrality in risk management. A seminal work is Hong (2009), which proposes a conditional infinitesimal perturbation analysis (IPA). Jiang and Fu (2015) avoid the need of computing the conditional expectation for some special cases. Liu and Hong (2009) propose a kernel-based estimator for estimating the quantile sensitivity, which can be extended to estimate sensitivities of conditional VaR (CVaR) (Hong and Liu 2009). A conditional Monte Carlo (CMC) method is applied to estimate quantile sensitivity in Fu et al. (2009). Heidergott and Volk-Makarewicz (2016) propose a measure-valued differentiation (MVD) estimator for quantile sensitivity. CMC and MVD for quantile sensitivity rely on estimating two distribution sensitivities, i.e., the derivatives of the distribution function with respect to both argument and parameters in the stochastic model.

The difficulty in distribution sensitivity estimation lies in the discontinuity introduced by the indicator function. Recently, Peng et al. (2018) propose a generalized likelihood ratio (GLR) method to handle discontinuities for a wide scope of sample performance in sensitivity analysis. Peng et al. (2020) develop GLR estimators for any distribution sensitivity, and use it to calculate maximum likelihood estimation for

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complex stochastic models. Peng et al. (2021) derive a GLR estimator for stochastic models with uniform random numbers as input, relaxing a requirement in Peng et al. (2018) that the tails of the input distribution of the stochastic model go smoothly to zero fast enough. Peng et al. (2017) apply GLR to estimate quantile sensitivity, and Peng et al. (2019) extend the GLR quantile sensitivity estimator to stochastic model with correlations and jumps. Glynn et al. (2021) use the GLR estimators for distribution sensitivities to estimate derivatives of distortion risk measures, which cover VaR and CVaR as special cases.

CMC methods can reduce the variance and smooth the performance function in simulation by conditioning on certain events or random variables and then integrating out the remaining randomness (Asmussen and Glynn 2007). Fu et al. (2009) uses CMC to smooth the discontinuity introduced by the indicator function, after which IPA is applied to differentiate the conditional expectation. GLR does not need smoothing to obtain an unbiased distribution sensitivity estimator, but CMC can be applied afterward to reduce the variance for GLR.

Randomized quasi-Monte Carlo (RQMC) methods, which can also reduce variance, replace the vectors of uniform random numbers that drive independent simulation runs by dependent vectors of uniform random numbers that cover the space more evenly. When estimating an expectation, they can provide an unbiased estimator with its variance converging to zero at a rate that is faster than with Monte Carlo (Lemieux 2009; Dick and Pillichshammer 2010). Such a faster rate can be proved when the estimator inside the expectation is sufficiently smooth as a function of the underlying uniform random numbers. When the estimator is not smooth (e.g., discontinuous), the convergence rate may not be improved, but RQMC could still reduce the variance by a constant factor. We show, through simulation experiments, that the variance of the quantile sensitivity estimator by GLR can be significantly reduced by appropriately combining the method with CMC and RQMC. Similar use of CMC and RQMC for reducing the variance of quantile estimation can be found in Nakayama et al. (2020).

The rest of the paper is organized as follows. Section 2 introduces the quantile sensitivity estimation problem and the GLR method. Variance reduction by CMC and RQMC is discussed in Section 3. Section 4 provide applications and simulation experiments. The last section offers conclusions.

## 2 QUANTILE SENSITIVITY ESTIMATION

For $0 \leq \alpha \leq 1$, the $\alpha$-quantile (also known as value-at-risk) of an output random variable $Y_\theta = h(X;\theta)$ with input random variables $X = (X_1, \ldots, X_n)$ and cumulative distribution function (cdf) $F(\cdot;\theta)$ is defined as

$$q_\alpha(\theta) := \arg\inf\{y : \ F(y;\theta) \geq \alpha\}. \tag{1}$$

When $F(\cdot;\theta)$ is continuous, $q_\alpha(\theta) = F^{-1}(\alpha;\theta)$. Our goal is to estimate the derivative of the $\alpha$-quantile with respect to parameter $\theta$, i.e., $dq_\alpha(\theta)/d\theta$, which is referred to as quantile sensitivity. Assume $h(X;\theta)$ is a continuous r.v. with a positive and continuous density $f(y;\theta)$ on $(q_\alpha(\theta)-\varepsilon, q_\alpha(\theta)+\varepsilon)$, $\varepsilon > 0$. Using the formula for the derivative of an implicit function, we have (Fu et al. 2009)

$$\frac{d}{d\theta} q_\alpha(\theta) = -\left.\frac{\partial F(y;\theta)}{\partial \theta}\right|_{y=q_\alpha(\theta)} \Big/ f(q_\alpha(\theta);\theta).$$

Notice that

$$\frac{\partial F(y;\theta)}{\partial \theta} = \frac{\partial \mathbb{E}\left[\mathbf{1}\{h(X;\theta) \leq y\}\right]}{\partial \theta}, \quad f(y;\theta) = \frac{\partial \mathbb{E}\left[\mathbf{1}\{h(X;\theta) \leq y\}\right]}{\partial y}\ . \tag{2}$$

From Peng et al. (2020), we have the GLR estimators for these distribution sensitivities: $G_{1,i}(X;y,\theta) := \mathbf{1}\{h(X;\theta) \leq y\}\psi_{1,i}(X;\theta)$ and $G_{2,i}(X;y,\theta) := \mathbf{1}\{h(X;\theta) \leq y\}\psi_{2,i}(X;\theta)$, $i = 1, \ldots, n$, such that

$$\frac{\partial F(y;\theta)}{\partial \theta} = \mathbb{E}\left[G_{1,i}(X;y,\theta)\right], \quad f(y;\theta) = \mathbb{E}\left[G_{2,i}(X;y,\theta)\right],$$

where

$$\psi_{1,i}(x;\theta) = \frac{\partial \log f_X(x;\theta)}{\partial \theta} - \left(\frac{\partial h(x;\theta)}{\partial x_i}\right)^{-1}\left[\frac{\partial h(x;\theta)}{\partial \theta}\left(\frac{\partial \log f_X(x;\theta)}{\partial x_i} - \frac{\partial^2 h(x;\theta)}{\partial x_i^2}\left(\frac{\partial h(x;\theta)}{\partial x_i}\right)^{-1}\right) + \frac{\partial^2 h(x;\theta)}{\partial \theta \partial x_i}\right],$$

$$\psi_{2,i}(x;\theta) = \left(\frac{\partial h(x;\theta)}{\partial x_i}\right)^{-1}\left(\frac{\partial \log f_X(x;\theta)}{\partial x_i} - \frac{\partial^2 h(x;\theta)}{\partial x_i^2}\left(\frac{\partial h(x;\theta)}{\partial x_i}\right)^{-1}\right),$$

with $f_X(\cdot;\theta)$ being the (joint) density of the vector of input random variables $X$. From Peng et al. (2021), we have the GLR estimators for distribution sensitivities when $X$ is a vector of uniform random numbers $U = (U_1,\dots,U_n)$:

$$\widetilde{G}_{1,i}(U;y,\theta) := \mathbf{1}\{h(\overline{U}_i;\theta) - y \le 0\} r_i(\overline{U}_i;\theta) - \mathbf{1}\{h(\underline{U}_i;\theta) - y \le 0\} r_i(\underline{U}_i;\theta) + \mathbf{1}\{h(U;\theta) - y \le 0\} d(U;\theta),$$

where

$$r_i(u;\theta) = \left(\frac{\partial h(u;\theta)}{\partial u_i}\right)^{-1}\frac{\partial h(u;\theta)}{\partial \theta},$$

$$d(u;\theta) = \left(\frac{\partial h(u;\theta)}{\partial u_i}\right)^{-1}\left[\left(\frac{\partial h(u;\theta)}{\partial u_i}\right)^{-1}\frac{\partial h(u;\theta)}{\partial \theta}\frac{\partial^2 h(u;\theta)}{\partial u_i^2} - \frac{\partial^2 h(u;\theta)}{\partial u_i \partial \theta}\right],$$

and

$$\widetilde{G}_{2,i}(U;z,\theta) := \mathbf{1}\{h(\overline{U}_i;\theta) - y \le 0\} \tilde{r}_i(\overline{U}_i;\theta) - \mathbf{1}\{h(\underline{U}_i;\theta) - y \le 0\} \tilde{r}_i(\underline{U}_i;\theta) + \mathbf{1}\{h(U;\theta) - y \le 0\} \tilde{d}(U;\theta),$$

where $\overline{U}_i := (U_1,\dots,\underbrace{1^-}_{i\text{th element}},\dots,U_n)$, $\underline{U}_i := (U_1,\dots,\underbrace{0^+}_{i\text{th element}},\dots,U_n)$ with $x^-$ and $x^+$ being the limits taken from left-hand side and right-hand side of $x$, respectively, and

$$\tilde{r}_i(u;\theta) = -\left(\frac{\partial h(u;\theta)}{\partial u_i}\right)^{-1} \qquad \tilde{d}(u;\theta) = -\left(\frac{\partial h(u;\theta)}{\partial u_i}\right)^{-2}\frac{\partial^2 h(u;\theta)}{\partial u_i^2}.$$

Let $X^{(j)}$, $j = 1,\dots,m$, be i.i.d. realizations of $X$, and $\hat{F}_m(\cdot;\theta)$ be the empirical distribution of $h(X^{(j)};\theta)$, $j = 1,\dots,m$. The empirical $\alpha$-quantile $\hat{F}_m^{-1}(\alpha;\theta)$ is the inverse of the empirical cdf evaluated at $\alpha$, defined as in (1) with $F^{-1}$ replaced by $\hat{F}_m^{-1}$. Then the quantile sensitivity can be estimated by

$$-\left.\frac{\sum_{j=1}^m \mathbf{1}\{h(X^{(j)};\theta) \le y\}\psi_{1,i}(X^{(j)};\theta)}{\sum_{j=1}^m \mathbf{1}\{h(X^{(j)};\theta) \le y\}\psi_{2,i}(X^{(j)};\theta)}\right|_{y=\hat{F}_m^{-1}(\alpha;\theta)}. \tag{3}$$

This is a ratio of averages as a function of $y$ evaluated at a random estimate, used to estimate a ratio of expectations. Even if each term in the ratio (3) is an unbiased estimator of the corresponding expectation, the ratio (3) is not an unbiased estimator of the ratio of expectations. There is also another source of bias for the sensitivity: the quantile estimator itself is also biased. Also, the two terms in (3) and quantile estimators are not independent because they are estimated by a same batch of inputs and outputs: $X^{(j)}$ and $h(X^{(j)};\theta)$, $j = 1,\dots,m$. The strong consistency and a central limit theorem for (3) have been established in Peng et al. (2017) using an empirical process theory. The asymptotic results for a CMC estimator of quantile sensitivity can also be found in Peng et al. (2017).

Here we present results for the GLR estimators of two more general stochastic models, the details of which are referred to previous work. Peng et al. (2018) consider sensitivity analysis for more general stochastic models:

$$\frac{\partial}{\partial\theta}\mathbb{E}[\varphi(g(X;\theta))], \tag{4}$$

where $\varphi:\ \mathbb{R}^n\to\mathbb{R}$ is a measurable function not necessarily continuous, $g(x;\theta)=(g_1(x;\theta),\dots,g_n(x;\theta))^T$ is a vector of functions with sufficient smoothness for $x\in\mathbb{R}^n$, and $X=(X_1,\dots,X_n)$ is a vector of input random variables with a joint density $f_X(x;\theta)$ supported on $\Omega\subseteq\mathbb{R}^n$. The Jacobian of $g(\cdot;\theta)$ is

$$J_g(x;\theta):=\begin{pmatrix} \frac{\partial g_1(x;\theta)}{\partial x_1} & \frac{\partial g_1(x;\theta)}{\partial x_2} & \cdots & \frac{\partial g_1(x;\theta)}{\partial x_n} \\ \frac{\partial g_2(x;\theta)}{\partial x_1} & \frac{\partial g_2(x;\theta)}{\partial x_2} & \cdots & \frac{\partial g_2(x;\theta)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n(x;\theta)}{\partial x_1} & \frac{\partial g_n(x;\theta)}{\partial x_2} & \cdots & \frac{\partial g_n(x;\theta)}{\partial x_n} \end{pmatrix}.$$

The Jacobian matrix is required to be invertible almost everywhere, which justifies local invertibility of function $g$ by the implicit function theory. The conditions for justifying unbiasedness are referred to Peng et al. (2018) and Peng et al. (2021). Let $e_i$ be the $i$-th unit vector. We define

$$\begin{aligned} d(x;\theta):=&\sum_{i=1}^{n} e_i^T J_g^{-1}(x;\theta)\,(\partial_{x_i}J_g(x;\theta))\,J_g^{-1}(x;\theta)\partial_\theta g(x;\theta)\\ &-\mathrm{trace}\left(J_g^{-1}(x;\theta)\partial_\theta J_g(x;\theta)\right)-\left(J_g^{-1}(x;\theta)\partial_\theta g(x;\theta)\right)^T\nabla_x\log f_X(x;\theta), \end{aligned}$$

where and $\partial_y M(y)$ is the matrix obtained by differentiating $M$ with respect to $y$ element-wise. From Peng et al. (2018), when the tails of $f_X(\cdot;\theta)$ go smoothly to zero fast enough, then, under certain regularity conditions, it can be shown that an unbiased GLR estimator for (4) is given by

$$G(X;\theta):=\varphi(g(X;\theta))w(X;\theta)\qquad\text{with}\quad w(x;\theta):=\frac{\partial\log f_X(x;\theta)}{\partial\theta}+d(x;\theta). \tag{5}$$

As a particular stochastic activity network (SAN) example, the output could be the maximum of the durations of activities on different paths. By the GLR method under general framework (5), we can consider quantile sensitivity for $\max_{i=1,\dots,n} g_i(X;\theta)$, which has a distribution function expressed as

$$\mathbb{E}\left[\mathbf{1}\left\{\max_{i=1,\dots,n} g_i(X;\theta)\le y\right\}\right]=\mathbb{E}\left[\prod_{i=1}^{n}\mathbf{1}\left\{g_i(X;\theta)-y\le 0\right\}\right],$$

with $\varphi(y)=\prod_{i=1}^n\mathbf{1}\{y_i\le 0\}$. Peng et al. (2021) consider the case when $X$ is a vector of uniform random numbers $U=(U_1,\dots,U_n)$ such that $\Omega=(0,1)^n$ and $\partial\log f_X(x;\theta)/\partial\theta=0$. Then under certain regularity conditions, we have the following unbiased GLR estimator for (4):

$$\widetilde{G}(U;\theta):=\sum_{i=1}^{n}\left[\varphi(g(\overline{U}_i;\theta))r_i(\overline{U}_i;\theta)-\varphi(g(\underline{U}_i;\theta))r_i(\underline{U}_i;\theta)\right]+\varphi(g(U;\theta))d(U;\theta), \tag{6}$$

where

$$r_i(x;\theta):=\left(J_g^{-1}(x;\theta)\,\partial_\theta g(x;\theta)\right)^T e_i,\quad i=1,\dots,n.$$

## 3 VARIANCE REDUCTION

CMC methods can reduce the variance and smooth the performance function in simulation by conditioning on certain events or random variables and then integrating out the remaining randomness (Asmussen and Glynn (2007)). For an estimator $H(Z)$, we have

$$\mathbb{E}[H(Z)] = \mathbb{E}[\widehat{H}(\widehat{Z})],$$

where $\widehat{H}(\widehat{Z}) := \mathbb{E}[H(Z)|\widehat{Z}]$, with $\widehat{Z}$ being a subset of input random variables in $Z$. The variance reduction for the conditional estimator $\widehat{H}(\widehat{Z})$ can be seen from the following variance decomposition formula:

$$\begin{aligned}\mathrm{Var}(H(Z)) &= \mathrm{Var}(\mathbb{E}[H(Z)|\widehat{Z}]) + \mathbb{E}[\mathrm{Var}(H(Z)|\widehat{Z})] \\ &\geq \mathrm{Var}(\widehat{H}(\widehat{Z})).\end{aligned}$$

Typically, $\widehat{H}(\widehat{Z})$ is smoother than $H(Z)$, due to the integration taken in the conditional expectation.

Quasi-Monte Carlo (QMC) refers to a class of deterministic numerical integration methods in which the integrand is evaluated at a fixed set of $m$ points, and the average is used as an approximation. One limitation of the method is that it is very hard to estimate the approximation error in practice. RQMC takes the QMC points and randomizes them in a way that each point has the uniform distribution over $(0,1)^n$, so that each randomized point represents a proper realization of $U$ while the set of $m$ points still covers the unit hypercube $(0,1)^n$ more uniformly than typical independent random points (so the points are not independent) (L'Ecuyer 2018). RQMC performs particularly well in the case when the effective dimension is low and the integrant is smooth (see Lemieux 2009). Thus, the smoothing effect of CMC can improve the performance of RQMC.

In general, for a given function $p$, RQMC estimates the integral $\mu = \int_{(0,1)^n} p(u)du$ by the average

$$\hat{\mu}_m := \frac{1}{m}\sum_{j=1}^{m} p(U^{(j)}),$$

where $U^{(1)},\ldots,U^{(m)}$ form an RQMC point set. The most common types of QMC point set constructions are lattice rules, polynomial lattices rules, and digital nets (Dick and Pillichshammer 2010; L'Ecuyer 2009). For lattice rules, an appropriate randomization is a random shift modulo 1, which adds a single uniform random point to all the lattice points, and retains the shifted points that are in $(0,1)^n$ as the $m$ RQMC points. This randomization preserves the lattice structure, and there are explicit expressions for $Var[\hat{\mu}_m]$ in terms of the Fourier coefficients of $h$, and computable bounds on this variance for certain classes of smooth functions (L'Ecuyer and Lemieux 2000; L'Ecuyer and Munger 2012; L'Ecuyer and Munger 2016). When the mixed derivatives of $p$ are sufficiently smooth, the variance can converge at a faster rate than $\mathscr{O}(m^{-1})$, sometimes nearly $\mathscr{O}(m^{-2})$ and even faster in some cases. When $p$ is not smooth (e.g., discontinuous), these convergence rate results do not apply, although weaker results do apply (He and Wang 2015), and even when the convergence rate is not improved, the variance is often reduced by a constant factor. For polynomial lattices rules and digital nets in general, which include Sobol' points, the random shift does not preserve the structure and net properties, but other appropriate randomizations do, including nested uniform scrambling, some affine scrambles, and random digital shifts. Variance bounds and convergence rate results are also available for these rules (Dick and Pillichshammer 2010; L'Ecuyer 2009).

Our model formulation (6) in terms of a function of independent $\mathscr{U}(0,1)$ random variables makes it an obvious candidate for the application of RQMC, which is designed exactly for this type of formulation. For formulation (5), suppose $X$ can be generated by $\Gamma(U)$, and RQMC can be applied. For example, when $X_1,\ldots,X_n$ are independent random variables with marginal distribution functions $F_1,\ldots,F_n$, then they can be generated by $X_i = F_i^{-1}(U_i)$, $i = 1,\ldots,n$. For the $\alpha$-quantile sensitivity estimation, the cdf $F(\cdot;\theta)$ of

$p(U)$ can be estimated by its empirical RQMC counterpart

$$\mathscr{F}_m(y) := \frac{1}{m}\sum_{j=1}^{m} \mathbf{1}\{p(U^{(j)}) \le y\},$$

where $\{U^{(1)}, \ldots, U^{(m)}\}$ is an RQMC point set, and the quantile $q_\alpha(\theta)$ can be estimated by the pseudo-inverse $\mathscr{F}_m^{-1}(\alpha)$. We can also use RQMC points in the sensitivity estimate (3).

## 4 NUMERICAL EXPERIMENTS

We apply the proposed method to two applications. GLR is compared with the finite difference method $\big(\hat{F}_m^{-1}(\alpha;\theta+\delta) - \hat{F}_m^{-1}(\alpha;\theta)\big)/\delta$ using common random numbers (FDC($\delta$)) in generating output random variables under perturbed parameter for estimating quantile. The GLR method together with CMC is called conditional GLR (CGLR), and CGLR together with RQMC is denoted as CGLRQ. For RQMC, we use the Sobol sequence scrambled by the algorithm of Matousek (1998) in Matlab. In the two examples, the sample size for estimating quantile sensitivities are set as $m = 2^{15}$ for the standard Monte Carlo and RQMC estimators, and the quantile senstivities and standard errors of the estimators are estimated by $10^4$ independent experiments.

### 4.1 TOY EXAMPLE

To illustrate, we estimate quantile sensitivity of a simple stochastic model $\theta X_1 + X_2 + U$, where $X_1, X_2$ are standard normal random variables with the common cdf $\Phi(\cdot)$, and $U$ is an independent uniform random variable $\mathscr{U}(0,1)$. In the stochastic model for deriving GLR estimators, the input random variable could either be taken as the normal random variables $X_1, X_2$ or uniform random number $U$. If we choose $U$ to be the input random variable, then $h(u;\theta,x_1,x_2) = \theta x_1 + x_2 + u$, $\partial h(u;\theta,x_1,x_2)/\partial u = 1$, $\partial h(u;\theta,x_1,x_2)/\partial\theta = x_1$, and other derivatives in (6) are zeros, so GLR estimators for distribution sensitivities are given by

$$\begin{aligned}
\widetilde{G}_{1,1}(U;\theta,X_1,X_2) &= \mathbf{1}\{\theta X_1 + X_2 + 1 \le y\}X_1 - \mathbf{1}\{\theta X_1 + X_2 \le y\}X_1,\\
\widetilde{G}_{2,1}(U;\theta,X_1,X_2) &= \mathbf{1}\{\theta X_1 + X_2 \le y\} - \mathbf{1}\{\theta X_1 + X_2 + 1 \le y\}.
\end{aligned}$$

By conditioning on $X_1$, we have

$$\begin{aligned}
\mathbb{E}[\widetilde{G}_{1,1}(U;y,\theta,X_1,X_2)] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\{\theta X_1 + X_2 + 1 \le y\}X_1 - \mathbf{1}\{\theta X_1 + X_2 \le y\}X_1 \middle| X_1\right]\right]\\
&= \mathbb{E}[(\Phi(y-\theta X_1 - 1) - \Phi(y - \theta X_1))X_1],\\
\mathbb{E}[\widetilde{G}_{2,1}(U;y,\theta,X_1,X_2)] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\{\theta X_1 + X_2 \le y\} - \mathbf{1}\{\theta X_1 + X_2 + 1 \le y\} \middle| X_1\right]\right]\\
&= \mathbb{E}[\Phi(y-\theta X_1) - \Phi(y-\theta X_1 - 1)].
\end{aligned}$$

We can take the expressions inside the expectations on the right-hand sides of the above equations as conditional GLR estimators. From this derivation, three quantile sensitivity estimators of the form (3) using GLR, CGLR, and CGLRQ to estimate distribution sensitivities in (2) are referred to as GLR-1, CGLR-1, and CGLRQ-1, respectively.

If we choose $X_1$ to be the input random variable, then $h(x_1,x_2;\theta,u) = \theta x_1 + x_2 + u$, $\partial h(x_1,x_2;\theta,u)/\partial x_1 = \theta$, $\partial h(x_1,x_2;\theta,u)/\partial\theta = x_1$, $\partial^2 h(x_1,x_2;\theta,u)/\partial\theta\partial x_1 = 1$, $\nabla_x \log f_X(x_1,x_2) = -(x_1,x_2)$, and the other derivatives in (5) are zeros, so GLR estimators for distribution sensitivities can be given by

$$\begin{aligned}
G_{1,1}(X_1,X_2;y,\theta,U) &= \frac{1}{\theta}\mathbf{1}\{\theta X_1 + X_2 + U \le y\}(X_1^2 - 1),\\
G_{2,1}(X_1,X_2;y,\theta,U) &= -\frac{1}{\theta}\mathbf{1}\{\theta X_1 + X_2 + U \le y\}X_1
\end{aligned}$$

Table 1: Sensitivity estimates for $\alpha$-quantile (mean $\pm$ standard error) based on $10^4$ independent experiments for numerical example in Section 4.1.

| | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 0.9$ |
|---|---|---|---|---|---|
| GLR-1 | $-0.888 \pm 1.4\times10^{-4}$ | $-0.363 \pm 1.0\times10^{-4}$ | $0 \pm 1.0\times10^{-4}$ | $0.363 \pm 1.0\times10^{-4}$ | $0.887 \pm 1.4\times10^{-4}$ |
| CGLR-1 | $-0.888 \pm 1.0\times10^{-4}$ | $-0.363 \pm 8\times10^{-5}$ | $0 \pm 7\times10^{-5}$ | $0.364 \pm 8\times10^{-5}$ | $0.888 \pm 1.0\times10^{-4}$ |
| CGLRQ-1 | $-0.888 \pm 4.7\times10^{-5}$ | $-0.363 \pm 4.1\times10^{-5}$ | $0 \pm 3.9\times10^{-5}$ | $0.363 \pm 4.1\times10^{-5}$ | $0.888 \pm 4.7\times10^{-5}$ |
| GLR-2 | $-0.888 \pm 2.3\times10^{-4}$ | $-0.363 \pm 1.8\times10^{-4}$ | $0 \pm 2.1\times10^{-4}$ | $0.363 \pm 2.9\times10^{-4}$ | $0.890 \pm 7.3\times10^{-4}$ |
| CGLR-2 | $-0.888 \pm 2.0\times10^{-4}$ | $-0.363 \pm 1.8\times10^{-4}$ | $0 \pm 2.0\times10^{-4}$ | $0.364 \pm 2.8\times10^{-4}$ | $0.889 \pm 7.2\times10^{-4}$ |
| CGLRQ-2 | $-0.888 \pm 4.9\times10^{-5}$ | $-0.363 \pm 4.1\times10^{-5}$ | $0 \pm 3.9\times10^{-5}$ | $0.363 \pm 4.1\times10^{-5}$ | $0.888 \pm 4.9\times10^{-5}$ |
| FDC(0.01) | $-0.888 \pm 1.2\times10^{-3}$ | $-0.364 \pm 8.5\times10^{-4}$ | $0 \pm 7.9\times10^{-4}$ | $0.365 \pm 8.6\times10^{-4}$ | $0.890 \pm 1.2\times10^{-3}$ |
| FDC(0.1) | $-0.911 \pm 3.9\times10^{-4}$ | $-0.373 \pm 2.7\times10^{-4}$ | $0 \pm 2.6\times10^{-4}$ | $0.372 \pm 2.8\times10^{-4}$ | $0.910 \pm 3.9\times10^{-4}$ |

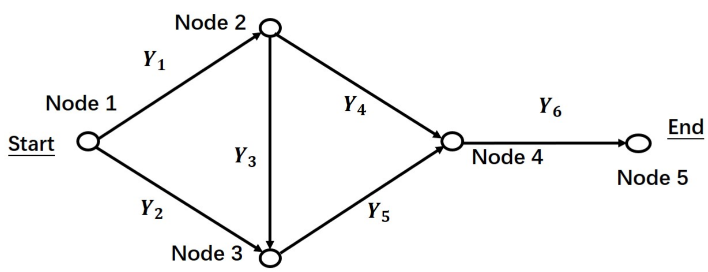


Figure 1: A SAN with seven activities.

By conditioning on $X_1, U$, we have

$$\mathbb{E}[G_{1,1}(X_1,X_2;y,\theta,U)] = \mathbb{E}\left[\mathbb{E}\left[\frac{1}{\theta}\mathbf{1}\{\theta X_1 + X_2 + U \le y\}X_1^2\middle|X_1,U\right]\right] = \frac{1}{\theta}\mathbb{E}[\Phi(y-\theta X_1 - U)X_1^2],$$
$$\mathbb{E}[G_{2,1}(X_1,X_2;y,\theta,U)] = \mathbb{E}\left[\mathbb{E}\left[-\frac{1}{\theta}\mathbf{1}\{\theta X_1 + X_2 + U \le y\}X_1\middle|X_1,U\right]\right] = -\frac{1}{\theta}\mathbb{E}[\Phi(y-\theta X_1 - U)X_1].$$

We can take the expressions inside the expectations on the right-hand sides of the above equations as conditional GLR estimators. From this derivation, three quantile sensitivity estimators of the form (3) using GLR, CGLR, and CGLRQ to estimate distribution sensitivities are referred to as GLR-2, CGLR-2, and CGLRQ-2, respectively.

For our experiments, we took $\theta = 1$ and five probability levels: $\alpha = 0.1$, $\alpha = 0.3$, $\alpha = 0.5$, $\alpha = 0.7$, and $\alpha = 0.9$. From Table 1, we can see that the quantile sensitivity estimates by GLR-1, CGLR-1, CGLRQ-1, GLR-2, CGLR-2, CGLRQ-2, and FDC(0.01) are statistically indistinguishable, whereas the quantile sensitivity estimates by FDC(0.1) lie outside of the 90% confidence intervals of other methods. GLR leads to smaller variance than FDC, the variances of GLR-1, CGLR-1, CGLRQ-1 are smaller than GLR-2, CGLR-2, CGLRQ-2, respectively, and the CMC and RQMC lead to significant variance reduction.

## 4.2 STOCHASTIC ACTIVITY NETWORK (SAN)

We estimate quantile sensitivity for a simple SAN. There are three different paths representing the tasks to reach the final stage of a project, i.e., $(1,4,6)$, $(2,5,6)$, $(1,3,5,6)$. The completion time for the entire project is $Y := \max(Y_1+Y_4+Y_6, Y_2+Y_5+Y_6, Y_1+Y_3+Y_5+Y_6)$, and the sample performance for the cdf of the completion time is

$$\mathbf{1}\{\max(Y_1+Y_4+Y_6, Y_2+Y_5+Y_6, Y_1+Y_3+Y_5+Y_6) \le y\}.$$

We assume that the first three activities follow independent exponential distributions: $Y_i = -\frac{1}{\lambda_i}\log(U_i)$, $i = 1,2,3$, and the other three activities follow independent log-normal distributions, $Y_i = \exp(\mu_i + \sigma_i X_i)$,

$i=4,5,6$, where $X_i$'s are standard normal random variables. Let $\lambda_1=\theta$. The distribution function $F(y;\theta)$ of the completion time $Y$ is

$$F(y;\theta)=\mathbb{E}[\mathbf{1}\{Y_1+\max(Y_4,Y_3+Y_5)+Y_6-y\leq 0\}\mathbf{1}\{Y_2+Y_5+Y_6-y\leq 0\}].$$

If we keep the original indicator above, then the image of $g$ is in $\mathbb{R}^3$ and we need three inputs. With the rewriting, the image of $g$ is in $\mathbb{R}^2$ and we select two inputs in a way that the Jacobian is invertible. To estimate the two distribution sensitivities, we can view $(U_1,U_2)$ as the input random variables in the stochastic model of Peng et al. (2021), and we have

$$\begin{aligned}
&g_1(U_1,U_2;\theta,y,U_3,X_4,X_5,X_6)=-\frac{1}{\lambda_1}\log U_1+\max(Y_4,Y_3+Y_5)+Y_6-y,\\
&g_2(U_1,U_2;\theta,y,U_3,X_4,X_5,X_6)=-\frac{1}{\lambda_2}\log U_2+Y_5+Y_6-y,\\
&\partial_\theta g(u_1,u_2;\theta,y,u_3,x_4,x_5,x_6)=\left(\log u_1/\lambda_1^2,0\right)^T,\quad \partial_y g(u_1,u_2;\theta,y,u_3,x_4,x_5,x_6)=-\left(1,1\right)^T.
\end{aligned}$$

The Jacobian matrix and its inverse are

$$J_g(u_1,u_2;\theta,y)=-\begin{pmatrix}\frac{1}{\lambda_1u_1} & 0\\ 0 & \frac{1}{\lambda_2u_2}\end{pmatrix},\quad J_g^{-1}(u_1,u_2;\theta,y)=-\begin{pmatrix}\lambda_1u_1 & 0\\ 0 & \lambda_2u_2\end{pmatrix},$$

$$\partial_{u_1}J_g(u_1,u_2;\theta,y)=\begin{pmatrix}\frac{1}{\lambda_1u_1^2} & 0\\ 0 & 0\end{pmatrix},\ \partial_{u_2}J_g(u_1,u_2;\theta,y)=\begin{pmatrix}0 & 0\\ 0 & \frac{1}{\lambda_2u_2^2}\end{pmatrix},\ \partial_\theta J_g(u_1,u_2;\theta,y)=\begin{pmatrix}\frac{1}{\lambda_1^2u_1} & 0\\ 0 & 0\end{pmatrix}$$

and

$$\begin{aligned}
&r_1(u_1,u_2;\theta)=\frac{1}{\lambda_1}u_1\log u_1,\quad r_2(u_1,u_2;\theta)=0,\quad d(u_1,u_2;\theta)=\frac{1}{\lambda_1}(\log u_1+1),\\
&\tilde{r}_1(u_1,u_2;\theta)=\lambda_1u_1,\quad \tilde{r}_2(u_1,u_2;\theta)=\lambda_2u_2,\quad \tilde{d}(u_1,u_2;\theta)=-\lambda_1-\lambda_2.
\end{aligned}$$

Then the GLR estimator of $\partial F(y;\theta)/\partial\theta$ is given by

$$G_1(U_1,U_2;\theta,y,U_3,X_4,X_5,X_6)=\left(\frac{1}{\lambda_1}-Y_1\right)\mathbf{1}\{\max(Y_1+Y_4+Y_6,Y_2+Y_5+Y_6,Y_1+Y_3+Y_5+Y_6)\leq y\},$$

and the GLR estimator of $\partial F(y;\theta)/\partial y$ is given by

$$\begin{aligned}
G_2(U_1,U_2;\theta,y,U_3,X_4,X_5,X_6)=&\lambda_1\mathbf{1}\{\max(Y_4+Y_6,Y_2+Y_5+Y_6,Y_3+Y_5+Y_6)\leq y\}\\
&+\lambda_2\mathbf{1}\{\max(Y_1+Y_4+Y_6,Y_5+Y_6,Y_3+Y_5+Y_6)\leq y\}\\
&-(\lambda_1+\lambda_2)\mathbf{1}\{\max(Y_1+Y_4+Y_6,Y_2+Y_5+Y_6,Y_1+Y_3+Y_5+Y_6)\leq y\}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\mathbb{E}[G_1(U_1,U_2;\theta,y,U_3,X_4,X_5,X_6)]=\mathbb{E}[\mathbb{E}[G_1(U_1,U_2;\theta,y,U_3,X_4,X_5,X_6)|Y_1,\ldots,Y_5]]\\
=&\mathbb{E}\left[\left(\frac{1}{\lambda_1}-Y_1\right)\Phi\left(\frac{1}{\sigma_6}\left(\log[(y-\widetilde{Y})^+]-\mu_6\right)\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}[G_2(U_1,U_2;\theta,y,U_3,X_4,X_5,X_6)] = \mathbb{E}[\mathbb{E}[G_2(U_1,U_2;\theta,y,U_3,X_4,X_5,X_6)|Y_1,\dots,Y_5]]\\
=&\lambda_1\mathbb{E}\left[\Phi\left(\frac{1}{\sigma_6}\big(\log[(y-\max(Y_4,Y_2+Y_5,Y_3+Y_5))^+]-\mu_6\big)\right)\right]\\
&+\lambda_2\mathbb{E}\left[\Phi\left(\frac{1}{\sigma_6}\big(\log[(y-\max(Y_1+Y_4,Y_5,Y_1+Y_3+Y_5))^+]-\mu_6\big)\right)\right]\\
&-(\lambda_1+\lambda_2)\mathbb{E}\left[\Phi\left(\frac{1}{\sigma_6}\big(\log[(y-\widetilde{Y})^+]-\mu_6\big)\right)\right],
\end{aligned}$$

where $\widetilde{Y}=\max(Y_1+Y_4,Y_2+Y_5,Y_1+Y_3+Y_5)$. We can take the expressions inside the expectations on the right-hand sides of the above equation as conditional GLR estimators. From this derivation, the three quantile sensitivity estimators of the form (3) using GLR, CGLR, and CGLRQ to estimate distribution sensitivities in (2) are referred to as GLR-1, CGLR-1, and CGLRQ-1, respectively.

On the other hand, we have

$$F(y;\theta)=\mathbb{E}[\mathbf{1}\{Y_4+Y_1+Y_6-y\le 0\}\mathbf{1}\{Y_5+\max(Y_2,Y_1+Y_3)+Y_6-y\le 0\}],$$

and we can also let $(X_4,X_5)$ be the input random variables in the stochastic model of Peng et al. (2018), leading to

$$\begin{aligned}
&g_1(X_4,X_5;\theta,y,U_1,U_2,U_3,X_6)=\exp(\mu_4+\sigma_4X_4)+Y_1+Y_6-y,\\
&g_2(X_4,X_5;\theta,y,U_1,U_2,U_3,X_6)=\exp(\mu_5+\sigma_5X_5)+\max(Y_2,Y_1+Y_3)+Y_6-y,\\
&\partial_\theta g(x_4,x_5;\theta,y,u_1,u_2,u_3,x_6)=\log u_1/\lambda_1^2\big(1,\mathbf{1}\{\log u_1/\lambda_1+\log u_3/\lambda_3<\log u_2/\lambda_2\}\big)^T,\\
&\partial_y g(x_4,x_5;\theta,y,u_1,u_2,u_3,x_6)=-\big(1,1\big)^T,\\
&\nabla\log f_{(X_4,X_5)}(x_4,x_5)=-\big(x_4,x_5\big)^T.
\end{aligned}$$

The Jacobian matrix and its inverse are

$$J_g(x_4,x_5)=\begin{pmatrix}\sigma_4e^{\mu_4+\sigma_4x_4} & 0\\ 0 & \sigma_5e^{\mu_5+\sigma_5x_5}\end{pmatrix},\quad J_g^{-1}(x_4,x_5)=\begin{pmatrix}\frac{1}{\sigma_4}e^{-\mu_4-\sigma_4x_4} & 0\\ 0 & \frac{1}{\sigma_5}e^{-\mu_5-\sigma_5x_5}\end{pmatrix},$$

$$\partial_{x_4}J_g(x_4,x_5)=\begin{pmatrix}\sigma_4^2e^{\mu_4+\sigma_4x_4} & 0\\ 0 & 0\end{pmatrix},\quad \partial_{x_5}J_g(x_4,x_5)=\begin{pmatrix}0 & 0\\ 0 & \sigma_5^2e^{\mu_5+\sigma_5x_5}\end{pmatrix},$$

and

$$\begin{aligned}
&d(x_4,x_5;\theta)=\frac{\log u_1}{\lambda_1^2}\left[\left(1+\frac{x_4}{\sigma_4}\right)e^{-\mu_4-\sigma_4x_4}+\mathbf{1}\left\{\frac{\log u_1}{\lambda_1}+\frac{\log u_3}{\lambda_3}<\frac{\log u_2}{\lambda_2}\right\}\left(1+\frac{x_5}{\sigma_5}\right)e^{-\mu_5-\sigma_5x_5}\right],\\
&\tilde{d}(x_4,x_5;\theta)=-\left(1+\frac{x_4}{\sigma_4}\right)e^{-\mu_4-\sigma_4x_4}-\left(1+\frac{x_5}{\sigma_5}\right)e^{-\mu_5-\sigma_5x_5}.
\end{aligned}$$

Then the GLR estimator of $\partial F(y;\theta)/\partial\theta$ is given by

$$G_1(X_4,X_5;\theta,y,U_1,U_2,U_3,X_6)=-\mathbf{1}\{Y\le z\}\frac{Y_1}{\lambda_1}\left[\left(1+\frac{X_4}{\sigma_4}\right)\frac{1}{Y_4}+\mathbf{1}\Big\{Y_1+Y_3>Y_2\Big\}\left(1+\frac{X_5}{\sigma_5}\right)\frac{1}{Y_5}\right].$$

Table 2: Sensitivity estimates for $\alpha$-quantile (mean $\pm$ standard error) based on $10^4$ independent experiments for numerical example in Section 4.2.

| | $\alpha = 0.1$ | $\alpha = 0.3$ | $\alpha = 0.5$ | $\alpha = 0.7$ | $\alpha = 0.9$ |
|---|---|---|---|---|---|
| GLR-1 | $-0.456 \pm 1.0\times10^{-4}$ | $-0.697 \pm 1.3\times10^{-4}$ | $-0.907 \pm 2.0\times10^{-4}$ | $-1.136 \pm 3.9\times10^{-4}$ | $-1.374 \pm 1.6\times10^{-3}$ |
| CGLR-1 | $-0.456 \pm 6.8\times10^{-5}$ | $-0.696 \pm 1.0\times10^{-4}$ | $-0.907 \pm 1.7\times10^{-4}$ | $-1.136 \pm 3.6\times10^{-4}$ | $-1.374 \pm 1.5\times10^{-3}$ |
| CGLRQ-1 | $-0.456 \pm 5.0\times10^{-5}$ | $-0.696 \pm 5.6\times10^{-5}$ | $-0.907 \pm 6.8\times10^{-5}$ | $-1.135 \pm 1.1\times10^{-4}$ | $-1.377 \pm 3.1\times10^{-4}$ |
| GLR-2 | $-0.456 \pm 1.0\times10^{-4}$ | $-0.700 \pm 6.7\times10^{-4}$ | $-0.911 \pm 1.1\times10^{-3}$ | $-1.163 \pm 2.9\times10^{-3}$ | $-0.094 \pm 7.4\times10^{-1}$ |
| CGLR-2 | $-0.456 \pm 3.0\times10^{-4}$ | $-0.698 \pm 5.4\times10^{-4}$ | $-0.913 \pm 1.0\times10^{-3}$ | $-1.161 \pm 2.5\times10^{-3}$ | $-1.026 \pm 2.5\times10^{-1}$ |
| CGLRQ-2 | $-0.456 \pm 2.6\times10^{-4}$ | $-0.697 \pm 4.4\times10^{-4}$ | $-0.910 \pm 7.9\times10^{-4}$ | $-1.145 \pm 1.7\times10^{-3}$ | $-1.957 \pm 2.0\times10^{-1}$ |
| FDC(0.01) | $-0.453 \pm 9.5\times10^{-4}$ | $-0.690 \pm 9.8\times10^{-4}$ | $-0.897 \pm 1.2\times10^{-3}$ | $-1.123 \pm 1.6\times10^{-3}$ | $-1.366 \pm 3.3\times10^{-3}$ |
| FDC(0.1) | $-0.423 \pm 2.9\times10^{-4}$ | $-0.639 \pm 3.0\times10^{-4}$ | $-0.826 \pm 3.6\times10^{-4}$ | $-1.024 \pm 5.0\times10^{-4}$ | $-1.223 \pm 1.0\times10^{-3}$ |

the GLR estimator of $\partial F(y;\theta)/\partial y$ is given by

$$G_2(X_4,X_5;\theta,y,U_1,U_2,U_3,X_6) = -\mathbf{1}\{Y \le z\}\left[\left(1+\frac{X_4}{\sigma_4}\right)\frac{1}{Y_4} + \left(1+\frac{X_5}{\sigma_5}\right)\frac{1}{Y_5}\right].$$

Furthermore, we have

$$\begin{aligned}&\mathbb{E}[G_1(X_4,X_5;\theta,y,U_1,U_2,U_3,X_6)] = \mathbb{E}[\mathbb{E}[G_1(X_4,X_5;\theta,y,U_1,U_2,U_3,X_6)|Y_1,\dots,Y_5]]\\ &= -\mathbb{E}\left[\frac{Y_1}{\lambda_1}\left[\left(1+\frac{X_4}{\sigma_4}\right)\frac{1}{Y_4} + \mathbf{1}\Big\{Y_1+Y_3 > Y_2\Big\}\left(1+\frac{X_5}{\sigma_5}\right)\frac{1}{Y_5}\right]\Phi\left(\frac{1}{\sigma_6}\big(\log[(y-\widetilde{Y})^+] - \mu_6\big)\right)\right].\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[G_2(X_4,X_5;\theta,y,U_1,U_2,U_3,X_6)] = \mathbb{E}[\mathbb{E}[G_2(X_4,X_5;\theta,y,U_1,U_2,U_3,X_6)|Y_1,\dots,Y_5]]\\ &= -\mathbb{E}\left[\left(\frac{1}{Y_4}+\frac{X_4}{\sigma_4 Y_4}+\frac{1}{Y_5}+\frac{X_5}{\sigma_5 Y_5}\right)\Phi\left(\frac{1}{\sigma_6}\big(\log[(y-\widetilde{Y})^+] - \mu_6\big)\right)\right].\end{aligned}$$

We can take the expressions inside the expectations on the right-hand sides of the above equations as conditional GLR estimators. From this derivation, the three quantile sensitivity estimators of the form (3) using GLR, CGLR, and CGLRQ to estimate distribution sensitivities in (2) are referred to as GLR-2, CGLR-2, and CGLRQ-2, respectively.

In the experiment, we set $\lambda_i = 1$, $i = 1,2,3$, $\mu_j = 0$, $\sigma_j = 1$, $j = 4,5,6$, and test five probability levels at $\alpha = 0.1$, $\alpha = 0.3$, $\alpha = 0.5$, $\alpha = 0.7$, and $\alpha = 0.9$. From Table 2, we can see that the quantile sensitivity estimates by GLR-1, CGLR-1, CGLRQ-1, GLR-2, CGLR-2, CGLRQ-2 are statistically indistinguishable, FDC(0.01) appears to be slightly biased, and FDC(0.1) exhibits a significant bias compared to the GLR method. The variances of GLR-1, CGLR-1, CGLRQ-1 are smaller than GLR-2, CGLR-2, CGLRQ-2, respectively. The variances of all quantile sensitivity estimators go up as the probability levels grow up, and in particular, the variances of GLR-2, CGLR-2, CGLRQ-2 go up dramatically when the probability levels grow up from $\alpha = 0.7$ to $\alpha = 0.9$. The variances of GLR-1, CGLR-1, CGLRQ-1 are smaller than those of FDC(0.1) and FDC(0.01), and the CMC and RQMC lead to significant variance reduction.

## 5 CONCLUSIONS

We combine GLR methods in Peng et al. (2018) and Peng et al. (2021) with CMC and RQMC to reduce the variance of quantile sensitivity estimation, and apply the proposed methods to a toy example and a SAN example. In both examples, the GLR method in Peng et al. (2021) leads to a better performance than the GLR method in Peng et al. (2018), and the variance reduction by CMC and RQMC is significant.

However, the variance reduction by RQMC for estimator (3) after smoothing the GLR estimators for two distribution sensitivities by CMC are not as substantial as that for distribution sensitivity estimators themselves (see Peng et al. 2021). The reason may be due to that the quantile estimator by order statistics is discontinuous with respect to the input random variables. This discontinuity issue can be treated by smoothing the empirical distribution function by certain kernel method, which could be a future work. Combining variance reduction technique for quantile estimate with the variance reduction techniques for distribution sensitivities could also reduce the variance for quantile sensitivity estimation.

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## REFERENCES

Asmussen, S., and P. W. Glynn. 2007. *Stochastic Simulation: Algorithms and Analysis*, Volume 57. Springer Science & Business Media.

Dick, J., and F. Pillichshammer. 2010. *Digital Nets and Sequences: Discrepancy Theory and Quasi-Monte Carlo Integration*. Cambridge University Press.

Fu, M. C., L. J. Hong, and J.-Q. Hu. 2009. "Conditional Monte Carlo Estimation of Quantile Sensitivities". *Management Science* 55(12):2019–2027.

Glynn, P. W., Y. Peng, M. C. Fu, and J. Hu. 2021. "Computing Sensitivities for Distortion Risk Measures". *INFORMS Journal on Computing, forthcoming*.

He, Z., and X. Wang. 2015. "On the Convergence Rate of Randomized Quasi–Monte Carlo for Discontinuous Functions". *SIAM Journal on Numerical Analysis* 53(5):2488–2503.

Heidergott, B., and W. Volk-Makarewicz. 2016. "A Measure-valued Differentiation Approach to Sensitivity Analysis of Quantiles". *Mathematics of Operations Research* 41(1):293–317.

Hong, L. J. 2009. "Estimating Quantile Sensitivities". *Operations Research* 57(1):118–130.

Hong, L. J., and G. Liu. 2009. "Simulating Sensitivities of Conditional Value At Risk". *Management Science* 55(2):281–293.

Jiang, G., and M. C. Fu. 2015. "Technical Note — On Estimating Quantile Sensitivities Via Infinitesimal Perturbation Analysis". *Operations Research* 63(2):435–441.

L'Ecuyer, P. 2009. "Quasi-Monte Carlo methods with applications in finance". *Finance and Stochastics* 42(6):926–938.

L'Ecuyer, P. 2018. "Randomized quasi-Monte Carlo: An introduction for practitioners". *Monte Carlo and Quasi-Monte Carlo Methods: MCQMC 2016 (Glynn P. W., Owen A. B. eds.)*:29–52.

L'Ecuyer, P., and C. Lemieux. 2000. "Variance Reduction via Lattice Rules". *Management Science* 46(9):1214–1235.

L'Ecuyer, P., and D. Munger. 2012. "On Figures of Merit for Randomly-Shifted Lattice Rules". *Monte Carlo and Quasi-Monte Carlo Methods 2010 (Wozniakowski H., Plaskota L., eds.)*:133–159.

L'Ecuyer, P., and D. Munger. 2016. "Algorithm 958: Lattice builder: A general software tool for constructing rank-1 lattice rules". *ACM Transactions on Mathematical Software* 42(2):1–30.

Lemieux, C. 2009. *Monte Carlo and Quasi-Monte Carlo Method*. New York: Springer.

Liu, G., and L. J. Hong. 2009. "Kernel Estimation of Quantile Sensitivities". *Naval Research Logistics* 56(6):511–525.

Matousek, J. 1998. "On the $L_2$-discrepancy for anchored boxes". *Journal of Complexity* 14(4):527–556.

Nakayama, M. K., Z. T. Kaplan, Y. Li, B. Tuffin, and P. L'Ecuyer. 2020. "Quantile Estimation Via a Combination of Conditional Monte Carlo and Randomized Quasi-Monte Carlo". In *Proceedings of Winter Simulation Conference*, edited by S. K. S. L.-M. Z. Z. T. R. K.-H. Bae, B. Feng and R. Thiesing, 301–312. Piscataway, New Jersey: IEEE.

Peng, Y., M. C. Fu, P. W. Glynn, and J.-Q. Hu. 2017. "On the Asymptotic Analysis of Quantile Sensivitiy Estimation by Monte Carlo Simulation". In *Proceedings of Winter Simulation Conference*, edited by G. Z. N. M. G. W. W. K. V. Chan, A. D'Ambrogio and E. Page, 2336–2347. Piscataway, New Jersey: IEEE.

Peng, Y., M. C. Fu, B. Heidergott, and H. Lam. 2020. "Maximum Likelihood Estimation By Monte Carlo Simulation: Towards Data-Driven Stochastic Modeling". *Operations Research* 68(6):1896–1912.

Peng, Y., M. C. Fu, J. Hu, P. L'Ecuyer, and B. Tuffin. 2021. "Generalized Likelihood Ratio Method for Stochastic Models with Uniform Random Numbers As Inputs". *archives ouvertes.fr, preprint in https://hal.inria.fr/hal-02652068/document, accessed 29th May 2020*.

Peng, Y., M. C. Fu, J. Hu, and L. Lei. 2019. "Estimating quantile sensitivity for financial models with correlations and jumps". In *Proceedings of Winter Simulation Conference*, edited by S. L.-M. M. R. C. S. P. H. N. Mustafee, K.-H.G. Bae and Y.-J. Son, 962–973. Piscataway, New Jersey: IEEE.

Peng, Y., M. C. Fu, J.-Q. Hu, and B. Heidergott. 2018. "A New Unbiased Stochastic Derivative Estimator for Discontinuous Sample Performances with Structural Parameters". *Operations Research* 66(2):487–499.

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