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IMPROVED INFINITESIMAL PERTURBATION ANALYSIS ESTIMATORS OF QUANTILE SENSITIVITY

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ABSTRACT

In this paper, we propose an approach that combines jackknifing and randomized quasi-Monte Carlo in infinitesimal perturbation analysis (IPA) to estimate quantile sensitivities, which improves the accuracy and precision of the classical IPA estimator. Theoretical properties of the new estimators are provided, and numerical examples are presented to illustrate the effectiveness of the new estimators.

1 INTRODUCTION

In many applications, quantile is a highly frequently used risk measure. For example, in finance, quantile, i.e., Value-at-Risk (VaR), is an important indicator of risk of financial portfolios. When solving quantile-based optimization problems, we need to estimate the quantile sensitivity. Recently, there are some works on estimating quantile sensitivity based on IPA methods, see Hong (2009), Jiang and Fu (2015), and Peng et al. (2018). However, those IPA-based estimators of α-quantile sensitivity do not perform well when the probability level α is closed to 0 and 1. To improve the accuracy of the IPA estimator, Jiang et al. (2014) proposed a two-fold jackknifing approach to reduce the bias but inflate the variance a little bit. If the bias dominates the mean squared error (MSE), the jackknifing estimator works well. But if the bias is not the main component of MSE, the jackknifing estimator will inflate the overall MSE despite of the bias reduction. In this paper we apply randomized qusai-Monte Carlo (QMC) combining jackknifing method to improve the IPA estimators of quantile sensitivity. For details of randomized QMC, refer to Owen (2003).

2 IPA ESTIMATOR VIA RANDOMIZED QMC AND JACKKNIFING

Let $h(X(\theta),\theta)$ be a performance function of a stochastic system, where $X(\theta)$ is the random variable. Let θ be the parameter of interest, and it can appear in the random variable or the performance function or both. The α-quantile of $h(X(\theta),\theta)$ is denoted by $q_\alpha(\theta)$ such that $\Pr\{h(X(\theta),\theta)\leq q_\alpha\}=\alpha$ for any preset $\alpha\in(0,1)$. The quantile sensitivity is defined by $q'_\alpha(\theta)=dq_\alpha(\theta)/d\theta$. The classical batched IPA estimator is given as follow. First, generate a batch (batch size m) of simulation samples (denoted by $h(X_i(\theta),\theta)$) of the performance function and the sample derivatives (denoted by $dh(X_i(\theta),\theta)/d\theta$) for $i=1,2,\ldots,m$. Abbreviate $h(X_i(\theta),\theta)$ to $h_i(\theta)$. Second, sort the simulation samples as $h_{(1)}(\theta)\leq h_{(2)}(\theta)\leq\cdots\leq h_{(\lceil\alpha m\rceil)}(\theta)\leq\cdots\leq h_{(m)}(\theta)$, where the subscript (l) denotes the lth order statistic from the simulation samples. Third, find the sample derivative of the $\lceil\alpha m\rceil$th order statistic, i.e., find $I_m\triangleq dh_{(\lceil\alpha m\rceil)}(\theta)/d\theta$. Fourth, replicate the first three steps k times and obtain the batched estimator $\hat{q}'^{m,k}_\alpha(\theta)\triangleq 1/k\sum_{i=1}^k I_{m,i}$.

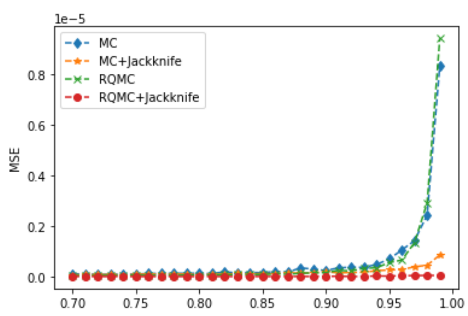
In this paper, the simulation samples in a batch are generated based on QMC method. We focus on the (t,d)-sequences or (t,r,d)-nets in base $b\geq 2$ for some integer $t\geq 0$, which are evenly dispersed over $[0,1)^d$ (d is the dimension), and we randomize the low-discrepancy point set to obtain a randomized (t,d)-sequences or (t,r,d)-nets (Owen 2003), which preserves certain benefits over QMC and MC. Besides, randomized QMC allows to establish a central limit theorem for estimator of quantile sensitivity, so the precision of the estimation can be quantified. According to Section 3 of He and Wang (2019), as sample size m grows, quantile estimator of QMC is consistent with true quantile as $m \to \infty$ and error decreases at rate $O(m^{-1/d})$ under mild circumstances. It is known that a randomized (t,r,d)-net is a (t,r,d)-net with probability 1 and all points in the net follow a uniform distribution $U[0,1)^d$, as detailed by Owen (2003). We sample points in randomized QMC setting, where $n = km$, and divide them into k batches with batch size m. Denote the batched randomized QMC-IPA estimator as $\tilde{q}'^{m,k}_{\alpha}(\theta) \triangleq 1/k\sum_{i=1}^{k} \tilde{I}_{m,i}$, and we summarize the properties of the estimator in Theorem 1.

Theorem 1 Under some mild circumstances, if $\sup_m E(|\tilde{I}|_m^{2+\gamma}) < \infty$, $\gamma > 0$, $m \to \infty$ and $k \to \infty$ as $n \to \infty$, then $\tilde{q}'^{m,k}_{\alpha}(\theta) \to q'_{\alpha}(\theta)$ converges in probability as $n = mk \to \infty$. Furthermore, if the conditions of *Lemmas* 2 and 3 in Hong (2009) hold, $m \to \infty$ and $k \to \infty$ as $n \to \infty$, $lim_{n\to\infty}\sqrt{k}/m^{1/d} = 0$, and $\sigma_m > 0$ for any $m > 0$, then $\sqrt{k}/\sigma_m[\tilde{q}'^{m,k}_{\alpha}(\theta) - q'^{m,k}_{\alpha}(\theta)] \Rightarrow N(0,1)$ as $n = mk \to \infty$, where σ_m is the variance of IPA estimator $\tilde{I}_m$.

To further reduce the bias, we apply jackknifing method into $\tilde{q}'_{\alpha}(\theta)$. Specifically, let $\tilde{J}_{m,i} = 2\tilde{I}_{m,i} - 1/2(\tilde{I}^1_{m,i} + \tilde{I}^2_{m,i})$, where $I^1_{m,i}$ and $I^2_{m,i}$ are randomized QMC-IPA estimators derived from the first $m/2$ and second $m/2$ samples in the ith batch. The batched randomized QMC-IPA estimator with jackknifing is given by $\bar{q}'_{\alpha}(\theta) \triangleq 1/k\sum_{i=1}^{k} \tilde{J}_{m,i}$. The pseudocode is provided in the online appendix (Yang 2019).

3 NUMERICAL EXAMPLES

We consider two examples, one is a European call option and the other is a portfolio return of two assets, which are similar to Jiang et al. (2014). We plot the MSE of the estimators in the following figures. Both figures indicate that the randomized QMC combining jackknifing performs very well.



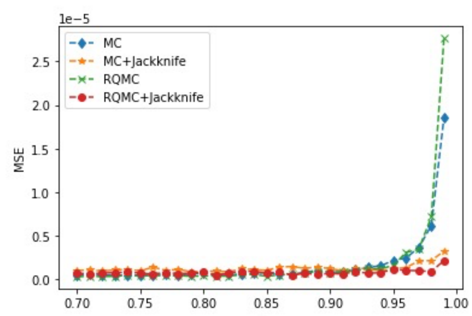


Figure 1: MSE of estimates in Ex.1 (left) and Ex.2 (right). $m = 800$, $k = 400$. In Ex. 1, the parameters are identical in Jiang et al. (2014). In Ex.2, $\mu = (0.06, 0.25)$, $\sigma = (0.02, 0.2)$, and the correlation $\rho = -0.2$.

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