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LIMITING DISTRIBUTIONAL FIXED POINTS IN SYSTEMIC RISK GRAPH MODELS

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ABSTRACT

We analyse the equilibrium behaviour of a large network of banks in presence of incomplete information, where inter-bank borrowing and lending is allowed, and banks suffer shocks to assets. In a two time period graphical model, we show that the equilibrium wealth distribution is the unique fixed point of a complex, high dimensional distribution-valued map. Fortunately, there is a dimension collapse in the limit as the network size increases, where the equilibriated system converges to the unique fixed point involving a simple, one dimensional distribution-valued operator, which, we show, is amenable to simulation. Specifically, we develop a Monte-Carlo algorithm that computes the fixed point of a general distribution-valued map and derive sample complexity guarantees for it. We numerically show that this limiting one-dimensional regime can be used to obtain useful structural insights and approximations for networks with as low as a few hundred banks.

1 MOTIVATION

We consider a graphical model of the banking network where vertices denote banks and edges spell out the inter-bank liability structure. A two time period framework is considered: In period one, banks borrow and lend capital to each other, and inter-bank linkages in the network are realised. In the next period, they receive income from their external assets modulo random shocks, and use it to clear external and inter-bank liabilities. The equilibrium wealth of the system is the resultant fixed point solution that balances the incoming and the outgoing wealth at each bank. Such graphical models have been extensively studied in literature, for example, by Allen and Gale (2000), Eisenberg and Noe (2001), Haldane and May (2011), Acemoglu et al. (2015) and Glasserman and Young (2016) among others. It can be shown that the wealth of banks in equilibrium is the unique fixed point of a vector valued map. However, one shortcoming in much of the existing literature is the assumption that the entire network wealth and liability structure is known to the modeller. This is often unrealistic in practice as observed by Anand et al. (2015): A modeller typically only gets access to balance sheets of banks in the network and only knows the aggregate assets and liabilities, i.e., how much each bank lends or borrows, but not from whom. A modeller may also know the average number of creditors and debtors of a typical bank, without access to further information granularity.

2 APPROXIMATION OF A LARGE BANKING NETWORK

To address this lack of complete information, we model the banking network as a random, weighted graph, where the randomness captures the modeller's incomplete information on exact assets and liabilities, as well as unobserved (both idiosyncratic and systemic) shocks to the bank's assets. We allow the inter-bank connectivities, liabilities and external assets to be random variables, whose distribution matches the observed statistical properties of the network. Other works, for example Amini et al. (2016), also consider random graphical models of banking networks . However in their models, the recoveries of a bank from a defaulting debtor are independent of the actual wealth possessed by the debtor. Our model departs from this by allowing recoveries on default to be state dependent, resulting in a more realistic view of the banking network. We consider a sparse graph regime where each bank has only a few counter-parties and show that the wealth possessed by banks in equilibrium is a random vector whose distribution is a fixed point of a distribution-valued map. A similar approach is also followed by Kavitha et al. (2018), however, they model the network as a dense graph, where the expected number of counter-parties of a bank go to infinity. The sparse graph model which we consider differs from theirs both in terms of the analysis and conclusions. Further, it has been observed empirically, for example, by Cont et al. (2010) that banking networks are sparse, and hence we expect our regime to more accurately capture real banking networks.

To infer the statistical properties of the network, it is essential to sample from this distributional fixed point. However, typical banking networks are quite large, often consisting of hundreds or thousands of banks. Sampling thus is computationally prohibitive due to the underlying high dimensionality. We show that as the size of the banking network grows, due to conditional independence amongst underlying random variables, there is a dimension collapse: the distributional fixed point of a large banking system converges to the product of the fixed point of a one-dimensional distribution valued map. Hence, the statistical properties of a large network are well approximated by the limiting fixed point distribution. This provides a great deal of structural insights and may be of use in conducting many *what if* analysis on a large network. Proving the existence and uniqueness of fixed points as well as distributional convergence for the growing random banking network is typically not straightforward. One of our contributions is to leverage the theory of optimal transport, which enables us to metrize the infinite dimensional space of distributions in which the fixed points and their limits lie. This provides a relatively simple approach to prove limit theorems.

The latter half of this paper focuses on simulation of the limiting fixed point and related computational issues. Let $T(\cdot)$ be a map on the space of probability measures on the line. Suppose $T(\cdot)$ is contractive (and hence has a unique fixed point), and satisfies certain moment bounds. Then, assuming that for any distribution P one can sample from $T(P)$, we develop a Monte Carlo algorithm which gives a distribution which is close to the fixed point of $T(\cdot)$, and derive probabilistic guarantees for it. To the best of our knowledge, the algorithm and its analysis are new. Applying this to the banking network, we show that to approximate a realistic system a small number of computations are required. Testing this algorithm on a simulated example where there are a few hundred banks in the network, we find that the limiting fixed point distribution approximates the global properties of the large network well.

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