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ON THE IMPACTS OF TAIL MODEL UNCERTAINTY IN RARE-EVENT ESTIMATION

Zhiyuan Huang

Department of IOE
University of Michigan
1205 Beal Avenue
Ann Arbor, MI 48109, USA

ABSTRACT

Rare-event probabilities and risk measures that quantify the likelihood of catastrophic or failure events can be sensitive to the accuracy of the underlying input models, especially regarding their tail behaviors. We investigate how the lack of tail information of the input can affect the output extremal measures, in relation to the level of data that are needed to inform the input tail. Using the basic setting of estimating the probability of the overshoot of an aggregation of i.i.d. input variables, we argue that heavy-tailed problems are much more vulnerable to input uncertainty than light-tailed problems. We explain this phenomenon via their large deviations behaviors, and substantiate with some numerical experiments.

1 INTRODUCTION

Assessing rare-event probabilities and extremal measures for the likelihood of catastrophic events is ubiquitous in risk analysis and management. In many cases, these extremal quantities are outputs that rely on underlying, granular stochastic components. Estimating these extremal quantities hinges on the provision of accurate probabilistic descriptions of these input components, with any deviations away from the reality leading to potential errors or even meaningless estimates. This issue has been studied and has gathered growing literature in recent years, generally known as the problem of model uncertainty or input uncertainty. See, e.g., Barton et al. (2002), Henderson (2003), Song et al. (2014) and Lam (2016). Its main focus is to develop methodologies that can quantify the impact of model misspecifications or errors that propagate to output estimation or decision-making.

In our paper, we address several validity questions that arise when, given input data, a modeler chooses to use standard approaches to obtain estimates and quantify uncertainty, namely:

1. By simply using the empirical distribution as my input model fit, would the rare-event estimate be reasonably close to the truth? (assuming computational or Monte Carlo noise is negligible)
2. Following the point estimate in Question 1, would it work if one runs a bootstrap to obtain a confidence interval that accounts for the input data noise?
3. If the bootstrap does not work, would incorporating extreme value theory in fitting the input tail helps with more reliable uncertainty quantification?

Our viewpoint is that the main source of uncertainty in determining the accuracy of rare-event estimation comes from the lack of knowledge of the tail of the input models. The main body (i.e., non-tail) part of the input distribution can be fit by both parametric and nonparametric techniques, where there are typically adequate data to perform such fit (and in Question 1 above, we simply use the empirical distribution as the fit). However, it is the portion beyond the scope of data that determines the distributional tail and in turn the rare-event behaviors. Thus, before we go to the above questions, we first focus on: "How does truncating the tail of the input model affect the rare-event estimate?"

In our paper, we discuss these questions with a basic setup on the overshoot of an aggregation of i.i.d. variables. We provide analysis to explain the impacts of tail truncation in light- versus heavy-tailed cases.

In numerical experiments, we discuss the comparisons between performance of the light and the heavy-tail cases in three tasks, i.e. estimation with tail-truncated distribution, estimation with empirical distribution and estimation of input uncertainty with bootstrapping.

2 THE IMPACT OF TAIL TRUNCATION

Our main contention is that heavy-tailed problems could be much more challenging than light-tailed counterparts regarding estimation and uncertainty quantification using the standard approaches in Questions 1-3. This challenge roots from Question 0 in that truncating the input tail in a heavy-tailed system exerts a huge effect on the rare-event estimate, when the truncation level represents the typical level of knowledge that the data informs (e.g., the top 1% or 0.1% of the data). To illustrate, we consider estimating $p = P(S_n > \gamma)$ where $S_n = X_1 + \cdots + X_n$ and $X_i \in \mathbb{R}$ are i.i.d. variables drawn from the distribution F. We let $\mu = E[X_i] < \infty$.

Our investigation pertaining to Question 0 is the following. Suppose we truncate the distribution $F(x)$ at the point u so that the density becomes 0 for $x > u$, i.e., consider the truncated distribution function given by

$$\tilde{F}_u(x) = \begin{cases} F(x)/F(u) & \text{for } \ x \leq u \\ 1 & \text{for } \ x > u. \end{cases}$$

For convenience, denote $p(G)$ as the probability $P_G(S_n > \gamma)$ where X_i's are governed by an arbitrary distribution G, and we simply denote $P(S_n > \gamma)$ if X_i's are governed by F. We consider the approximation error $p(\tilde{F}_u) - p(F)$.

Our analysis shows how the approximation error varies with n asymptotically in heavy- and light-tail cases under reasonable conditions on γ and u. In heavy-tail cases, the challenge in approximating $p(F)$ with truncated distribution is revealed. For X_i's with Pareto tail, as $n \to \infty$ the approximation error is given by $p(\tilde{F}_u) - p(F) = -p(F)(1 + o(1))$. This indicates that due to the small magnitude of $p(\tilde{F}_u)$, the approximation error is too large relative to $p(F)$. On the other hand, in light-tail cases, the approximation error is asymptotically negligible, which is given by $p(\tilde{F}_u) - p(F) = -o(p(F))$ as $n \to \infty$.

As a consequence of the approximation challenge revealed in heavy-tail cases, estimation tasks are less robust to a lack of tail information. More specifically, using empirical distribution, or bootstrap on the empirical distribution in heavy-tail cases, which significantly ignores the tail content, would fail to estimate the rare-event quantity and vastly under-estimate the uncertainty. Using extreme value theory in Question 3 to extrapolate tail (such as the peak-over-threshold method) helps to an extent, but could introduce extra bias, at least using our fitting methods (though we should point out that better techniques are available). On the other hand, the effect of missing tails on light-tailed estimation is relatively milder.

REFERENCES

Barton, R., S. Chick, R. Cheng, S. Henderson, A. Law, B. Schmeiser, L. Leemis, L. Schruben, and J. Wilson. 2002. “Panel Discussion on Current Issues in Input Modeling”. In *Proceedings of the 2002 Winter Simulation Conference*, edited by E. Ycesan, C.-H. Chen, J. L. Snowdon, and J. M. Charnes, 353–369. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers, Inc.

Henderson, S. G. 2003. “Input Model Uncertainty: Why Do We Care and What Should We Do About It?”. In *Proceedings of the 2003 Winter Simulation Conference*, edited by S. Chick, P. J. Snchez, D. Ferrin, and D. J. Morrice, 90–100. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers, Inc.

Lam, H. 2016. “Advanced Tutorial: Input Uncertainty and Robust Analysis in Stochastic Simulation”. In *Proceedings of the 2016 Winter Simulation Conference*, edited by T. M. K. Roeder, P. I. Frazier, R. Szechtman, and E. Zhou, 178–192. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers, Inc.

Song, E., B. L. Nelson, and C. D. Pegden. 2014. “Advanced Tutorial: Input Uncertainty Quantification”. In *Proceedings of the 2014 Winter Simulation Conference*, edited by A. Tolk, S. Y. Diallo, I. O. Ryzhov, L. Yilmaz, S. Buckley, and J. A. Miller, 162–176. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers, Inc.