Proceedings of the 2019 Winter Simulation Conference
N. Mustafee, K.-H.G. Bae, S. Lazarova-Molnar, M. Rabe, C. Szabo, P. Haas, and Y.-J. Son, eds.

ESTIMATING QUANTILE SENSITIVITY FOR FINANCIAL MODELS WITH CORRELATIONS AND JUMPS

Yijie Peng

Department of Industrial Engineering and Management
College of Engineering
Peking University
Beijing 100084, CHINA

Michael C. Fu

Robert H. Smith School of Business
Institute for Systems Research
University of Maryland
College Park, MD 20742, USA

Jian-Qiang Hu

Department of Management Science
School of Management
Fudan University
Shanghai, 200433, CHINA

Lei Lei

School of Economics and Business Administration
Chongqing University
Chongqing, 400044, CHINA

ABSTRACT

We apply a generalized likelihood ratio (GLR) derivative estimation method in previous works to estimate quantile sensitivity of financial models with correlations and jumps. Examples illustrate the wide applicability of the GLR method by providing several practical settings where other techniques are difficult to apply, and numerical results demonstrate the effectiveness of the new estimator.

1 INTRODUCTION

Quantile, also known as the value at risk (VaR), has become a standard benchmark for measuring financial risk, which can be translated directly into a minimum capital requirement. Since J.P. Morgan launched the RiskMetrics system in October 1994, VaR has been applied initially to market risk and then extended ubiquitously to credit risk, operational risk and enterprise-wide risk. See Jorion (2007) for an overview.

In applications, the quantile of a stochastic system is rarely available in closed form. Therefore, statistical sampling (simulation) is a commonly used technique to estimate the quantile. The asymptotic properties of quantile estimators have been studied extensively in a large body of statistics literature, e.g., David and Nagaraja (1970) and Serfling (2009). Simulation techniques to enhance the efficiency of quantile estimation can be found in Jin et al. (2003) and Alexopoulos et al. (2019).

Quantile sensitivity estimates are important because they provide information on the variation of the quantile w.r.t. parameters of the underlying stochastic model, which can be used for hedging and gradient-based optimization. Quantile sensitivity estimation is pioneered by Hong (2009), a seminal work leading to a series of work on sensitivity estimation for financial risk measures (Liu and Hong 2009, Fu et al. 2009, Hong and Liu 2009, Heidergott, Volk-Makarewicz, and Vázquez-Abad 2014, Hong et al. 2014, Jiang and Fu 2015, Heidergott and Volk-Makarewicz 2016, Lei et al. 2018). Peng et al. (2017) and Glynn et al. (2019) establish asymptotic results for several quantile sensitivity estimators in a unified manner using functional limit theory.

Recently, Peng et al. (2018) proposed a generalized likelihood ratio (GLR) method which can deal with sensitivity analysis for sample performance with a large scope of discontinuities, and Peng et al.

(2019) provided the GLR estimator for any distribution sensitivity under simpler and more interpretable conditions. Distribution sensitivities are pivotal in quantile sensitivity estimation studied in this work. The GLR estimator for quantile sensitivity does not require batching or binning but is easier to implement and has broader applicability than the previously proposed estimators.

Early work in sensitivity analysis in finance engineering and risk management can be found in Fu and Hu (1995), Broadie and Glasserman (1996), Glasserman (2004), Kaniel et al. (2008), Lyuu and Teng (2011), Liu and Hong (2011), Wang et al. (2012), and Chen and Liu (2014). Sensitivity analysis work on financial models with jumps include Glasserman and Liu (2010), Glasserman and Liu (2011), Peng et al. (2014), Peng et al. (2016), and sensitivity analysis of portfolio credit derivatives with correlated default times can be found in Joshi and Kainth (2004), Chen and Glasserman (2008), and Hong et al. (2014). Recently, Lei et al. (2019) consider sensitivity analysis of portfolio credit derivatives with respect to systemic parameters with correlated default times modeled by copulas.

The focus of this work is to apply the GLR method to estimate quantile sensitivity of financial models with correlations and jumps, which has not been studied in previous work. To fill the gap in the literature, we estimate the quantile sensitivity of a linear model with dependence modeled by an Archimedean copula and the quantile sensitivity of the option price under a Lévy-driven stochastic volatility model. Examples illustrate the wide applicability of the GLR method by providing several practical settings where other techniques are difficult to apply, and numerical results demonstrate the effectiveness of the new estimator.

The rest of the paper is organized as follows. In Section 2, we formulate the quantile sensitivity estimation problem and present the GLR estimator. Applications and numerical experiments are provided in Section 3. The last section offers conclusions.

2 QUANTILE SENSITIVITY ESTIMATION

Our setting considers stochastic models with the parameter appearing both in the performance function and in the underlying input distributions:

$$Y_\theta = g(X;\theta), \tag{1}$$

where $g(\cdot;\theta)$ is a performance function, $X=(X_1,\dots,X_n)$ is the input random vector with a joint density $f_X(\cdot;\theta)$, θ is the parameter of interest. Our goal is to estimate the derivative of the α-quantile with respect to parameter θ, $dq^\alpha_\theta/d\theta$, defined such that $F(q^\alpha_\theta;\theta)=\alpha$ where F is the distribution function of Y_θ. Assume Y_θ is a continuous r.v. with a positive and continuous density $f(y;\theta)$ on $(q^\alpha_\theta-\varepsilon, q^\alpha_\theta+\varepsilon)$, $\varepsilon>0$. Using the formula for the derivative of an implicit function, we have (Fu et al. 2009)

$$\frac{d}{d\theta}q^\alpha_\theta = -\left.\frac{\partial F(y;\theta)}{\partial \theta}\right|_{y=q^\alpha_\theta} \Big/ f(q^\alpha_\theta;\theta)\ .$$

Notice that

$$\frac{\partial F(y;\theta)}{\partial\theta} = \frac{\partial \mathbb{E}\left[\mathbf{1}\{g(X;\theta)\le y\}\right]}{\partial\theta}, \quad f(y;\theta) = \frac{\partial \mathbb{E}\left[\mathbf{1}\{g(X;\theta)\le y\}\right]}{\partial y}\ .$$

The derivatives are called distribution sensitivities. Discontinuity is introduced by the indicator function. Because of the existence of structural parameters in the discontinuous sample performance for distribution sensitivity estimation, classic gradient estimation techniques such as infinitesimal perturbation analysis (IPA) and likelihood ratio (LR) method do not apply (Ho and Cao 1991, Rubinstein and Shapiro 1993, Pflug 1996). From Peng et al. (2019), we have the GLR estimators for distribution sensitivities $\mathbf{1}\{Y_\theta\le y\}\Psi_{1,i}(X;\theta)$ and $\mathbf{1}\{Y_\theta\le y\}\Psi_{2,i}(X;\theta)$, $i=1,\dots,n$, such that

$$\frac{\partial F(y;\theta)}{\partial\theta} = \mathbb{E}\left[\mathbf{1}\{g(X;\theta)\le y\}\Psi_{1,i}(X;\theta)\right], \quad f(y;\theta) = \mathbb{E}\left[\mathbf{1}\{g(X;\theta)\le y\}\Psi_{2,i}(X;\theta)\right],$$

where

$$\begin{aligned}\Psi_{1,i}(x;\theta) =& \frac{\partial \log f_X(x;\theta)}{\partial \theta} \\ &- \left(\frac{\partial g(x;\theta)}{\partial x_i}\right)^{-1} \left[\frac{\partial g(x;\theta)}{\partial \theta}\left(\frac{\partial \log f_X(x;\theta)}{\partial x_i} - \frac{\partial^2 g(x;\theta)}{\partial x_i^2}\left(\frac{\partial g(x;\theta)}{\partial x_i}\right)^{-1}\right) + \frac{\partial^2 g(x;\theta)}{\partial \theta \partial x_i}\right], \\ \Psi_{2,i}(x;\theta) =& \left(\frac{\partial g(x;\theta)}{\partial x_i}\right)^{-1}\left(\frac{\partial \log f_X(x;\theta)}{\partial x_i} - \frac{\partial^2 g(x;\theta)}{\partial x_i^2}\left(\frac{\partial g(x;\theta)}{\partial x_i}\right)^{-1}\right).\end{aligned}$$

The GLR estimators for distribution sensitivities may not be unique. Different weight functions with $i = 1,\ldots,n$ could lead to different variances (Cui et al. 2019), though all unbiased under certain conditions.

We allow input random variables $X_1,\ldots,X_n$ to be dependent. Copulas are used widely in finance and actuarial science to quantify the dependence between assets (Nelsen 2013). Archimedean copulas are an important family of copulas that have explicit formulas and the potential to capture heavy tails of extreme events that cannot be captured by a Gaussian copula. Except for the Gaussian family, it is difficult in general to simulate dependent random variables and their corresponding pathwise derivative estimates, because there are not explicit formulas for the conditional distribution functions. For a special case of Archimedean copulas, pathwise simulation of dependent random variables and their corresponding derivatives is relatively simple, but still requires Laplace transform inversion, which is time consuming if the inverse is not available in analytical form (Marshall and Olkin 1988).

The quantile sensitivity of Archimedean copulas can be efficiently addressed by GLR using change of measure. We consider smooth Archimedean copulas that have the following form:

$$C(u;\theta) = h^{-1}\left(h(u_1;\theta) + \cdots + h(u_n;\theta);\theta\right),$$

where $u = (u_1,\ldots,u_n)$, $h\colon [0,1]\times\Theta \to [0,\infty)$ is the generator function that is continuous, strictly decreasing, convex, and $h(1;\theta) = 0$ for all $\theta \in \Theta$. For marginal distribution functions $F_1(\cdot;\theta),..,F_n(\cdot;\theta)$, a joint distribution function with $F_i(\cdot)$ as its marginal distribution function can be defined through a copula:

$$F_Z(z;\theta) = C(F_1(z_1;\theta),\ldots,F_n(z_n;\theta);\theta),$$

where $z = (z_1,\ldots,z_n)$. Assuming sufficient smoothness of the copula, the copula density is given by

$$c(u;\theta) = \frac{\partial^n C(u;\theta)}{\partial u_1 \cdot\cdot \partial u_n},$$

and the density of the joint distribution is

$$f_Z(z;\theta) = c\left(F_1(z_1;\theta),\ldots,F_n(z_n;\theta);\theta\right)\ \prod_{i=1}^n f_i(z_i;\theta),$$

where $f_i(z_i;\theta)$, $i = 1,..,n$, are the densities of marginal distributions. For $g(Z;\theta)$ where $Z = (Z_1,\ldots,Z_n)$ follows a joint distribution $F_Z(\cdot)$, we have

$$F(y;\theta) = \mathbb{E}\left[\mathbf{1}\{g(Z;\theta)\le y\}\right] = \mathbb{E}\left[\mathbf{1}\{g(X;\theta)\le y\}\ c\left(F_1(X_1;\theta),\ldots,F_n(X_n;\theta);\theta\right)\right],$$

where X_i, $i = 1,..,n$, are independent and follow distribution $F_i(\cdot;\theta)$. Essentially, we avoid directly simulating dependent random variables by change of measure. Then, the GLR method in Peng et al. (2018), whose derivation includes function smoothing, integration by parts, and taking limits, can be easily extended to handle this problem if the copula density $c(\cdot;\theta)$ has sufficient smoothness: for $i = 1,\ldots,n$,

$$\frac{\partial F(y;\theta)}{\partial \theta} = \mathbb{E}\left[\mathbf{1}\{g(Z;\theta)\le y\}\widehat{\Psi}_{1,i}(X;\theta)\right], \quad f(y;\theta) = \mathbb{E}\left[\mathbf{1}\{g(Z;\theta)\le y\}\widehat{\Psi}_{2,i}(X;\theta)\right],$$

where

$$\begin{aligned}
\widehat{\Psi}_{1,i}(x;\theta) =& \sum_{i=1}^{n} \frac{\partial \log f_i(x_i;\theta)}{\partial \theta} + \frac{\partial c(u;\theta)}{\partial \theta}\bigg|_{u_i=F_i(x_i;\theta)} \\
&+ \sum_{i=1}^{n} \frac{\partial c(u;\theta)}{\partial u_i}\bigg|_{u_i=F_i(x_i;\theta)} \frac{\partial F_i(x_i;\theta)}{\partial \theta} - \frac{\partial g(x;\theta)}{\partial \theta} \frac{\partial c(u;\theta)}{\partial u_i}\bigg|_{u_i=F_i(x_i;\theta)} f_i(x_i;\theta) \\
&+ c(F_1(x_1;\theta),..,F_n(x_1;\theta);\theta)\left(\frac{\partial g(x;\theta)}{\partial \theta}\frac{\partial \log f_i(x_i;\theta)}{\partial x_i} + \frac{\partial^2 g(x;\theta)}{\partial \theta \partial x_i}\right)\left(\frac{\partial g(x;\theta)}{\partial x_i}\right)^{-1} \\
&+ c(F_1(x_i;\theta),..,F_n(x_i;\theta);\theta)\frac{\partial g(x;\theta)}{\partial \theta}\frac{\partial^2 g(x;\theta)}{\partial x_i^2}\left(\frac{\partial g(x;\theta)}{\partial x_i}\right)^{-2}, \\
\widehat{\Psi}_{2,i}(x;\theta) =& \left(\frac{\partial c(u;\theta)}{\partial u_i}\bigg|_{u_i=F_i(x_i;\theta)} f_i(x_i;\theta) + c(F_1(x_1;\theta),..,F_n(x_n;\theta);\theta)\frac{\partial \log f_i(x_i;\theta)}{\partial x_i}\right)\left(\frac{\partial g(x;\theta)}{\partial x_i}\right)^{-1} \\
&- c(F_1(x_1;\theta),..,F_n(x_n;\theta);\theta)\frac{\partial^2 g(x;\theta)}{\partial x_i^2}\left(\frac{\partial g(x;\theta)}{\partial x_i}\right)^{-2}.
\end{aligned}$$

To deal with jumps in the financial models, we consider a generalization of (1):

$$g_N(X(N);\theta),$$

where $g_n(\cdot)$ is a performance function, $X(n) = (X_1,\ldots,X_n)$ and N is a discrete random variable with probability mass function $p(n;\theta) = Pr(N=n)$ for $n \in \mathbb{Z}^+$. A simple example illustrating how the introduction of N allows jumps in a financial model is the following jump-diffusion model:

$$dS(t)/S(t) = \mu dt + \sigma dW(t) + dJ(t),$$

where μ and σ are drift and volatility parameters, respectively, W is a standard Brownian motion, and J is the jump term given by

$$J(t) = \sum_{i=1}^{N(t)} Y_i,$$

where $\{N(t)\}_{t=0}^{\infty}$ is a counting process, e.g., a Poisson process, and Y_i, $i=1,..,N$, are jump size r.v.s. For a fixed t, define

$$G_N(X_{1:N};\theta) = \mu t + \sigma X_1 + \sum_{i=2}^{N} X_i,$$

where θ is a generic parameter that can be μ, σ, or a parameter governing the jump intensity of $N(t)$, $N = N(t)+1$, X_1 is a standard normal r.v., $X_i = Y_{i-1}$, $i=2,..,N$. The GLR estimators for distribution sensitivities can be generalized straightforwardly by

$$\frac{\partial F(y;\theta)}{\partial \theta} = \mathbb{E}\left[\mathbf{1}\{g_N(X(N);\theta) \le y\}\widetilde{\Psi}_{1,i}(X(N);\theta)\right], \quad f(y;\theta) = \mathbb{E}\left[\mathbf{1}\{g_N(X(N);\theta) \le y\}\widetilde{\Psi}_{2,i}(X(N);\theta)\right],$$

where

$$\widetilde{\Psi}_{j,i}(X(N);\theta) = \frac{\partial \log p(n;\theta)}{\partial \theta}\bigg|_{n=N} + \Psi_{j,i}(X(N);\theta), \quad i=1,\ldots,n,\ j=1,2.$$

3 APPLICATIONS AND EXPERIMENTS

In this section, two numerical examples are provided: a summation of two random variables with dependence modeled by a bivariate Archimedean copula and option pricing under a stochastic volatility model driven by a pure jump Lévy process. In these two examples where IPA in Hong 2009 and CMC in Fu et al. 2009 are biased or computationally inefficient to apply, GLR is compared with the finite difference (FD) method and FD method with common random number (CRN) (FDC).

Example 1. Archimedean copula

We test the quantile sensitivity of $g(Z_1,Z_2)=Z_1+Z_2$, where (Z_1,Z_2) follows a distribution corresponding to a bivariate Archimedean copula with generator function $\log[(1-\theta(1-x))/x]$, given by

$$C(u_1,u_2;\theta)=\frac{u_1u_2}{1-\theta(1-u_1)(1-u_2)},$$

where $\theta\in[-1,1)$, governing the strength of dependence. Its copula density is given by

$$c(u_1,u_2;\theta)=\frac{1-\theta}{[1-\theta(1-u_1)(1-u_2)]^2}+\frac{2\theta u_1u_2}{[1-\theta(1-u_1)(1-u_2)]^3},$$

and the derivatives of the copula density are given by

$$\begin{aligned}
\frac{\partial c(u_1,u_2;\theta)}{\partial\theta} &= -\frac{1}{[1-\theta(1-u_1)(1-u_2)]^2}+\frac{2[(1-\theta)(1-u_1)(1-u_2)+u_1u_2]}{[1-\theta(1-u_1)(1-u_2)]^3}\\
&\quad+\frac{6\theta u_1u_2(1-u_1)(1-u_2)}{[1-\theta(1-u_1)(1-u_2)]^4},\\
\frac{\partial c(u_1,u_2;\theta)}{\partial u_1} &= \frac{2\theta[u_2-(1-\theta)(1-u_2)]}{[1-\theta(1-u_1)(1-u_2)]^3}-\frac{6\theta^2u_1u_2(1-u_2)}{[1-\theta(1-u_1)(1-u_2)]^4},\\
\frac{\partial c(u_1,u_2;\theta)}{\partial u_2} &= \frac{2\theta[u_1-(1-\theta)(1-u_1)]}{[1-\theta(1-u_1)(1-u_2)]^3}-\frac{6\theta^2u_1u_2(1-u_1)}{[1-\theta(1-u_1)(1-u_2)]^4}\ .
\end{aligned}$$

We assume the marginal distributions of Z_1 and Z_2 are standard normal, and set $\theta=0$ and $i=1$ in $\widehat{\Psi}_{j,i}(X;\theta)$, $j=1,2$. Because it is computationally intensive to simulate dependent random variables with a joint distribution given by the Archimedean copula, IPA, CMC, and FD or FDC based on direct simulation of dependent input random variables would be computationally inefficient. We apply FDC to approximate the numerator and denominator of quantile sensitivity, i.e.

$$-\frac{\sum_{\ell=1}^M \mathbf{1}\{g(X_1^{(\ell)},X_2^{(\ell)})\le q_\theta^\alpha\}\left[c\left(F_1(X_1^{(\ell)}),F_2(X_2^{(\ell)});\theta+\delta\right)-c\left(F_1(X_1^{(\ell)}),F_2(X_2^{(\ell)});\theta\right)\right]}{\sum_{\ell=1}^M\left[\mathbf{1}\{g(X_1^{(\ell)},X_2^{(\ell)})\le q_\theta^\alpha+\delta\}-\mathbf{1}\{g(X_1^{(\ell)},X_2^{(\ell)})\le q_\theta^\alpha\}\right]c\left(F_1(X_1^{(\ell)}),F_2(X_2^{(\ell)});\theta\right)},$$

$(X_1^{(\ell)},X_2^{(\ell)})$, $\ell=1,\dots,M$, are i.i.d. copies of (X_1,X_2), and set the perturbed size $\delta=0.01$. When the perturbation size δ is small, the denominator is likely to be zero if the sample size M is not large enough. Quantile in this example is estimated by:

$$\widehat{q}_\theta^\alpha=\arg\min_y\left[\frac{1}{M}\sum_{\ell=1}^M\mathbf{1}\{g(X_1^{(\ell)},X_2^{(\ell)})\le y\}c\left(F_1(X_1^{(\ell)}),F_2(X_2^{(\ell)});\theta\right)-\alpha\right]^2\ .$$

From Table 1, the variance of GLR is smaller than that of FDC.

Table 1: Means and standard errors of α-quantile sensitivity of a Archimedean copula based on 10^2 independent experiments using a sample size of $M = 10^4$ in the upper table and $M = 10^6$ in the lower table.

$M = 10^4$	GLR	FDC
$\alpha = 0.2$	$-0.235 \pm 8\times10^{-4}$	$-0.229 \pm 6\times10^{-3}$
$\alpha = 0.4$	$-0.079 \pm 8\times10^{-4}$	$-0.077 \pm 2\times10^{-3}$
$\alpha = 0.6$	0.077 ± 10^{-3}	$0.076 \pm 2\times10^{-3}$
$\alpha = 0.8$	$0.235 \pm 2\times10^{-3}$	$0.227 \pm 6\times10^{-3}$

$M = 10^6$	GLR	FDC
$\alpha = 0.2$	$-0.236 \pm 8\times10^{-5}$	$-0.235 \pm 5\times10^{-4}$
$\alpha = 0.4$	$-0.077 \pm 8\times10^{-5}$	$-0.077 \pm 2\times10^{-4}$
$\alpha = 0.6$	0.077 ± 10^{-4}	$0.078 \pm 2\times10^{-4}$
$\alpha = 0.8$	$0.235 \pm 2\times10^{-4}$	$0.237 \pm 5\times10^{-4}$

Example 2. Lévy-Driven Ornstein-Uhlenbeck (OU) Process

Barndorff-Nielsen and Shephard (2001) introduced the following Lévy-driven OU process:

$$dV(t) = -\lambda V(t)dt + dZ(t), \quad V(0) > 0,$$

where $\lambda > 0$ and $\{Z(t)\}_{t=0}^{\infty}$ is called the background driving Lévy process (BDLP), a subordinator without drift and $Z(0) = 0$. The economic interpretation of stochastic volatility is random intensity of business activities in the market. The Bandorff-Nielsen and Shepard (BNS) stochastic volatility model is given by

$$dR(t) = (\mu + \beta V(t))dt + \sqrt{V(t)}dW(t) + \rho dZ(t), \quad R(0) = 0,$$

which is a generalization of the Black-Scholes model. The solution of (3) is

$$V(t) = e^{-\lambda t}V(0) + \int_0^t e^{-\lambda(t-t')}dZ(t'), \quad V(0) > 0,$$

and conditional on $Z(t)$, $V(t)$ and $V(0)$, the return process is given by

$$R(t) = \left(\mu t + \rho Z(t) - \beta \int_0^t V(t')dt'\right) + \left(\int_0^t V(t')dt'\right)^{\frac{1}{2}} X_1$$

where $X_1 \sim N(0,1)$. From the stochastic differential equation (3),

$$\int_0^t V(t')dt' = \frac{1}{\lambda}\left[Z(t) - (V(t) - V(0))\right] \ .$$

We assume the dynamics of the return process of the underlying asset follow a BNS model. The stochastic model of interest is the European call option price at time t, denoted by $C_{t,T}^{BNS}(S(t),V(t))$, which can be priced via Fourier transforms as shown in the Appendix. The simplest BNS model is $\Gamma(a,b)$-OU, where the marginal distribution of $V(t)$ is a Gamma distribution with parameters a and b (see the Appendix) and the BDLP is a compound Poisson process with

$$Z(t) = \sum_{i=1}^{N(t)} J_i, \qquad V(t) = e^{-\lambda t}V(0) + \sum_{i=1}^{N(t)} e^{\lambda(T_i - t)}J_i, \quad V(0) > 0,$$

where $\{N(t)\}_{t=0}^{\infty}$ is the Poisson process with intensity rate λa, T_i, $i = 1,..,N(t)$, are the jump times, and J_i, $i = 1,..,N(t)$, are the jump sizes which follow an exponential distribution with mean $1/b$. We define

$$g_N(X(N);\theta) = C_{t,T}^{BNS}(S(t),V(t)),$$

Table 2: The means and standard errors of α-quantile sensitivity of option price under the Lévy-driven stochastic volatility model based on 10^4 independent experiments using a sample size of $M = 10^3$ in the upper table and $M = 10^4$ in the lower table.

$M = 10^3$	GLR	FDC	FD
$\alpha = 0.2$	$2.495 \pm 1.6 \times 10^{-3}$	$2.513 \pm 0.7 \times 10^{-3}$	$2.495 \pm 6.5 \times 10^{-3}$
$\alpha = 0.4$	$2.532 \pm 3.1 \times 10^{-3}$	$2.539 \pm 0.4 \times 10^{-3}$	$2.524 \pm 5.0 \times 10^{-3}$
$\alpha = 0.6$	$2.554 \pm 1.5 \times 10^{-3}$	$2.556 \pm 0.2 \times 10^{-3}$	$2.561 \pm 4.9 \times 10^{-3}$
$\alpha = 0.8$	$2.595 \pm 0.8 \times 10^{-3}$	$2.578 \pm 0.5 \times 10^{-3}$	$2.616 \pm 6.5 \times 10^{-3}$

$M = 10^4$	GLR	FDC	FD
$\alpha = 0.2$	$2.496 \pm 0.5 \times 10^{-3}$	$2.513 \pm 0.2 \times 10^{-3}$	$2.493 \pm 2.0 \times 10^{-3}$
$\alpha = 0.4$	$2.531 \pm 1.1 \times 10^{-3}$	$2.539 \pm 0.1 \times 10^{-3}$	$2.528 \pm 1.6 \times 10^{-3}$
$\alpha = 0.6$	$2.555 \pm 0.5 \times 10^{-3}$	$2.556 \pm 0.1 \times 10^{-3}$	$2.556 \pm 1.5 \times 10^{-3}$
$\alpha = 0.8$	$2.600 \pm 0.3 \times 10^{-3}$	$2.578 \pm 0.1 \times 10^{-4}$	$2.604 \pm 2.1 \times 10^{-3}$

where $S(t) = S(0)\exp(R(t))$, $N = 2N(t)+1$, $X_i = J_{i-1}$, $i = 2,..,N(t)+1$, $X_{N(t)+1+j} = T_j$, $j = 1,..,N(t)$, and θ can be any parameter in $(\lambda,\mu,\beta,\rho,a,b,r)$.

We can see the parameter λ governs the jump intensity (together with a) and decay speed of the stochastic volatility, and for the call option price at time t under the Lévy-driven stochastic volatility model, IPA fails because of jumps in $(R(t),V(t))$. CMC requires an inversion of the option price formula $C_t^{BNS}(S(t),V(t))$, which is computationally intensive.

We test the quantile sensitivity with respect to the parameter λ of the option price. The sensitivity is estimated at $\lambda = 0.2$, $\mu = 0$, $\beta = 0$, $\rho = 0$, $a = 2$, $b = 100$, $r = 0.03$, $t = 1$, $T = 2$, $S(0) = 1$ and $K = 1$. Set $i = 1$ in $\widetilde{\Psi}_{j,i}(X;\theta)$, $j = 1,2$. The derivatives used to calculate the quantile sensitivity estimator are given in the Appendix. To calculate the Fourier transform, we use the trapezoidal rule (Feng and Lin 2013) and set the discretization size to 0.1, the truncation length to 100, and the auxiliary parameter σ to 1.

Because there is no analytical form for the quantile sensitivity, we implement FD and FDC with perturbation size 0.01 to compare with GLR. Random variables X_1 and $V(0)$ are shared in the simulation of the stochastic model under different parameter values. For the simulation of the compound Poisson process, we first generate the Poisson process $\{N(t)\}_{t=0}^{\infty}$ with rate $a(\lambda+\delta)$ and the corresponding jump sizes; then for the sample path of parameter $a(\lambda-\delta)$, we discard each jump with probability $2\delta/(\lambda+\delta)$.

In Table 2, we can see that the standard errors of GLR are smaller than FD but larger than FDC. The means of the quantile sensitivity estimates of the three methods are close, although there are three sources of biases: ratio bias, the finite number of samples and perturbation size, and the numerical calculation of the Fourier transform (with finite discretization and truncation sizes) when we calculate the option price. These sources of biases affect the three methods differently.

4 CONCLUSIONS

We provide a GLR estimator for quantile sensitivities. The proposed estimator does not require batching or binning, and has broad applicability. The numerical examples illustrate how the GLR estimator can deal with stochastic models that have discontinuous sample paths and discontinuous sample performance measures, and dependent input random variables that are computationally intensive to generate.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation of China (NSFC) under Grants 71901003, 71571048, 71720107003, 71690232, 91846301, and 71790615, by the National Science Foundation under Grant CMMI-1462787, and by the Defense Advanced Research Projects Agency (DARPA) under Grant N660011824024. The views, opinions and/or findings expressed are those of the authors and should not be interpreted as representing the official views or policies of the Department of Defense or the U.S. Government.

APPENDIX

Option Pricing Formula in Example 2 of Section 3.

In the risk-neutral case, $\mu = r - l(\rho)$ and $\beta = -1/2$, where r is the interest rate and $l(\cdot)$ is the Laplace exponent of Z under the risk-neutral probability. For the call option price at time t,

$$C^{BNS}_{t,T}(S(t),V(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\log K}\zeta_{t,T}(u)du + C^{BS}_{t,T}(S(t),\sigma^2),$$

where T is the maturity time, $C^{BS}_{t,T}(S(t),\sigma^2)$ is the Black-Scholes price of a call option with volatility σ, strike price K, underlying value $e^{R(t)}$ and interest rate r, and

$$\zeta_{t,T}(u) = e^{iu(r(T-t)+\log S(t))}\frac{\Lambda_{t,T}(u-i) - \Lambda^{\sigma}_{t,T}(u-i)}{iu(1+iu)},$$

where i is the unit imaginary number, $\Lambda_{t,T}$ is the characteristic function of $R(T)-R(t)$ conditional on $V(t)$ and $\Lambda^{\sigma}_{t,T} = \exp\left(-\frac{\sigma^2(T-t)}{2}(u^2+iu)\right)$. See Carr and Madan (1999) and Carr et al. (2003) for details.

When the marginal distribution of $V(t)$ is the $\Gamma(a,b)$-distribution with PDF

$$\frac{b^a x^{a-1}}{\Gamma(a)}\exp(-bx)\mathbf{1}\{x>0\}$$

where $\Gamma(a) = \int_0^\infty e^{-y}y^{a-1}dy$ denotes the gamma function, the BDLP is a compound Poisson process. For the $\Gamma(a,b)$-OU stochastic volatility model, the characteristic function of $R(T)-R(t)$ conditional on $V(t)$ can be given explicitly by

$$\begin{aligned}\log\Lambda_{t,T}(u) =& i\,(r-l(\rho))\,(T-t) - V(t)\kappa(T-t)\frac{u^2+iu}{2}\\ &+ \frac{ab}{b-\Gamma_2(u)}\ln\frac{b-\Gamma_1(u)}{b-iu\rho} + \frac{\lambda a(T-t)\Gamma_2(u)}{b-\Gamma_2(u)},\end{aligned}$$

where $l(\rho) = \lambda a\rho/(\lambda-\rho)$, $\kappa(T-t) = (1-e^{-\lambda(T-t)})/\lambda$ and

$$\begin{aligned}\Gamma_1(u) &= iu\rho - \frac{u^2+iu}{2}\kappa(T-t),\\ \Gamma_2(u) &= iu\rho - \lambda^{-1}\frac{u^2+iu}{2}\ .\end{aligned}$$

See Tankov (2004) for details. To simulate Lévy-driven processes $Z(t)$ and $V(t)$ generally requires the inversion of characteristic functions (Peng et al. 2016), which is computationally intensive. A compound Poisson approximation for a Lévy process can be found in Glasserman and Liu (2011).

Derivatives in Example 2 of Section 3:

$$\frac{\partial}{\partial\lambda}\zeta_{t,T}(u) = \frac{e^{iu(r(T-t)+R(t))}}{1+iu}\left((\Lambda_{t,T}(u-i)-\Lambda^{\sigma}_{t,T}(u-i))\frac{\partial R(t)}{\partial\lambda} + \frac{1}{iu}\frac{\partial\Lambda_{t,T}}{\partial\lambda}\right),$$

$$\frac{\partial}{\partial x_1}\zeta_{t,T}(u) = e^{iu(r(T-t)+R(t))}\frac{\Lambda_{t,T}(u-i)-\Lambda^{\sigma}_{t,T}(u-i)}{1+iu}\frac{\partial R(t)}{\partial x_1},$$

$$\frac{\partial^2}{\partial\lambda\partial x_1}\zeta_{t,T}(u) = \frac{e^{iu(r(T-t)+R(t))}}{1+iu}\left\{(\Lambda_{t,T}-\Lambda^{\sigma}_{t,T})\left(iu\frac{\partial R(t)}{\partial\lambda}\frac{\partial R(t)}{\partial x_1} + \frac{\partial^2 R(t)}{\partial\lambda\partial x_1}\right) + \frac{\partial R(t)}{\partial x_1}\frac{\partial\Lambda_{t,T}}{\partial\lambda}\right\},$$

$$\frac{\partial^2}{\partial x_1^2}\zeta_{t,T}(u) = \frac{iue^{iu(r(T-t)+R(t))}(\Lambda_{t,T}(u-i)-\Lambda^{\sigma}_{t,T}(u-i))}{1+iu}\left(\frac{\partial R(t)}{\partial x_1}\right)^2;$$

$$\begin{aligned}\frac{\partial}{\partial\lambda}\ln\Lambda_{t,T}(\vartheta) =& -i(T-t)\frac{\partial l(\rho)}{\partial\lambda} - \kappa(T-t)\frac{\vartheta^2+i\vartheta}{2}\frac{\partial V(t)}{\partial\lambda} - V(t)\frac{\vartheta^2+i\vartheta}{2}\frac{\partial\kappa(T-t)}{\partial\lambda}\\ &+\frac{ab}{(b-\Gamma_2)^2}\ln\frac{b-\Gamma_1}{b-i\vartheta\rho}\frac{\partial\Gamma_2}{\partial\lambda} - \frac{ab}{(b-\Gamma_1)(b-\Gamma_2)}\frac{\partial\Gamma_1}{\partial\lambda} + \frac{a(T-t)\Gamma_2}{b-\Gamma_2}\\ &+\frac{\lambda a(T-t)}{b-\Gamma_2}\frac{\partial\Gamma_2}{\partial\lambda} + \frac{\lambda a(T-t)\Gamma_2}{(b-\Gamma_2)^2}\frac{\partial\Gamma_2}{\partial\lambda},\end{aligned}$$

$$\frac{\partial}{\partial x_1}\Lambda_{t,T}(\vartheta) = \frac{\partial^2}{\partial\lambda\partial x_1}\Lambda_{t,T}(\vartheta) = \frac{\partial^2}{\partial x_1^2}\Lambda_{t,T}(\vartheta) = 0,$$

where $\vartheta = u-i$;

$$\frac{\partial R(t)}{\partial\lambda} = -\frac{1}{2\sqrt{\lambda}}\left\{\frac{1}{\lambda}[Z(t)-(V(t)-V(0))] + \frac{\partial V(t)}{\partial\lambda}\right\}[Z(t)-(V(t)-V(0))]^{-\frac{1}{2}}X_1,$$

$$\frac{\partial R(t)}{\partial x_1} = \left\{\frac{1}{\lambda}[Z(t)-(V(t)-V(0))]\right\}^{\frac{1}{2}}, \qquad \frac{\partial^2 R(t)}{\partial x_1^2} = 0,$$

$$\frac{\partial^2 R(t)}{\partial\lambda\partial x_1} = -\frac{1}{2\sqrt{\lambda}}\left\{\frac{1}{\lambda}[Z(t)-(V(t)-V(0))] + \frac{\partial V(t)}{\partial\lambda}\right\}[Z(t)-(V(t)-V(0))]^{-\frac{1}{2}};$$

$$\frac{\partial V(t)}{\partial\lambda} = -te^{-\lambda t}V(0) + \sum_{i=1}^{N(t)}(T_i-t)J_ie^{\lambda(T_i-t)},$$

$$\frac{\partial V(t)}{\partial x_1} = \frac{\partial^2 V(t)}{\partial\lambda\partial x_1} = \frac{\partial^2 V(t)}{\partial x_1^2} = \frac{\partial Z(t)}{\partial\lambda} = \frac{\partial Z(t)}{\partial x_1} = \frac{\partial^2 Z(t)}{\partial\lambda\partial x_1} = \frac{\partial^2 Z(t)}{\partial x_1^2} = 0;$$

$$\begin{aligned}&\frac{\partial\ln p(n)}{\partial\lambda} = \frac{n}{\lambda} - at, \quad \frac{\partial l(\rho)}{\partial\lambda} = \frac{a\rho}{\lambda-\rho} - \frac{\lambda a\rho}{(\lambda-\rho)^2},\\ &\frac{\partial\kappa(T-t)}{\partial\lambda} = \frac{(T-t)e^{-\lambda(T-t)}}{\lambda} - \frac{(1-e^{-\lambda(T-t)})}{\lambda^2},\\ &\frac{\partial\Gamma_1(\vartheta)}{\partial\lambda} = -\frac{\vartheta+i\vartheta^2}{2}\frac{\partial\kappa(T-t)}{\partial\lambda}, \quad \frac{\partial\Gamma_2(\vartheta)}{\partial\lambda} = \frac{\vartheta+i\vartheta^2}{2\lambda^2};\end{aligned}$$

$$C^{BS}_{t,T}(R(t),\sigma^2) = C_{1t} - C_{2t};$$

$$\begin{aligned}\frac{\partial C_{1t}}{\partial\lambda} = e^{R(t)}\frac{\partial R(t)}{\partial\lambda}\Bigg\{&F_{\mathcal{N}}\left(\frac{R(t)+(r+\frac{\sigma^2}{2})(T-t)-\ln K}{\sigma\sqrt{T-t}}\right)\\ &+\frac{1}{\sigma\sqrt{T-t}}f_{\mathcal{N}}\left(\frac{R(t)+(r+\frac{\sigma^2}{2})(T-t)-\ln K}{\sigma\sqrt{T-t}}\right)\Bigg\},\end{aligned}$$

$$\frac{\partial C_{1t}}{\partial x_1} = e^{R(t)} \frac{\partial R(t)}{\partial x_1} \left\{ F_{\mathcal{N}} \left(\frac{R(t) + (r + \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) + \frac{1}{\sigma\sqrt{T-t}} f_{\mathcal{N}} \left(\frac{R(t) + (r + \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \right\},$$

$$\begin{aligned}
\frac{\partial^2 C_{1t}}{\partial \lambda \partial x_1} = e^{R(t)} \Bigg\{ & F_{\mathcal{N}} \left(\frac{R(t) + (r + \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \left(\frac{\partial^2 R(t)}{\partial \lambda \partial x_1} + \frac{\partial R(t)}{\partial \lambda} \frac{\partial R(t)}{\partial x_1} \right) \\
& + \frac{1}{\sigma\sqrt{T-t}} f_{\mathcal{N}} \left(\frac{R(t) + (r + \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \left(\frac{\partial^2 R(t)}{\partial \lambda \partial x_1} + 2 \frac{\partial R(t)}{\partial \lambda} \frac{\partial R(t)}{\partial x_1} \right) \\
& + \frac{1}{\sigma^2(T-t)} f'_{\mathcal{N}} \left(\frac{R(t) + (r + \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \frac{\partial R(t)}{\partial \lambda} \frac{\partial R(t)}{\partial x_1} \Bigg\}, \\
\frac{\partial^2 C_{1t}}{\partial x_1^2} = e^{R(t)} \left(\frac{\partial R(t)}{\partial x_1} \right)^2 \Bigg\{ & F_{\mathcal{N}} \left(\frac{R(t) + (r + \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \\
& + \frac{2}{\sigma\sqrt{T-t}} f_{\mathcal{N}} \left(\frac{R(t) + (r + \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \\
& + \frac{1}{\sigma^2(T-t)} f'_{\mathcal{N}} \left(\frac{R(t) + (r + \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \Bigg\},
\end{aligned}$$

where $F_{\mathcal{N}}$ and $f_{\mathcal{N}}$ are the distribution function and density of the standard normal distribution;

$$\begin{aligned}
\frac{\partial C_{2t}}{\partial \lambda} &= \frac{Ke^{-r(T-t)}}{\sigma\sqrt{T-t}} f_{\mathcal{N}} \left(\frac{R(t) + (r - \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \frac{\partial R(t)}{\partial \lambda}, \\
\frac{\partial C_{2t}}{\partial x_1} &= \frac{Ke^{-r(T-t)}}{\sigma\sqrt{T-t}} f_{\mathcal{N}} \left(\frac{R(t) + (r - \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \frac{\partial R(t)}{\partial x_1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 C_{2t}}{\partial \lambda \partial x_1} = \frac{Ke^{-r(T-t)}}{\sigma\sqrt{T-t}} \Bigg\{ & f_{\mathcal{N}} \left(\frac{R(t) + (r - \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \frac{\partial^2 R(t)}{\partial \lambda \partial x_1} \\
& + \frac{1}{\sigma\sqrt{T-t}} f'_{\mathcal{N}} \left(\frac{R(t) + (r - \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \frac{\partial R(t)}{\partial \lambda} \frac{\partial R(t)}{\partial x_1} \Bigg\}, \\
\frac{\partial^2 C_{2t}}{\partial x_1^2} = \frac{Ke^{-r(T-t)}}{\sigma^2(T-t)} & f'_{\mathcal{N}} \left(\frac{R(t) + (r - \frac{\sigma^2}{2})(T-t) - \ln K}{\sigma\sqrt{T-t}} \right) \left(\frac{\partial R(t)}{\partial x_1} \right)^2 .
\end{aligned}$$

REFERENCES

Alexopoulos, C., D. Goldsman, A. Mokashi, K. Tien, and J. Wilson. 2019. "Sequest: A Sequential Procedure for Estimating Quantiles in Steady-state Simulations". *Operations Research, to appear*.

Barndorff-Nielsen, O. E., and N. Shephard. 2001. "Non-Gaussian Ornstein-Uhlenbeck Based Models and Some of Their Uses in Financial Economics". *Journal of the Royal Statistical Society: Series B* 63(2):167–241.

Broadie, M., and P. Glasserman. 1996. "Estimating Security Price Derivatives Using Simulation". *Management Science* 42(2):269–285.

Carr, P., H. Geman, D. B. Madan, and M. Yor. 2003. "Stochastic Volatility for Lévy Processes". *Mathematical Finance* 13(3):345–382.

Carr, P., and D. B. Madan. 1999. "Option Valuation Using the Fast Fourier Transform". *Journal of Computational Finance* 2(4):61–73.

Chen, N., and Y. Liu. 2014. "American Option Sensitivities Estimation Via A Generalized Infinitesimal Perturbation Analysis Approach". *Operations Research* 62(3):616–632.

Chen, Z., and P. Glasserman. 2008. "Sensitivity Estimates for Portfolio Credit Derivatives Using Monte Carlo". *Finance and Stochastics* 12(4):507–540.

Cui, Z., M. C. Fu, J. Hu, Y. Liu, Y. Peng, and L. Zhu. 2019. "On the Variance of Single-run Unbiased Stochastic Derivative Estimators". *INFORMS Journal on Computing, accepted, preprint in ResearchGate with DOI: 10.13140/RG.2.2.27015.04007*.

David, H. A., and H. N. Nagaraja. 1970. *Order Statistics*. Wiley Online Library.

Feng, L., and X. Lin. 2013. "Inverting Analytic Characteristic Functions and Financial Applications". *SIAM Journal on Financial Mathematics* 4(1):372–398.

Fu, M. C., L. J. Hong, and J.-Q. Hu. 2009. "Conditional Monte Carlo Estimation of Quantile Sensitivities". *Management Science* 55(12):2019–2027.

Fu, M. C., and J.-Q. Hu. 1995. "Sensitivity Analysis for Monte Carlo Simulation of Option Pricing". *Probability in the Engineering and Informational Sciences* 9(03):417–446.

Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. Springer.

Glasserman, P., and Z. Liu. 2010. "Sensitivity Estimates from Characteristic Functions". *Operations Research* 58(6):1611–1623.

Glasserman, P., and Z. Liu. 2011. "Sensitivity Estimates for Lévy-driven Models in Finance". *Journal of Computational Finance* 14(2):3–56.

Glynn, W. P., L. Fan, M. C. Fu, J. Hu, and Y. Peng. 2019. "Technical note—Central Limit Theorems for Estimated Functions at Estimated Points". *Operations Research, accepted, preprint in Researchgate with DOI: 10.13140/RG.2.2.28902.37447*.

Heidergott, B., and W. Volk-Makarewicz. 2016. "A Measure-valued Differentiation Approach to Sensitivity Analysis of Quantiles". *Mathematics of Operations Research* 41(1):293–317.

Heidergott, B., W. Volk-Makarewicz, and F. J. Vázquez-Abad. 2014. "Gradient Estimation for Quantiles of Stationary Waiting Times.". In *WODES*, 241–246.

Ho, Y.-C., and X.-R. Cao. 1991. *Discrete Event Dynamic Systems and Perturbation Analysis*. Kluwer Academic Publishers, Boston, MA.

Hong, L. J. 2009. "Estimating Quantile Sensitivities". *Operations Research* 57(1):118–130.

Hong, L. J., Z. Hu, and G. Liu. 2014. "Monte Carlo Methods for Value-at-risk and Conditional Value-at-risk: A Review". *ACM Transactions on Modeling and Computer Simulation* 24(4):1–37.

Hong, L. J., and G. Liu. 2009. "Simulating Sensitivities of Conditional Value At Risk". *Management Science* 55(2):281–293.

Hong, L. J., J. Sandeep, and L. Jun. 2014. "Estimating Sensitivities of Portfolio Credit Risk Using Monte Carlo". *INFORMS Journal on Computing* 26(4):848–865.

Jiang, G., and M. C. Fu. 2015. "Technical Note — On Estimating Quantile Sensitivities Via Infinitesimal Perturbation Analysis". *Operations Research* 63(2):435–441.

Jin, X., M. C. Fu, and X. Xiong. 2003. "Probabilistic Error Bounds for Simulation Quantile Estimators". *Management Science* 49(2):230–246.

Jorion, P. 2007. *Value at Risk: The New Benchmark for Managing Financial Risk*, Volume 2. McGraw-Hill, New York.

Joshi, M., and D. Kainth. 2004. "Rapid and Accurate Development of Prices and Greeks for nth-to-default Credit Swaps in the Li Model". *Quantitative Finance* 4(3):266–275.

Kaniel, R., S. Tompaidis, and A. Zemlianov. 2008. "Efficient Computation of Hedging Parameters for Discretely Exercisable Options". *Operations Research* 56(4):811–826.

Lei, L., Y. Peng, M. C. Fu, and J.-Q. Hu. 2018. "Applications of Generalized Likelihood Ratio Method to Distribution Sensitivities and Steady-state Simulation". *Journal of Discrete Event Dynamic Systems* 28(1):109–125.

Lei, L., Y. Peng, M. C. Fu, and J.-Q. Hu. 2019. "Sensitivity Analysis of Portfolio Credit Derivatives by Conditional Monte Carlo Simulation". *submitted to INFORMS Journal on Computing, preprint in ResearchGate with DOI: 10.13140/RG.2.2.21951.97444*.

Liu, G., and L. J. Hong. 2009. "Kernel Estimation of Quantile Sensitivities". *Naval Research Logistics* 56(6):511–525.

Liu, G., and L. J. Hong. 2011. "Kernel Estimation of the Greeks for Options with Discontinuous Payoffs". *Operations Research* 59(1):96–108.

Lyuu, Y.-D., and H.-W. Teng. 2011. "Unbiased and Efficient Greeks of Financial Options". *Finance and Stochastics* 15(1):141–181.

Marshall, A. W., and I. Olkin. 1988. "Families of Multivariate Distributions". *Journal of the American Statistical Association* 83(403):834–841.

Nelsen, R. B. 2013. *An Introduction to Copulas*, Volume 139. Springer Science & Business Media.

Peng, Y., M. C. Fu, P. W. Glynn, and J.-Q. Hu. 2017. "On the Asymptotic Analysis of Quantile Sensivitiy Estimation by Monte Carlo Simulation". In *Proceedings of 2017 Winter Simulation Conference*, edited by W. K. V. Chan, A. D' Ambrogio, G. Zacharewicz, N. Mustafee, G. Wainer, and E. Page, 2336–2347. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers, Inc.

Peng, Y., M. C. Fu, B. Heidergott, and H. Lam. 2019. "Maximum Likelihood Estimation by Monte Carlo Simulation: Data Driven Stochastic Modeling". *submitted to Operations Research, under second review, preprint in Researchgate with DOI: 10.13140/RG.2.2.34695.85925*.

Peng, Y., M. C. Fu, and J. Hu. 2014. "Gradient-based Simulated Maximum Likelihood Estimation for Lévy-driven Ornstein-Uhlenbeck Stochastic Volatility Models". *Quantitative Finance* 14(8):1399–1414.

Peng, Y., M. C. Fu, and J.-Q. Hu. 2016. "Gradient-based Simulated Maximum Likelihood Estimation for Stochastic Volatility Models Using Characteristic Functions". *Quantitative Finance* 16(9):1393–1411.

Peng, Y., M. C. Fu, J.-Q. Hu, and B. Heidergott. 2018. "A New Unbiased Stochastic Derivative Estimator for Discontinuous Sample Performances with Structural Parameters". *Operations Research* 66(2):487–499.

Pflug, G. C. 1996. *Optimization of Stochastic Models*. Kluwer Academic, Boston.

Rubinstein, R. Y., and A. Shapiro. 1993. *Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization by the Score Function Method*, Volume 346. Wiley New York.

Serfling, R. J. 2009. *Approximation Theorems of Mathematical Statistics*. John Wiley & Sons.

Tankov, P. 2004. *Financial Modelling with Jump Processes*. CRC Press.

Wang, Y., M. C. Fu, and S. I. Marcus. 2012. "A New Stochastic Derivative Estimator for Discontinuous Payoff Functions with Application to Financial Derivatives". *Operations Research* 60(2):447–460.

AUTHOR BIOGRAPHIES

YIJIE PENG received the B.S. degree in mathematics from Wuhan University, Wuhan, China, in 2007, and the Ph.D. degree in management science from Fudan University, Shanghai, China, in 2014, respectively. He was a research fellow with Fudan University and George Mason University. He is currently an Assistant Professor at the Department of Industrial Engineering and Management, Peking University, Beijing, China. His research interests include stochastic modeling and analysis, simulation optimization, machine learning, data analytics, and healthcare. His email address is pengyijie@pku.edu.cn.

MICHAEL C. FU holds the Smith Chair of Management Science in the Robert H. Smith School of Business, with a joint appointment in the Institute for Systems Research and affiliate faculty appointment in the Department of Electrical and Computer Engineering, all at the University of Maryland. His research interests include simulation optimization and applied probability. He served as WSC2011 Program Chair, NSF Operations Research Program Director, Management Science Stochastic Models and Simulation Department Editor, and Operations Research Simulation Area Editor. He is a Fellow of INFORMS and IEEE. His e-mail addresses is mfu@umd.edu.

JIAN-QIANG HU is the Hongyi Professor of Management Science in School of Management, Fudan University. He was an Associate Professor with the Department of Mechanical Engineering and the Division of Systems Engineering at Boston University before joining Fudan University. His research interests include discrete-event stochastic systems, simulation, stochastic optimization, with applications towards supply chain management, financial engineering, and healthcare. His e-mail addresses is hujq@fudan.edu.cn.

LEI LEI received the B.S. degree in mathematics from Sun Yat-sen University, China and the Ph.D. degree in management science from Fudan University, China, in 2012 and 2018, respectively. She joined the School of Economics and Business Administration at Chongqing University, Chongqing, China, in December, 2018. Her research interests include discrete-event stochastic systems, sensitivity analysis and simulation, with applications towards supply chain management and financial engineering. Her email address is leilei312@cqu.edu.cn.