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THE ASYMPTOTIC VALIDITY OF SEQUENTIAL STOPPING RULES FOR CONFIDENCE INTERVAL CONSTRUCTION USING STANDARDIZED TIME SERIES

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ABSTRACT

We establish the asymptotic validity of a class of sequential stopping rules when applying standardized time series (STS) to construct fixed-width confidence intervals (CI). The STS CI construction avoids requiring a consistent variance estimator, which is attractive to a class of steady-state simulation problems in which variance estimation is difficult. We quantify the asymptotic distribution of STS at stopping times as the prescribed half-width of the CI approaches zero. This provides us with the appropriate scaling parameter for the CI in the sequential stopping setting.

1 INTRODUCTION

In this paper, we study sequential stopping rules for steady-state simulation problems. In particular, we are interested in using simulation to estimate $\alpha = \int h(x)d\pi(x)$, where π is the stationary distribution of the stochastic process $X = \{X(t) : t \geq 0\}$ and h is a real-valued function defined on the state space of $X(t)$. (Discrete processes can be handled in the usual way, by setting $X(t) = X(\lfloor t \rfloor)$.) One natural candidate estimator for α is the time average:

$$\alpha(t) := \frac{1}{t}\int_0^t h(X(s))ds$$

for t large enough. The goal is to have a procedure to choose t so that one can control both the reliability (confidence level or coverage probability of the confidence interval (CI)) and the precision (half-width of the CI) of the estimator.

Under certain regularity conditions, $\alpha(t)$ satisfies a Central Limit Theorem, i.e. there exists $\sigma > 0$, such that for t large, $\alpha(t) \overset{D}{\approx} \alpha + \sigma/\sqrt{t}N(0,1)$, where $N(0,1)$ is a normal random variable (rv) with mean 0 and variance 1 and $\overset{D}{\approx}$ denote "approximately distributed as". Then the asymptotic $100(1-\delta)\%$ CI for α takes the form $\left[\alpha(t) - z_{\delta/2}\sigma/\sqrt{t}, \alpha(t) + z_{\delta/2}\sigma/\sqrt{t}\right]$, where $z_{\delta/2}$ is the $\delta/2$-th upper quantile of $N(0,1)$, i.e. $\mathbb{P}(N(0,1) > z_{\delta/2}) = \delta/2$. If we know the value of σ, then to obtain a CI with prescribed half-width ε, we would need $t = \lceil z_{\delta/2}^2 \sigma^2/\varepsilon^2 \rceil$. However, in steady-state simulation problems, σ is never known and is in general very difficult to estimate. In particular, σ^2 is known as the time-average variance constant (TAVC), and it reflects the complicated auto-correlation structure of the underlying stochastic process. In fact, the problem of consistently estimating the TAVC continues to be an active research area; see, for example, Alexopoulos et al. (2007), Wu (2009), Meterelliyoz et al. (2012).

Standardized time series methods are an alternative class of asymptotically valid CI procedures for steady-state simulation problems (Schruben 1983; Glynn and Iglehart 1990). The key idea is to cancel out the unknown variance parameter, which appears as a common factor in the numerator and the denominator of the constructed statistics. One commonly used standardized time series method is the batch means method. We next explain the basic idea of STS in the context of the batch means method.

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For batch means with m batches, we divide a single simulation run, $[0,t]$, into m equally sized non-overlapping batches, and define the batch mean of the k-th batch, $k = 1,2,\ldots,m$, as

$$\hat{\alpha}_k(t) := \frac{m}{t}\int_{(k-1)t/m}^{kt/m} h(X(s))ds$$

Then $\alpha(t) = \frac{1}{m}\sum_{k=1}^{m}\hat{\alpha}_k(t)$. We also write

$$S_m(t) := \sqrt{\frac{1}{m(m-1)}\sum_{k=1}^{m}\left(\hat{\alpha}_k(t)-\alpha(t)\right)^2}$$

For large values of t, $\sqrt{t/m}(\hat{\alpha}_k(t)-\alpha) \overset{D}{\approx} \sigma N(0,1)$, $k = 1,2,\ldots,m$ and $\sqrt{t}S_m(t) \overset{D}{\approx} \sigma\sqrt{\chi^2_{m-1}/(m-1)}$, where χ^2_{m-1} denotes a chi-squared rv with $m-1$ degrees of freedom. Here, the $N(0,1)$'s and the χ^2_{m-1} rv are independent. Thus, $(\alpha(t)-\alpha)/S_m(t) \overset{D}{\approx} t_{m-1}$, where t_{m-1} denote a Student t-distribution with $m-1$ degrees of freedom. For a fixed value of t, the asymptotic $100(1-\delta)\%$ CI for α takes the form $\left[\alpha(t)-\tilde{z}_{\delta/2}S_m(t), \alpha(t)+\tilde{z}_{\delta/2}S_m(t)\right]$, where $\tilde{z}_{\delta/2}$ is the $\delta/2$-th upper quantile of t_{m-1}. If we want to achieve a prescribed precision level ε, we may naively apply a sequential stopping scheme where we keep sampling until $\tilde{z}_{\delta/2}S_m(t) < \varepsilon$. However, such a sequential stopping scheme is not appropriate in the sense that $\lim_{\varepsilon\to 0}\mathbb{P}(\alpha \in [\alpha(T(\varepsilon))-\varepsilon, \alpha(T(\varepsilon))+\varepsilon]) < 1-\delta$ where $T(\varepsilon)$ is the (random) time at which $\tilde{z}_{\delta/2}S_m(t)$ is first less than ε. Empirical evidence of the under-coverage failure for such sequential procedures was observed a number of years ago (Adam 1983).

One remedy to the under-coverage problem is to construct strongly consistent estimators of σ (Glynn and Whitt 1992). Damerdji (1994) proves that if the number of batches m increases with t (sample size) at a suitably slower rate than the batch size, then $S_m(t)$ is a strongly consistent estimator of the TVAC. This suggests that one way to construct a sequential stopping procedure for batch means is to increase the number of batches as the run length increases. There are subsequent works that build on this idea, including the LBATCH procedure (Fishman 1996), the ABATCH procedure (Fishman and Yarberry 1997), ASAP3 (Steiger et al. 2005), and more recently developed SKART (Tafazzoli and Wilson 2011) and SPSTS (Alexopoulos et al. 2016). The basic idea for this class of procedures is the following. At each iteration, a statistical test is performed to check for correlation between batches. If significant correlation is detected, then we increase the batch size in the next iteration. Otherwise (if no significant correlation is detected), we increase the number of batches in the next iteration. These procedures also adjust for the finite sample bias by properly inflate the scaling parameter, see also Singham and Shruben (2012) for a recent development on finite sample adjustment. However, it is in general difficult to establish validity, even asymptotically, of these procedures in general steady-state simulation problems, and the performance can vary across problem instances.

More generally, there is a trade-off between the coverage probability and the expected width of the CI when choosing the number of batches with a finite sample size (Schmeiser 1982). Specifically, on the one hand, the smaller the number of batches, the larger the batch size and the closer we are to the independence and normality required by batch means, which leads to more accurate coverage probabilities. On the other hand, the smaller the number of batches, the larger the expected width of the CI. Schmeiser (1982) conducted a detailed study on the tradeoff between the number of batches and the batch size. The main message is that the effects of too few batches may be large but additional batches have diminishing effect and it doesn't add much benefit to go beyond 30 batches.

In this paper, we propose a simple remedy to the naive sequential stopping scheme. As we shall explain, if we replace the scaling parameter, $\tilde{z}_{\delta/2}$, with a new scaling parameter $u_{\delta/2}$ (see Table 1), then we can achieve the asymptotically accurate coverage probability. More generally, in this paper, we introduce a framework to construct asymptotically valid sequential stopping procedures for standardized time series where we do not require a strongly consistent variance estimator. Similar asymptotically valid sequential stopping procedure has been developed for the sectioning (replication) method with a fixed number of sections in Dong and Glynn (2019). Standardized time series methods lead to more complicated asymptotic behavior than the sectioning method.

2 STANDARDIZED TIME SERIES

Denote

$$Y(t) = \int_0^t h(X(s))ds,$$

i.e. $Y(t) = t\alpha(t)$. We also write $C[0,\infty)$ as the space of all continuous functions from $[0,\infty)$ to $\mathbb{R}$. The basis of STS is the introduction of a class of measurable functions $g : \mathbb{R}^+ \times C[0,\infty) \to \mathbb{R}$, satisfying the following properties:

(i) For fixed $t > 0$, $g(t,x)$ can be written as a function of $\{x(s) : 0 \le s \le t\}$ and is Lipschitz continuous.
(ii) For fixed $x \in C[0,\infty)$, $g(t,x)$ is continuous in t.
(iii) $g(t,x) = \frac{1}{h} g\left(\frac{t}{h}, x_h\right)$ for any $t > 0$, $x \in C[0,\infty)$, where $x_h(s) = x(hs)$ for $s \ge 0$.
(iv) $g(t,ax) = ag(t,x)$ for any $t > 0$, $a > 0$, $x \in C[0,\infty)$
(v) $g(t,x-bI) = g(t,x)$ for any $t > 0$, $b \in \mathbb{R}$ and $x \in C[0,\infty)$, where $I(t) = t$ is the identity function.
(vi) $\mathbb{P}(g(t,B) > 0) = 1$ for any $t > 0$, where B denotes a standard Brownian motion.
(vii) $\mathbb{P}(B \in D(g)) = 0$, where $D(g)$ denote the discontinuities of g.

We denote this class of functions as $\mathscr{M}$. The definition of $\mathscr{M}$ follows from that used in Glynn and Iglehart (1990). We add some extra properties, (i)-(iii), to accommodate the sequential stopping setting. Property (i) allows one to write $g(t,x)$ as $g(t,\{x(s) : 0 \le s \le t\})$. Thus, for a fixed value of t, $g(t,\cdot) : C[0,t] \to \mathbb{R}$. Property (iii) allows one to apply time scaling to g.

We next introduce a specific feature of $g \in \mathscr{M}$, which leads to an alternative definition of $\mathscr{M}$ in Glynn and Iglehart (1990). Let $\Psi_t : C[0,t] \to C[0,t]$ be defined as $(\Psi_t x)(s) := x(s) - \frac{s}{t}x(t)$. Then by Property (i) and (v),

$$g(t,x) = g\left(t, \{x(s) : 0 \le s \le t\} - \frac{x(t)}{t} I\right) = g(t, \Psi_t(\{x(s) : 0 \le s \le t\})).$$

As $B(t)$ is independent of $B(s) - \frac{s}{t}B(t)$, it is easy to see from this alternative representation that for fixed value of t, $B(t)$ is independent of $g(t,B)$.

In the special case of batch means, $S_m(t)$ can be written in the form of

$$g_b(t,Y;m) = \sqrt{\frac{m}{t^2}\frac{1}{m-1}\sum_{k=1}^{m}\left(\left(Y\left(\frac{k}{m}t\right) - \frac{k}{m}Y(t)\right) - \left(Y\left(\frac{k-1}{m}t\right) - \frac{k-1}{m}Y(t)\right)\right)^2}.$$

It is straightforward to see that $g_b(\cdot;m) \in \mathscr{M}$. Other examples of STS include the standardized sum process and the standardized maximum interval process (Schruben 1983; Glynn and Iglehart 1990).

To establish the asymptotic validity of the sequential stopping procedure, we impose the following strong approximation assumption on the process $\{X(t) : t \ge 0\}$.

Assumption 1 There exist constants $\sigma \in (0,\infty)$ and $\gamma > 1/2$, and probability space supporting both $\{X(t) : t \ge 0\}$ and a standard Brownian motion $B = \{B(t) : t \ge 0\}$, such that

$$\int_0^t h(X(s))ds = \alpha t + \sigma B(t) + O(t^{1-\gamma}) \text{ almost surely (a.s.),}$$

i.e. $|\int_0^t h(X(s))ds - \alpha t - \sigma B(t)| \le Ct^{1-\gamma}$ where C is a finite-valued random variable.

Note that under Assumption 1, $\alpha(t) \Rightarrow \alpha$ and $\sqrt{t}(\alpha(t)-\alpha) \Rightarrow N(0,1)$ as $t \to \infty$, where $\Rightarrow$ denotes weak convergence or convergence in distribution. Moreover, by continuous mapping theorem, $\sqrt{n}g(nt,Y) \Rightarrow \sigma g(t,B)$ and $\sqrt{t}g(t,Y) \Rightarrow \sigma g(1,B)$ as $t \to \infty$.

Remark 1 It has been noted in Dong and Glynn (2019) that Harris recurrent chains with appropriate moment conditions satisfy Assumption 1 (see Theorem 3 to 5 in Dong and Glynn (2019)); see also Csaki and Csorgo (1995) and Merlevede and Rio (2015). Merlevede and Rio (2012) obtained strong approximation results (Assumption 1) for partial sums of strictly stationary strongly mixing sequences of random variables. See also Berkes et al. (2014) for strong approximation results for stationary processes that are functions of independent and identically distributed innovations.

2.1 Sequential Stopping for STS

For the sequential stopping procedure to achieve an absolute precision level, ε, we consider a sequence of stopping times indexed by ε as follows. Let $\beta \in (1/2,\gamma)$ be a fixed constant, and set

$$\tau_g(\varepsilon,u) := \inf\left\{t > \varepsilon^{-1/\beta} : ug(t,Y) < \varepsilon\right\},$$

where $t > \varepsilon^{-1/\beta}$ is required to avoid early stopping.

Our goal is to characterize the limiting distribution of $(\alpha(\tau_g(\varepsilon,u))-\alpha)/g(\tau_g(\varepsilon,u),Y)$ as $\varepsilon \to 0$. This will provide us with the appropriate scaling parameter, $u_{\delta/2}$, such that $\lim_{\varepsilon\to 0}\mathbb{P}(|\alpha(\tau_g(\varepsilon,u_{\delta/2}))-\alpha| \le \varepsilon) = 1-\delta$.

We start by analyzing the asymptotic behavior of $\tau_g(\varepsilon,u)$. Define

$$T_g(a) := \inf\{t > 0 : ag(t,B) < 1\}.$$

for Brownian motion $B = \{B(t) : t \ge 0\}$.

Proposition 1 Under Assumption 1, and for $g \in \mathscr{M}$, $\varepsilon^2\tau_g(\varepsilon,u) \Rightarrow T_g(u\sigma)$ as $\varepsilon \to 0$.

Proof of Proposition 1. Let $r(t) = g(t,Y) - \sigma g(t,B)$ where Y and B are constructed under a suitably enlarged probability space as in Assumption 1. Then under Assumption 1, $r(t) = O(t^{-\gamma})$. We first notice that

$$\begin{aligned}
\varepsilon^2\tau_g(\varepsilon,u) &= \varepsilon^2\inf\left\{t > \varepsilon^{-1/\beta} : ug(t,Y) < \varepsilon\right\} \\
&= \inf\left\{\varepsilon^2 t > \varepsilon^{-1/\beta+2} : \frac{1}{\varepsilon}\left(u\sigma g(t,B) + r(t)\right) < 1\right\} \\
&= \inf\left\{t > \varepsilon^{-1/\beta+2} : u\sigma\frac{1}{\varepsilon}g\left(\frac{t}{\varepsilon^2},B\right) + \frac{1}{\varepsilon}r\left(\frac{t}{\varepsilon^2}\right) < 1\right\} \\
&= \inf\left\{t > \varepsilon^{-1/\beta+2} : u\sigma g\left(t,\frac{1}{\varepsilon}B_{\varepsilon^{-2}}\right) + \frac{1}{\varepsilon}r\left(\frac{t}{\varepsilon^2}\right) < 1\right\} \text{ by Property (iii) and (iv)} \\
&\overset{D}{=} \inf\left\{t > \varepsilon^{-1/\beta+2} : u\sigma g(t,B) + \frac{1}{\varepsilon}r\left(\frac{t}{\varepsilon^2}\right) < 1\right\} \text{ where } \overset{D}{=} \text{ denotes equal in distribution} \\
&\Rightarrow \inf\{t > 0 : u\sigma g(t,B) < 1\} \text{ as } \varepsilon \downarrow 0,
\end{aligned}$$

The equality in distribution is due to the fact that $(\frac{1}{c}B(c^2t) : t \ge 0) \overset{D}{=} (B(t) : t \ge 0)$. The last convergence follows as for $t > \varepsilon^{-1/\beta+2}$, $\frac{1}{\varepsilon}r\left(\frac{t}{\varepsilon^2}\right) \to 0$ a.s. as $\varepsilon \downarrow 0$. □

Proposition 1 indicates that $\tau_g(\varepsilon,u)$ scales as ε^{-2}. We also notice that $T_g(u\sigma) \overset{D}{=} u^2\sigma^2T_g(1)$, as

$$\begin{aligned}
\frac{T_g(a)}{a^2} = \inf\left\{t > 0 : ag(a^2t,B) < 1\right\} &= \inf\left\{t > 0 : g\left(t,\frac{1}{a}B_{a^2}\right) < 1\right\} \text{ by (iii)} \\
&\overset{D}{=} \inf\{t > 0 : g(t,B) < 1\} = T_g(1).
\end{aligned}$$

The following theorem is the main result of the paper. It establishes the asymptotic validity of the sequential stopping procedure by characterizing the limiting distribution of the STS estimator at stopping times as the half-width of the CI (error size), ε, goes to zero.

Theorem 1 Under Assumption 1, and for $g \in \mathscr{M}$, if $T_g(1) > 0$ with probability 1, then

$$\frac{\alpha(\tau_g(\varepsilon,u)) - \alpha}{g(\tau_g(\varepsilon,u),Y)} \Rightarrow \frac{N(0,1)}{\sqrt{T_g(1)}} \text{ as } \varepsilon \downarrow 0,$$

where $N(0,1)$ is independent of $T_g(1)$. Furthermore, if we set $u_{\delta/2}$ as the $\delta/2$-th upper quantile of $N(0,1)/\sqrt{T_g(1)}$, then

$$\mathbb{P}(\alpha \in [\alpha(\tau_g(\varepsilon,u_{\delta/2})) - \varepsilon, \alpha(\tau_g(\varepsilon,u_{\delta/2})) + \varepsilon]) \to 1-\delta \text{ as } \varepsilon \downarrow 0.$$

Before we prove Theorem 1, we comment that the condition that $T_g(1) > 0$ with probability 1 is very hard to check. Indeed, we currently can not check this condition analytically. We leave this as a future task. In Section 3 and 4, we talk about special examples of g. There, we conduct extensive simulation experiments to check whether $T_g(1) > 0$, and summarize the observations as two conjectures, see Conjecture 1 and 2

Proof of Theorem 1. Let $\bar{Y}_n(t) := \frac{1}{n}Y(nt)$. We first notice that

$$\frac{Y(\tau_g(\varepsilon,u))/\tau_g(\varepsilon,u) - \alpha}{g(\tau_g(\varepsilon,u),Y)} = \frac{\bar{Y}_{\varepsilon^{-2}}(\varepsilon^2\tau_g(\varepsilon,u)) - \alpha\varepsilon^2\tau_g(\varepsilon,u)}{\varepsilon^2\tau_g(\varepsilon,u)g(\varepsilon^2\tau_g(\varepsilon,u),\bar{Y}_{\varepsilon^{-2}})} = \frac{\left(\bar{Y}_{\varepsilon^{-2}}(\varepsilon^2\tau_g(\varepsilon,u)) - \alpha\varepsilon^2\tau_g(\varepsilon,u)\right)/\varepsilon}{\varepsilon^2\tau_g(\varepsilon,u)g\left(\varepsilon^2\tau_g(\varepsilon,u),(\bar{Y}_{\varepsilon^{-2}} - \alpha I)/\varepsilon\right)}.$$

We also note that under Assumption 1, $\frac{1}{\varepsilon}(\bar{Y}_{\varepsilon^{-2}}(t) - \alpha t) \Rightarrow \sigma B(t)$ in $(D(0,\infty),J_1)$ as $t \to \infty$. Let $h(t,x) = x(t)/(tg(t,x))$ for $g(t,x) \neq 0$. As $\frac{t}{\varepsilon}(\bar{Y}_\varepsilon - \alpha) \Rightarrow \sigma B$ and $P(\sigma B \in D(h)) = 0$, by continuous mapping theorem, we have

$$\frac{\frac{1}{\varepsilon}(\bar{Y}_{\varepsilon^{-2}}(t) - \alpha t)}{tg\left(t,(\bar{Y}_{\varepsilon^{-2}} - \alpha I)/\varepsilon\right)} \Rightarrow \frac{B(t)}{tg(t,B)} \text{ in } (D(0,\infty),J_1) \text{ as } t \to \infty,$$

where $(D(0,\infty),J_1)$ is the space of right continuous functions with left limit endowed with Skorokhod J_1 topology. Then, applying continuous mapping theorem (the composition map is continuous (Whitt 1980)), we have

$$\frac{\left(\bar{Y}_{\varepsilon^{-2}}(\varepsilon^2\tau_g(\varepsilon,u)) - \alpha\varepsilon^2\tau_g(\varepsilon,u)\right)/\varepsilon}{\varepsilon^2\tau_g(\varepsilon,u)g\left(\varepsilon^2\tau_g(\varepsilon,u),(\bar{Y}_{\varepsilon^{-2}} - \alpha I)/\varepsilon\right)} \Rightarrow \frac{B(T_g(u\sigma))}{T_g(u\sigma)g(T_g(u\sigma),B)} = \frac{B(T_g(u\sigma))}{T_g(u\sigma)/(u\sigma)} = \frac{B(T_g(u\sigma))/\sqrt{T_g(u\sigma)}}{\sqrt{T_g(u\sigma)/(u\sigma)^2}}.$$

The first equality follows as $g(T_g(u\sigma),B) = (u\sigma)^{-1}$.

We next show that $B(T_g(a))/\sqrt{T_g(a)}$ is independent of $T_g(a)$, and $B(T_g(a))/\sqrt{T_g(a)} \stackrel{D}{=} N(0,1)$. We first define a sequence of discretized approximations of $T_g(a)$. Let $\Delta_n = 10^{-n}$ and define $T_n(a) := \min\{k\Delta_n : k\Delta_n > T_g(a)\}$. Then $T_n(a) \downarrow T_g(a)$ almost surely as $n \to \infty$ (Revuz and Yor 1999 page 45). For $T_n(a)$, we have for any $z \in \mathbb{R}$,

$$\begin{aligned}
& \mathbb{P}\left(B(T_n(a))/\sqrt{T_n(a)} \leq z, T_n(a) = k\Delta_n\right) \\
= \;& \mathbb{P}\left(B(T_n(a))/\sqrt{T_n(a)} \leq z | T_n(a) = k\Delta_n\right) P(T_n(a) = k\Delta_n) \\
= \;& \mathbb{P}\left(B(k\Delta_n)/\sqrt{k\Delta_n} \leq z | \inf_{0<t<(k-1)\Delta_n} \sigma g(t,B) \geq 1, \inf_{(k-1)\Delta_n \leq t < k\Delta_n} \sigma g(t,B) < 1\right) \mathbb{P}(T_n(a) = k\Delta_n). \quad (1)
\end{aligned}$$

As $B(k\Delta_n)$ is independent of $g(t,B)$ for $t \le k\Delta_n$,

$$(1) = \mathbb{P}\left(B(k\Delta_n)/\sqrt{k\Delta_n} \le z\right)\mathbb{P}(T_n(a) = k\Delta_n) = \mathbb{P}(N(0,1) \le z)\mathbb{P}(T_n(a) = k\Delta_n).$$

Thus, $B(T_n(a))/\sqrt{T_n(a)} \stackrel{D}{=} N(0,1)$ and is independent of $T_g^n(a)$. As $B(t)/\sqrt{t}$ is continuous in t,

$$\left(T_n(a), B(T_n(a))/\sqrt{T_n(a)}\right) \Rightarrow \left(T_g(a), B(T_g(a))/\sqrt{T_g(a)}\right) \text{ as } n \to \infty.$$

Then $B(T_g(a))/\sqrt{T_g(a)}$ is distributed as $N(0,1)$, and for any $z \in \mathbb{R}$, $t \in \mathbb{R}^+$,

$$\begin{aligned}\mathbb{P}\left(\frac{B(T_g(a))}{\sqrt{T_g(a)}} \le z, T_g(a) \le t\right) &= \lim_{n\to\infty} \mathbb{P}\left(\frac{B(T_n(a))}{\sqrt{T_n(a)}} \le z, T_n(a) \le t\right) \\ &= \lim_{n\to\infty} \mathbb{P}(N(0,1) \le z)\mathbb{P}(T_n(a) \le t) = \mathbb{P}(N(0,1) \le z)\mathbb{P}(T_g(a) \le t).\end{aligned}$$

Therefore, $B(T_g(a))/\sqrt{T_g(a)}$ is independent of $T_g(a)$.

Lastly, as $T_g(a)/a^2 \stackrel{D}{=} T_g(1)$,

$$\frac{B(T_g(u\sigma))/\sqrt{T_g(u\sigma)}}{\sqrt{T_g(u\sigma)/(u\sigma)^2}} \stackrel{D}{=} \frac{N(0,1)}{\sqrt{T_g(1)}}.$$

Now if we select $u_{\delta/2}$ such that $P(|N(0,1)/\sqrt{T_g(1)}| \le u_{\delta/2}) = 1-\delta$, then

$$\begin{aligned}\mathbb{P}\left(\left|\alpha(\tau_g(\varepsilon, u_{\delta/2})) - \alpha\right| \le \varepsilon\right) = \mathbb{P}\left(\left|\frac{\alpha(\tau_g(\varepsilon, u_{\delta/2})) - \alpha}{\varepsilon/u_{\delta/2}}\right| < u_{\delta/2}\right) &\to \mathbb{P}\left(\left|\frac{\sigma B(T_g(\sigma))}{T_g(\sigma)}\right| < u_{\delta/2}\right) \text{ as } \varepsilon \downarrow 0 \\ &= \mathbb{P}\left(\left|\frac{N(0,1)}{\sqrt{T_g(1)}}\right| < u_{\delta/2}\right) = 1-\delta.\end{aligned}$$

□

2.2 Relative Error

In the previous subsection, we analyzed the sequential stopping rule to achieve an absolute precision ε. There is another commonly used precision criteria called relative error (Asmussen and Glynn 2007 page 158), where we want to achieve a precision level relative to the size of $|\alpha|$. For the case of relative error, we define a new sequence of stopping times, indexed by ε, as $\tilde{\tau}_g(\varepsilon, u) := \inf\{t > \varepsilon^{-1/\beta} : ug(t,Y) < \varepsilon|\alpha(t)|\}$.

Following the same line of analysis as in the absolute precision case, we can show that if we choose the scaling parameter $u_{\delta/2}$ as the $\delta/2$-th upper quantile of $N(0,1)/\sqrt{T_g(1)}$, then

$$\mathbb{P}(\alpha \in [\alpha(\tilde{\tau}_g(\varepsilon, u_{\delta/2}))(1-\varepsilon), \alpha(\tilde{\tau}_g(\varepsilon, u_{\delta/2}))(1+\varepsilon)]) \to 1-\delta \text{ as } \varepsilon \downarrow 0.$$

3 THE BATCH MEANS METHOD

The batch means method with a fixed number of batches is the most commonly used class of STS methods. In this section, we analyze the sequential stopping procedure for batch means with m batches.

One important assumption in Theorem 1 is the assumption that $T_g(1) > 0$ a.s. This is very hard to verify in practice, and we in general cannot rule out the possibility that $T_g(1) = 0$. In fact, for batch means, when $m = 2$, $T_g(1) = 0$ as shown in the following lemma.

Lemma 1 When $m = 2$, $T_g(1) = 0$ with probability 1.

Proof of Lemma 1. By the reversibility of Brownian motion,

$$\begin{aligned}
T_g(1) &\stackrel{D}{=} \inf\left\{t > 0 : \sqrt{\sum_{k=1}^{m}\left(B\left(\frac{k}{m}t\right) - B\left(\frac{k-1}{m}t\right) - \frac{1}{m}B(t)\right)^2} < \sqrt{\frac{(m-1)}{m}t}\right\} \\
&\stackrel{D}{=} \inf\left\{t > 0 : \sqrt{\sum_{k=1}^{m}\left(\frac{k}{m}B\left(\frac{m}{kt}\right) - \frac{k-1}{m}B\left(\frac{m}{(k-1)t}\right) - \frac{1}{m}B\left(\frac{1}{t}\right)\right)^2} < \sqrt{\frac{(m-1)}{m}}\right\} \\
&\stackrel{D}{=} \frac{1}{\sup\left\{t > 0 : \sqrt{\sum_{k=1}^{m}\left(\frac{k}{m}B\left(\frac{m}{k}t\right) - \frac{k-1}{m}B\left(\frac{m}{k-1}t\right) - \frac{1}{m}B(t)\right)^2} < \sqrt{\frac{(m-1)}{m}}\right\}}.
\end{aligned}$$

When $m = 2$,

$$\sum_{k=1}^{m}\left(\frac{k}{m}B\left(\frac{m}{k}t\right) - \frac{k-1}{m}B\left(\frac{m}{k-1}t\right) - \frac{1}{m}B(t)\right)^2 = \frac{1}{2}\left(B(2t) - B(t)\right)^2.$$

Let $t_n = 2^n$, and $H_n = \{B(2t_n) - B(t_n) > 0\}$, $L_n = \{B(2t_n) - B(t_n) < 0\}$. As $\{H_n : n \geq 0\}$ is a sequence of independent events and $\sum_{n=0}^{\infty}\mathbb{P}(H_n) = \sum_{n=0}^{\infty}\mathbb{P}\left(B(2t_n) - B(t_n) > 0\right) = \infty$, the event H_n occurs infinitely often (i.o.) by Borel-Cantelli Lemma. Similarly, we can show that L_n occurs i.o.. The intermediate value theorem then ensures that there are infinitely many times at which $B(2t) - B(t) = 0$. Therefore, $T_g(1) = 0$. □

For other values of m, we conducted an extensive numerical analysis (see Figure 1), and put forth the following conjecture.

Conjecture 1 For batch means, when $m = 3$, $T_g(1) = 0$ a.s.; When $m \geq 4$, $T_g(1) > 0$ a.s.

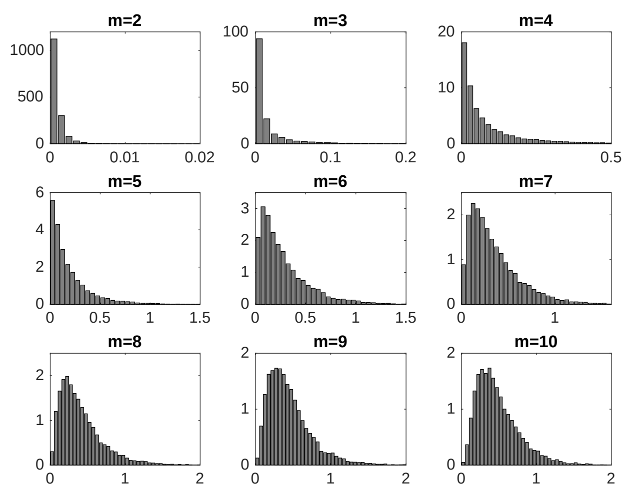


Figure 1: Comparison of the distribution of $T_g(1)$ for different values of m. (The normalized histogram are constructed based on 10^4 i.i.d. samples with a step-size 10^{-5}.)

Remark 2 When conducting the numerical experiments, we cannot check $T_g(1)$ continuously. Instead, we use the following discretization scheme. For $\Delta_n = 10^{-n}$, $n \in \mathbb{Z}^+$, we set $\tilde{T}_n = \min\{k\Delta_n : g(k\Delta_n, B) < 1\}$ We notice that $\tilde{T}_n$ is monotonically decreasing in n and $\tilde{T}_n \to T_g(1)$ as $n \to \infty$ path by path (Revuz and Yor 1999 page 46). We refer to Δ_n as the step-size in our numerical experiments.

Under the conjecture, we would require $m \geq 4$ when applying sequential stopping to the batch means method. Note that for the batch means method with a fixed sample size t, we only require $m \geq 2$. We believe that the extra constraint on m for the sequential stopping procedure has to do with the transience of the limiting functional of Brownian motion in dimensions 3 or greater (see Dong and Glynn (2019) for similar results for the sectioning method).

The actual sequential stopping algorithm goes as follows.

Algorithm 1

Input: the number of batches m, the confidence level $1-\delta$, the error bound ε, the scaling parameter u (See Table 1 for some of the commonly used scaling parameters), the step size Δ, and β.

Output: a $100(1-\delta)\%$ CI with half width ε.

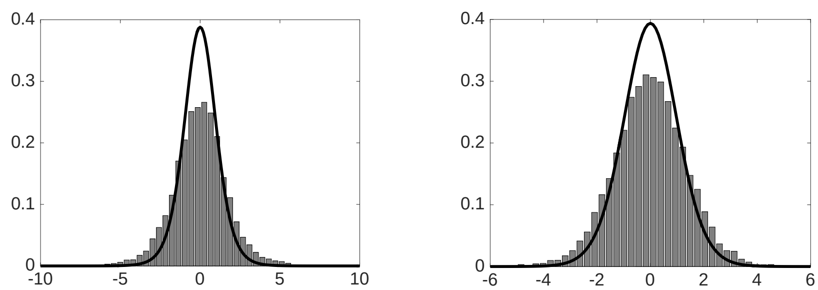
(i) Sample $X(t)$ for $0 \leq t \leq \varepsilon^{-1/\beta}$. Initialize $T = \varepsilon^{-1/\beta} + \Delta$.

(ii) Sample $X(t)$ between $T-\Delta$ and T. Calculate $\alpha(T)$ and $S_m(T)$.

(iii) If $u \times S_m(T) \geq \varepsilon$, set $T = T + \Delta$, go back to step (ii); otherwise, output $[\alpha(T)-\varepsilon, \alpha(T)+\varepsilon]$.

The Distribution of $Z/\sqrt{T_g(1)}$

As discussed in Section 1, for fixed sample-size procedure, we would use t_{m-1} to calibrate the scaling parameter for the CI when applying batch means with m batches. To understand the failure of the naive implementation of the sequential stopping procedure discussed in Section 1, we shall first compare the distribution of $Z/\sqrt{T_g(1)}$ to t_{m-1}. Figure 2 compares the normalized histogram of $Z/\sqrt{T_g(1)}$ and the pdf of t_{m-1} for $m = 10$ and 20. We observe that $Z/\sqrt{T_g(1)}$ has a heavier tail than the corresponding t_{m-1}. We also notice that, as m increase (form 10 to 20), the tail of $Z/\sqrt{T_g(1)}$ becomes lighter.



(a) $m = 10$ (b) $m = 20$

Figure 2: Comparison of the two limiting distributions for different values of m. (The normalized histograms are based on 10^4 i.i.d. samples with a step-size 10^{-5}.)

Table 1: Sample percentile of $Z/\sqrt{T_g(1)}$ (10^6 i.i.d. samples with step size $m \times 10^{-5}$).

	0.9	0.925	0.95	0.975	0.995
$m=5$	4.351 ± 0.005	5.198 ± 0.006	6.323 ± 0.006	8.897 ± 0.007	15.996 ± 0.062
$m=10$	2.048 ± 0.005	2.342 ± 0.005	2.748 ± 0.006	3.447 ± 0.007	5.070 ± 0.025
$m=15$	1.772 ± 0.004	2.009 ± 0.005	2.327 ± 0.005	2.850 ± 0.006	3.998 ± 0.021
$m=20$	1.662 ± 0.004	1.882 ± 0.005	2.165 ± 0.005	2.623 ± 0.005	3.603 ± 0.019
$m=25$	1.598 ± 0.004	1.807 ± 0.004	2.080 ± 0.004	2.509 ± 0.005	3.395 ± 0.016
$m=30$	1.556 ± 0.004	1.757 ± 0.004	2.021 ± 0.004	2.436 ± 0.005	3.278 ± 0.014

Table 1 lists the some useful sample quantiles of $Z/\sqrt{T_g(1)}$ (with its corresponding 95% CI) for the batch means method with m batches.

4 COMPARISON AMONG DIFFERENT STS METHODS

The batch means (STSb) method is the mostly commonly used STS method. Other popular STS methods include the standardized sum process (STSs) and the standardized maximum interval process (STSm) (see Glynn and Iglehart (1990) for details).

For STSs with m batches, let

$$I_k(t,Y) := \int_0^t \left\{ \left(Y\left(\frac{(k-1)t+s}{m}\right) - Y\left(\frac{(k-1)t}{m}\right)\right) - \frac{s}{t}\left(Y\left(\frac{kt}{m}\right) - Y\left(\frac{(k-1)t}{m}\right)\right)\right\} ds,$$

for $k = 1,2,\ldots,m$, and

$$g_s(t,Y;m) := \sqrt{12\sum_{k=1}^{m}\left(\frac{1}{t^2}I_k(t,Y)\right)^2}.$$

For STSm with m batches, let

$$M_k(t,Y) = \max\left\{ \left(Y\left(\frac{(k-1)t+s}{m}\right) - Y\left(\frac{(k-1)t}{m}\right)\right) - \frac{s}{t}\left(Y\left(\frac{kt}{m}\right) - Y\left(\frac{(k-1)t}{m}\right)\right) : 0 \le s \le t\right\},$$

$$\gamma_k(t,Y) = \inf\left\{ s \ge 0 : \left(Y\left(\frac{(k-1)t+s}{m}\right) - Y\left(\frac{(k-1)t}{m}\right)\right) - \frac{s}{t}\left(Y\left(\frac{kt}{m}\right) - Y\left(\frac{(i-1)t}{m}\right)\right) = M_k(t,Y)\right\},$$

for $k = 1,2,\ldots m$, and

$$g_m(t,Y;m) = \sqrt{\frac{1}{3}\sum_{k=1}^{m}\frac{M_k(t,Y)^2}{\gamma_k(t,Y)(t-\gamma_k(t,Y))}}.$$

For these two cases, we conduct extensive numerical analysis on $T_g(1)$ and put forth the following conjecture.

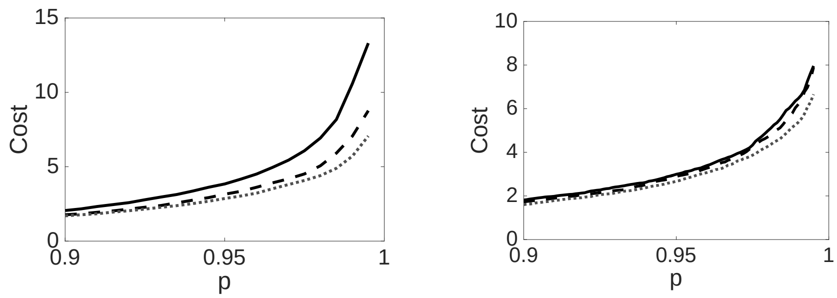
Conjecture 2 For STSs, when $m \ge 3$, $T_g(1) > 0$ a.s. For STSm, when $m \ge 2$. $T_g(1) > 0$ a.s.

The three methods have different requirements on m because their corresponding $\sqrt{t}g(t,Y)$ converges to functions of chi-squared distributions with different degrees of freedom. Table 2 provides $u_{0.025}$ (to construct 95% CI) for the three methods with different values of m.

Table 2: Sample percentile of $Z/\sqrt{T_g(1)}$ (10^4 i.i.d. samples with step size $m \times 10^{-5}$ for $m = 10,20,30$).

	STSb	STSs	STSm
$m = 10$	3.45 ± 0.01	3.47 ± 0.09	2.60 ± 0.05
$m = 20$	2.62 ± 0.01	2.77 ± 0.09	2.31 ± 0.05
$m = 30$	2.44 ± 0.01	2.53 ± 0.03	2.25 ± 0.04

From the analysis in Section 2, we have $\varepsilon^2\tau_g(\varepsilon,u) \Rightarrow u^2\sigma^2 T_g(1)$ as $\varepsilon \downarrow 0$. In Figure 3, we compare $C_g := u^2_{\delta/2}\mathbb{E}[T_g(1)]$ for the three different methods (STSb, STSs, and STSm). The quantity C_g roughly measures the required number of samples for the procedures. We observe that for fixed value of m and δ ($p = 1-\delta$), STSb requires more samples on average than STSs, which requires more samples than STSm. As m increases, the three costs are getting closer (Figure 3 (a) v.s. 3 (b)). We also comment that in actual implementations, STSs and STSm takes much more time and storage space to calculate $g(t,Y)$ at each iteration than STSb.



(a) $m = 10$ (b) $m = 20$

Figure 3: Comparison of $u_{\delta/2}^2 E[T_g(1)]$ among STSb (solid), STSs (dashed) and STSm (dotted) for different values of $p = 1-\delta$. (Estimated values are based on 10^4 i.i.d. samples.)

5 SIMULATION EXPERIMENTS

In this section, we demonstrate the performance of our sequential stopping procedures through some simulation experiments.

We consider an M/M/1 queue, which has a Poisson arrival process with rate λ and independent identically distributed (i.i.d.) exponential service times with rate μ. Let $X(t)$ denote the number of customers in the system at time t. We are interested in estimating $\alpha = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)dt = \rho/(1-\rho)$ (the steady-state average number of people in the system), where $\rho = \lambda/\mu$, is the traffic intensity. All estimations for expectations and coverage probabilities reported in this section are based on 10^3 independent replications of the sequential stopping algorithm. We also report the corresponding 95% CI for the estimates.

5.1 Batch Means Method with Absolute Precision

We first fix $\lambda = 0.8$, $\mu = 1$ and apply the sequential batch means method to different values of m and ε to construct the 95% CIs. The results are summarized in Table 3. When $m = 10$, we set the scaling parameter $u = 3.45$, which is the 0.975 quantile of $Z/\sqrt{T_g(1)}$ (see Table 1), whereas the 0.975-quantile of t_9 is 2.26. When $m = 20$, we set $u = 2.62$, whereas the 0.975-quantile of t_{19} is 2.09. We observe that the algorithm on average achieves the correct coverage probability (95%) for small values of ε ($\varepsilon \leq 0.1$) if we use the right scaling parameters (see the "Coverage probability" column). Plugging in the wrong scaling parameters suggested by the corresponding Student t-distribution will lead to under-coverage (see the "Coverage for t_{m-1}" column). For a fixed value of m, $\varepsilon^2\mathbb{E}[\tau_g(\varepsilon,u)]$ is of about the same value when ε is small enough. This confirms the results of Proposition 1. We also observed that $\varepsilon^2\mathbb{E}[\tau_g(\varepsilon,u)]$ is decreasing in the number of batches, m.

We also check for different traffic intensities, ρ. Table 4 summarizes the results with $m = 10$ and $\varepsilon = 0.1$ for the 95% CIs. We make two observations here. The first one is that the estimated coverage probabilities are around 95% for all values of ρ tested. The second observation is that as ρ increases, $\mathbb{E}[\tau_g(\varepsilon,u)]$ increases. This is expected, as Proposition 1 suggests that $\mathbb{E}[\tau_g(\varepsilon,u)]$ is increasing in σ.

5.2 Batch Means Method with Relative Error

In this section, we apply the sequential batch means method to construct 95% relative CIs. For $\lambda = 0.8$ and $\mu = 1$, Table 5 summarizes the results for different values of m and ε. Similar to Table 3, we observe that our algorithm achieves the correct coverage probability for small values of ε. Plugging in the wrong scaling parameter suggested by the corresponding Student t-distribution will lead to under-coverage.

Table 3: Performance of the sequential batch means procedure with different values of m and ε: $M/M/1$ queue ($\lambda = 0.8$, $\mu = 1$). $\Delta = 10$, $\beta = 2.9/4$.

	ε	Coverage probability	$\mathbb{E}[\tau_g(\varepsilon, u_{0.025})]$	$\varepsilon^2\mathbb{E}[\tau_g(\varepsilon, u_{0.025})]$	Coverage for t_{m-1}
$m = 10$	0.5	0.904 ± 0.018	$(3.96 \pm 0.17) \times 10^4$	9.90×10^3	0.715 ± 0.028
	0.1	0.961 ± 0.012	$(1.05 \pm 0.05) \times 10^6$	1.05×10^4	0.838 ± 0.023
	0.05	0.950 ± 0.014	$(4.06 \pm 0.15) \times 10^6$	1.02×10^4	0.852 ± 0.022
	0.01	0.952 ± 0.013	$(1.02 \pm 0.04) \times 10^8$	1.02×10^4	0.836 ± 0.023
$m = 20$	0.5	0.851 ± 0.022	$(2.73 \pm 0.11) \times 10^4$	6.83×10^3	0.705 ± 0.028
	0.1	0.953 ± 0.013	$(7.72 \pm 0.18) \times 10^5$	7.72×10^3	0.897 ± 0.019
	0.05	0.952 ± 0.013	$(3.14 \pm 0.07) \times 10^6$	7.85×10^3	0.878 ± 0.020
	0.01	0.949 ± 0.014	$(7.89 \pm 0.18) \times 10^7$	7.89×10^3	0.887 ± 0.020

Table 4: Performance of the sequential batch means procedure with $m = 10$ and different values of ρ: $M/M/1$ queue ($\mu = 1$, $\lambda = \rho$). $\Delta = 10$, $\beta = 2.9/4$.

	Coverage probability	$\mathbb{E}[\tau_g(\varepsilon, u_{0.025})]$	Coverage for t_{m-1}
$\rho = 0.85$	0.948 ± 0.014	$(3.73 \pm 0.13) \times 10^6$	0.842 ± 0.023
$\rho = 0.9$	0.949 ± 0.014	$(2.00 \pm 0.07) \times 10^7$	0.843 ± 0.023
$\rho = 0.95$	0.946 ± 0.014	$(3.44 \pm 0.13) \times 10^8$	0.818 ± 0.024

Table 5: Performance of the sequential batch means procedure with $m = 10$ and different values of ε (relative error): $M/M/1$ queue ($\lambda = 0.8$, $\mu = 1$). $\Delta = 10$, $\beta = 2.9/4$.

ε	Coverage probability	$E[\tau_g(\varepsilon, u_{0.025})]$	$\varepsilon^2 E[\tau_g(\varepsilon, u_{0.025})]$	Coverage for t_{m-1}
0.5	0.794 ± 0.025	$(1.39 \pm 0.11) \times 10^3$	3.48×10^2	0.587 ± 0.031
0.1	0.951 ± 0.013	$(7.07 \pm 0.25) \times 10^4$	7.07×10^2	0.847 ± 0.022
0.05	0.956 ± 0.013	$(2.73 \pm 0.09) \times 10^5$	6.83×10^2	0.836 ± 0.023
0.01	0.959 ± 0.012	$(6.54 \pm 0.23) \times 10^6$	6.54×10^2	0.828 ± 0.023

5.3 Comparison between Different STS Methods

We also compare the performance of the three STS methods: STSb, STSs, and STSm. We set $m = 10$ for all three methods. Our goal is to achieve a 95% confidence level. For STSb, we set the scaling parameter $u_{0.025} = 3.45$, for STSs, we set $u_{0.025} = 3.56$, for STSm, we set $u_{0.025} = 2.65$ (For STSs and STSm, we pick the upper confidence bound in Table 2 as there is more estimation error when simulating these quantiles). The results are summarized in Table 6. We observe that both STSs and STSm requires a smaller expected number of samples than STSb, with STSm being the smallest. However, in our actual implementations, calculating $g(t,Y)$ for STSs and STSm takes much more time than STSb.

Table 6: Performance of different sequential STS procedures with $m = 10$ and $\varepsilon = 0.1$: $M/M/1$ queue ($\lambda = 0.8$, $\mu = 1$)

Method	Coverage probability	$\mathbb{E}[\tau_g(\varepsilon, u_{0.025})]$
STSb	0.961 ± 0.012	$(1.05 \pm 0.05) \times 10^6$
STSs	0.947 ± 0.014	$(7.43 \pm 0.21) \times 10^5$
STSm	0.941 ± 0.015	$(5.95 \pm 0.12) \times 10^5$

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