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EXACT SIMULATION OF THE QUEUE-HAWKES PROCESS

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ABSTRACT

The Queue-Hawkes process is a generalization of the self-exciting Hawkes process in which the counting process is coupled with an infinite server queue. That is, when a service is completed the process intensity jumps downwards. Thus, the influence an arrival brings to the intensity is ephemeral, as it expires upon the entity's departure. In this poster we provide an exact simulation procedure for this new process via an update of methodology from the traditional Hawkes process literature and discuss the model in general.

1 MODEL DEFINTION

The Queue-Hawkes process is an ephemerally self-exciting process. This structure follows from a coupling of a self-exciting process and a queue: arrivals increase the likelihood of future arrivals, departures decrease it, and it decays between. This is modeled through down-jumps of size equal to the gap between the excitement and the baseline intensity, divided by the number in system. The formal definition of this process is given below in Definition 1, which we originally stated in a prior, submitted work.

Definition 1 (The Queue-Hawkes Process) Let $t \geq 0$ and let $\nu^* > 0$ be the baseline intensity, $\alpha > 0$ be the intensity jump size, $\beta \geq 0$ be the intensity decay rate, and $\mu \geq 0$ be the service rate. Then, define ν_t, N_t, and D_t such that:

i) N_t is an arrival process driven by the intensity ν_t,
ii) D_t is the associated departure process with i.i.d. Exp(μ) service for each entity,
iii) ν_t is governed by

$$\mathrm{d}\nu_t = \beta(\nu^* - \nu_t)\mathrm{d}t + \alpha \mathrm{d}N_t - \frac{\nu_t - \nu^*}{Q_t}\mathrm{d}D_t$$

where $Q_t = N_t - D_t$.

Then, we say that the intensity-queue pair (ν_t, Q_t) is a **Queue-Hawkes process** with baseline intensity ν^*, intensity jump size α, decay rate β, and rate of exponential service μ.

2 EXACT SIMULATION PROCEDURE

We now give an exact simulation procedure for the Queue-Hawkes process. Using the parameter $T > 0$ for the duration of the experiment and the Queue-Hawkes process parameters ν^*, α, β, and μ, this procedure is given in pseudocode as follows in Algorithm 1. This procedure is an adaptation of methodology given by Dassios and Zhao (2013) for simulating the traditional Hawkes process. In the manner of that work, this approach is based around finding the minimum of three potential inter-event times: S_1 for the time until a new arrival is generated by the process excitement, S_2 for the time until a new arrival is generated by the baseline intensity, and S_3 for the time until the next service completion. Because of the decay and the down-jumps, there may be some arrivals that do not cause further arrivals from their excitement. Hence, S_1 may be a defective random variable and thus the quantity D is used to mitigate this. These inter-event times are handled in Lines 5-18. In Lines 19-25 the newly generated event time is recorded as either an arrival or a service completion, and the intensity and the queue are updated accordingly.

Algorithm 1: Exact Simulation of the Queue-Hawkes Process

1 ***input*** : $\nu^* > 0$, $\mu > 0$, $\beta > 0$, $\alpha > 0$, $T > 0$, $Q_0 \geq 0$, $\nu_0 \geq \nu^*$
2 $t = 0$, $\mathscr{A} = \emptyset$, $\mathscr{S} = \emptyset$;
3 ***while*** $t < T$
4   U_1 , U_2 , $U_3 \sim \text{Uni}(0,1)$, $upJump = \text{true}$;
5   ***if*** $Q > 0$
6         $D = 1 + \frac{\beta \log(U_1)}{\nu - \nu^*}$, $S_2 = -\frac{\log(U_2)}{\nu^*}$, $S_3 = -\frac{log(U_3)}{\mu Q}$;
7   ***if*** $D > 0$
8             $S_1 = -\frac{\log(D)}{\beta}$, $S = \min\{S_1, S_2, S_3\}$;
9   ***else***
10             $S = \min\{S_2, S_3\}$;
11   ***end***
12   ***if*** $S \equiv S_3$
13             $upJump = \text{false}$;
14   ***end***
15   ***else***
16             $S = -\frac{\log(U_2)}{\nu^*}$;
17   ***end***
18     $t = t + S$;
19   ***if*** $t < T$
20   ***if*** $upJump$
21             $\mathscr{A} = \mathscr{A} \cup \{t\}$, $\nu = (\nu - \nu^*)e^{-\beta S} + \nu^* + \alpha$, $Q = Q + 1$;
22   ***else***
23             $\mathscr{S} = \mathscr{S} \cup \{t\}$, $\nu = (\nu - \nu^*)e^{-\beta S} + \nu^* - \frac{(\nu - \nu^*)e^{-\beta S}}{Q}$, $Q = Q - 1$;
24   ***end***
25   ***end***
26 ***end***
27 ***output*** : $\mathscr{A}$, $\mathscr{S}$

Algorithm 1 outputs a single replication of the Queue-Hawkes process on the interval $[0, T)$. Specifically, it gives a list of arrival times $\mathscr{A}$ and service times $\mathscr{S}$ for the sample path over that time interval. Using these quantities, one can directly calculate the queue, intensity, and counting processes across time. It is worth noting that if one can experiment with generalizations of this process by modifying the up- and down-jump sizes on Lines 21 and 23 of Algorithm 1.

REFERENCES

Dassios, A., and H. Zhao. 2013. "Exact Simulation of Hawkes Process with Exponentially Decaying Intensity". *Electronic Communications in Probability* 18:1–13.