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MULTIPLY REFLECTED VARIANCE ESTIMATORS FOR SIMULATION

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ABSTRACT

In a previous article, we studied a then-new class of standardized time series (STS) estimators for the asymptotic variance parameter of a stationary simulation output process. Those estimators invoke the well-known reflection principle of Brownian motion on the suitably standardized original output process to compute several "reflected" realizations of the STS, each of which is based on a single reflection point. We then calculated variance- and mean-squared-error-optimal linear combinations of the estimators formed from the singly reflected realizations. The current paper repeats the exercise except that we now examine the efficacy of employing multiple reflection points on each reflected realization of the STS. This scheme provides additional flexibility that can be exploited to produce estimators that are superior to their single-reflection-point predecessors with respect to mean-squared error. We illustrate the enhanced performance of the multiply reflected estimators via exact calculations and Monte Carlo experiments.

1 INTRODUCTION

An archetypal task in discrete-event computer simulation output analysis is that of estimating the mean μ of a stationary stochastic process, $X_1, X_2, \ldots$. The usual point estimator for μ is the sample mean based on n consecutive observations, $\bar{X}_n \equiv \sum_{j=1}^{n} X_j/n$. In order to make a statement on the precision of the sample mean, the experimenter must also provide a point estimator for $\mathrm{Var}(\bar{X}_n)$ or, almost equivalently, the *variance parameter*, $\sigma^2 \equiv \lim_{n\to\infty} n\mathrm{Var}(\bar{X}_n)$; such a variance estimator can then be used to construct confidence intervals (CIs) for μ and σ^2.

Over the years, a great deal of work has appeared on the topic of steady-state variance estimation for simulation output. The benchmark in the literature is the method of non-overlapping batch means (Schmeiser 1982), which divides the time series $\{X_1, X_2, \ldots, X_n\}$ into b adjacent batches of size m (where $n = bm$); the batch sample means are then treated as approximately independent and identically distributed (i.i.d.) normal random variables, so that we estimate σ^2 by the sample variance of the batch means multiplied by m. The method of overlapping batch means (Meketon and Schmeiser 1984) generalizes non-overlapping batch means by using all $n-m+1$ overlapping size-m batches; the resulting estimator for σ^2 has only 2/3 the variance of the non-overlapping batch means estimator. The theory of standardized time series (STS) (Schruben 1983) provides a rich class of estimators for σ^2, all of which incorporate some form of batching. Many STS estimators for σ^2 can be shown to converge as the batch size $m \to \infty$ to random variables related to the weighted area under a Brownian bridge; and some of these area estimators turn out to have low bias and variance. One can achieve further variance reductions by reusing the underlying data in various ways to produce multiple estimators for σ^2 which are then averaged; see, for instance, Foley and Goldsman (1999), Calvin and Nakayama (2006), Calvin (2007), Batur et al. (2009), Alexopoulos et al. (2010), and Meterelliyoz et al. (2012), among others.

One interesting idea involving data reuse stems from what is known as the Reflection Principle of Brownian motion (BM), which states that a reflected BM (RBM) is itself a BM (Karlin and Taylor 1975). Figure 1 illustrates the concept with four sample paths. In that figure, the original BM, $\{\mathscr{W}(t) : 0 \le t \le 1\}$, is depicted in black. We reflect this realization at time $t = 0.25$ about the horizontal line $y = \mathscr{W}(0.25)$, producing the blue reflected path. Then at time $t = 0.50$, we reflect the blue path to obtain the red path; and at time $t = 0.75$, a reflection of the red path yields the green path. There are obviously infinite possibilities.

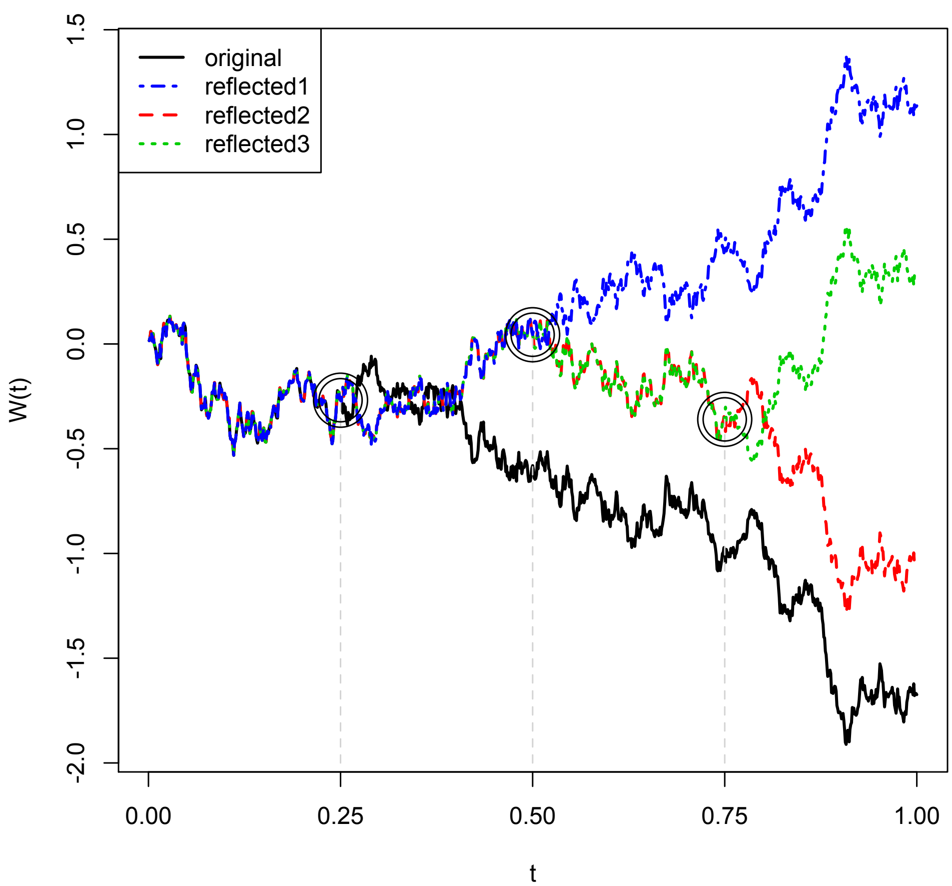

Figure 1: Several reflected sample paths of Brownian motion.

Meterelliyoz et al. (2015) studied STS estimators for σ^2 based on reflected realizations of Brownian motion, each of which used a single reflection at a different point t. The authors then proposed linear combinations of the singly reflected estimators having minimum variance and mean squared error (MSE). In the current paper, we propose and evaluate a natural generalization of the singly reflected estimators from Meterelliyoz et al. (2015). This generalization is motivated by a return to Figure 1, where we see that the blue path is the result of one reflection, the red path of two reflections, and the green path of three reflections. Indeed, our goal herein is to take advantage of estimators resulting from multiple reflections — not just one as before; and so the current paper repeats the exercise from Meterelliyoz et al. (2015) except that we now examine the efficacy of employing multiple reflection points on any realization of the STS. This scheme provides additional flexibility that can be exploited to produce variance estimators with smaller MSE than their single-reflection-point predecessors.

The current paper is organized as follows. In §2, we provide background material on STS and previous work on reflected variance estimators. §3 introduces the new multiply-reflected estimators. §4 gives analytical results on the expected values and variances of the estimators, along with a short Monte Carlo study demonstrating their performance. §5 wraps up the paper with conclusions and suggestions for future work in the area.

2 BACKGROUND

This section comprises background material that is intended to keep the paper self-contained. We begin in §2.1 with a primer on classic STS basics, and then §2.2 reviews the analogous results from Meterelliyoz et al. (2015) for reflected STS estimators.

2.1 Basics for STS Area Estimators

This subsection gives assumptions, definitions, and results for the baseline standardized time series and the resulting area estimator for σ^2.

Assumptions A:

1. $\{X_j\}$ is stationary with mean μ and covariance function $R_j \equiv \mathrm{Cov}(X_1, X_{1+j})$, $j = 0, \pm 1, \pm 2, \ldots$, with $|R_j| = O(\delta^j)$, for some $\delta \in (0,1)$.
2. Let $S_i \equiv \sum_{j=1}^{i} X_j$, $i = 1, 2, \ldots, n$. For $\sigma^2 > 0$, $n = 1, 2, \ldots$, and $t \in [0,1]$, define

$$Y_n(t) \equiv \frac{\lfloor nt \rfloor (\bar{X}_{\lfloor nt \rfloor} - \mu)}{\sigma\sqrt{n}} = \frac{S_{\lfloor nt \rfloor} - \lfloor nt \rfloor \mu}{\sigma\sqrt{n}}, \tag{1}$$

   where $\lfloor \cdot \rfloor$ is the floor function. We assume $Y_n(\cdot) \underset{n\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} \mathscr{W}(\cdot)$, where $\underset{n\to\infty}{\overset{\mathscr{D}}{\longrightarrow}}$ denotes weak convergence (as $n \to \infty$) (Billingsley 1968).
3. The function $f(t)$ is twice differentiable and bounded on $[0,1]$ with $\mathrm{Var}\left[\int_0^1 f(t)\mathscr{B}(t)\,\mathrm{d}t\right] = 1$, where $\mathscr{B}(t) \equiv \mathscr{W}(t) - t\mathscr{W}(1)$ is a standard Brownian bridge process on $[0,1]$ that is independent of $\mathscr{W}(1)$. Let $F(t) \equiv \int_0^t f(s)\,\mathrm{d}s$, $\bar{F}(t) \equiv \int_0^t F(s)\,\mathrm{d}s$, $F \equiv F(1)$, and $\bar{F} \equiv \bar{F}(1)$.

Under Assumption A.1, Aktaran-Kalaycı et al. (2007) show that

$$\sum_{i=m}^{\infty} i^{\ell}|R_i| = O(m^{\ell}\delta^m) \quad \text{and} \quad \sum_{i=1}^{m} i^{\ell} R_i = \frac{\gamma_\ell}{2} + O(m^{\ell}\delta^m) \quad \text{for } \ell = 0, 1, 2, \ldots,$$

where $\gamma_\ell \equiv 2\sum_{i=1}^{\infty} i^{\ell} R_i$, $\ell = 0, 1, 2, \ldots$. Moreover, Assumption A.2 can be used to show that the *standardized time series*, $T_n(t)$, of the sample $X_1, \ldots, X_n$ converges weakly to $\mathscr{B}(t)$, i.e.,

$$T_n(t) \equiv \frac{\lfloor nt \rfloor (\bar{X}_{\lfloor nt \rfloor} - \bar{X}_n)}{\sigma\sqrt{n}} \underset{n\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} \mathscr{B}(t), \quad t \in [0,1]$$

(Glynn and Iglehart 1990, Schruben 1983). Under Assumption A.3, the *weighted area estimator* for σ^2, $\mathscr{A}(f;n)$, and its limiting functional, $\mathscr{A}(f)$, are

$$\mathscr{A}(f;n) \;\equiv\; \left[\frac{1}{n}\sum_{j=1}^{n} f\big(\tfrac{j}{n}\big)\sigma T_n\big(\tfrac{j}{n}\big)\right]^2 \;\underset{n\to\infty}{\overset{\mathscr{D}}{\longrightarrow}}\; \mathscr{A}(f) \;\equiv\; \left[\int_0^1 f(t)\sigma\mathscr{B}(t)\,\mathrm{d}t\right]^2 \sim \sigma^2\chi_1^2,$$

where χ_ν^2 denotes a chi-squared random variable with ν degrees of freedom. Further,

$$\mathrm{E}[\mathscr{A}(f;n)] \;=\; \sigma^2 - \frac{[(F-\bar{F})^2+\bar{F}^2]\gamma_1}{2n} + O(1/n^2);$$

and if $\{\mathscr{A}^2(f;n) : n = 1,2,\ldots\}$ is uniformly integrable (Billingsley 1968), then $\lim_{n\to\infty}\mathrm{Var}[\mathscr{A}(f;n)] = \mathrm{Var}[\mathscr{A}(f)] = 2\sigma^4$ (Goldsman et al. 1990).

2.2 Reflected Area Estimators

For any point $c \in [0,1]$, the Reflection Principle establishes that the process

$$\mathscr{W}_c(t) \;\equiv\; \begin{cases} \mathscr{W}(t), & \text{if } 0 \le t < c \\ 2\mathscr{W}(c) - \mathscr{W}(t), & \text{if } c \le t \le 1 \end{cases}$$

is also a BM (see Figure 1). In order to obtain reflected versions of the STS area estimator based on $\mathscr{W}_c(t)$, Meterelliyoz et al. (2015) begin by defining a reflected version of the original sample path, namely,

$$X_{c,j} \;\equiv\; \begin{cases} X_j, & \text{if } 1 \le j \le \lfloor nc \rfloor \\ -X_j, & \text{if } \lfloor nc \rfloor + 1 \le j \le n. \end{cases}$$

Continuing the analogy, let $S_{c,k} \equiv \sum_{j=1}^{k} X_{c,j}$ for $k = 1,2,\ldots,n$, so that

$$S_{c,\lfloor nt \rfloor} \;=\; \begin{cases} S_{\lfloor nt \rfloor}, & \text{if } 0 \le t < c \\ 2S_{\lfloor nc \rfloor} - S_{\lfloor nt \rfloor}, & \text{if } c \le t \le 1. \end{cases}$$

As in Meterelliyoz et al. (2015), we temporarily assume that the mean $\mu = 0$. Then we define $Y_{c,n}(t) \equiv S_{c,\lfloor nt \rfloor}/(\sigma\sqrt{n})$, and note that, by Equation (1) with $\mu = 0$,

$$Y_{c,n}(t) \underset{n\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} \mathscr{W}_c(t) \quad \text{for } 0 \le t \le 1.$$

For $t \in [0,1]$, the corresponding reflected STS with reflection point c is

$$T_{c,n}(t) \;\equiv\; \frac{\lfloor nt \rfloor(\bar{X}_{c,\lfloor nt \rfloor} - \bar{X}_{c,n})}{\sigma\sqrt{n}} \;=\; \frac{S_{c,\lfloor nt \rfloor} - tS_{c,n} - (\lfloor nt \rfloor - nt)\bar{X}_{c,n}}{\sigma\sqrt{n}},$$

where $\bar{X}_{c,k} \equiv S_{c,k}/k$ for $k = 1,2,\ldots,n$. It can be shown that this reflected STS converges to the corresponding reflected Brownian bridge, $\mathscr{B}_c(t) \equiv \mathscr{W}_c(t) - t\mathscr{W}_c(1)$,

$$T_{c,n}(t) \underset{n\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} \mathscr{B}_c(t) \;=\; \begin{cases} \mathscr{W}(t) - t[2\mathscr{W}(c) - \mathscr{W}(1)], & \text{if } 0 \le t < c \\ t\mathscr{W}(1) - \mathscr{W}(t) + 2(1-t)\mathscr{W}(c), & \text{if } c \le t \le 1. \end{cases}$$

The *reflected area estimator* for σ^2, $\mathscr{A}_c(f;n)$, and its limiting functional, $\mathscr{A}_c(f)$, with weight function $f(t)$ satisfying Assumption A.3, are

$$\mathscr{A}_c(f;n) \equiv \left[\frac{1}{n}\sum_{j=1}^{n} f\left(\tfrac{j}{n}\right)\sigma T_{c,n}\left(\tfrac{j}{n}\right)\right]^2 \xrightarrow[n\to\infty]{\mathscr{D}} \mathscr{A}_c(f) \equiv \left[\int_0^1 f(t)\sigma \mathscr{B}_c(t)\,\mathrm{d}t\right]^2 \sim \sigma^2\chi_1^2. \tag{2}$$

Further, if we define

$$h_i(f) \;\equiv\; \sum_{\ell=1}^{n} \tfrac{\ell}{n} f\left(\tfrac{\ell}{n}\right) - \sum_{\ell=i}^{n} f\left(\tfrac{\ell}{n}\right), \quad \text{for } i = 1,2,\ldots,n,$$

then

$$\mathrm{E}[\mathscr{A}_c(f;n)] \;=\; \mathrm{E}[\mathscr{A}(f;n)] - \frac{4}{n^3}\sum_{i=1}^{nc}\sum_{j=nc+1}^{n} h_i(f)h_j(f)R_{j-i}.$$

Remark 1 For processes with $\mu \neq 0$, Meterelliyoz et al. (2015) propose a corrected estimator for σ^2 in their online appendix. Alternatively, we can use the difference of two independent realizations to estimate σ^2 (at double the sampling cost), in which case the mean and the variance of the sample mean of the difference are 0 and about $2\sigma^2/n$, respectively. Another approach is to subtract the grand sample mean from the X_i's to obtain a new process with mean 0 and variance parameter $\approx \sigma^2$. $\triangleleft$

Example 1 Aktaran-Kalaycı et al. (2007) and Meterelliyoz et al. (2015) compare the expected values of various vanilla and reflected area estimators, e.g., those having the constant and quadratic weight functions $f_0(t) \equiv \sqrt{12}$ and $f_2(t) \equiv \sqrt{840}(3t^2 - 3t + 1/2)$, $t \in [0,1]$, both of which satisfy Assumption A.3. After applying a little algebraic elbow grease, we have

$$\begin{aligned}
\mathrm{E}[\mathscr{A}(f_0;n)] &= \sigma^2 - \frac{3\gamma_1}{n} + O(n^{-2}); \\
\mathrm{E}[\mathscr{A}_c(f_0;n)] &= \sigma^2 - 3(8c^2 - 8c + 3)\frac{\gamma_1}{n} + O(n^{-2}); \\
\mathrm{E}[\mathscr{A}(f_2;n)] &= \sigma^2 + \frac{7(\sigma^2 - 6\gamma_2)}{2n^2} + O(n^{-3}); \quad \text{and} \\
\mathrm{E}[\mathscr{A}_c(f_2;n)] &= \sigma^2 - 420[c(c-1)(2c-1)]^2\frac{\gamma_1}{n} + \frac{7(\sigma^2 - 6\gamma_2)}{2n^2} + O(n^{-3}).
\end{aligned}$$

Thus, as estimators of σ^2, $\mathscr{A}(f_0;n)$ and $\mathscr{A}_c(f_0;n)$ are always first-order biased, that is, the bias is of order $O(1/n)$. Fortuitously, $\mathscr{A}(f_2;n)$ is first-order unbiased (FOU); and $\mathscr{A}_c(f_2;n)$ is FOU for $c = 0$, 1/2, and 1. $\triangleleft$

3 MULTIPLY REFLECTED AREA ESTIMATORS

The goal of this section is to introduce our new multiply reflected area variance estimators. We start in §3.1 by noting that the weighted areas under the Brownian bridges arising from RBMs can be written as Itô integrals. §3.2 uses the Itô representation to show that these functionals are uncorrelated for certain selections of reflection points. In §3.3, we provide examples of reflection points that assure uncorrelated areas for different weight functions. In fact, since these uncorrelated area functionals are jointly normal, they also have the bonus of being independent. These area functionals can be squared in the manner of $\mathscr{A}_c(f)$ from Equation (2), leading to independent χ_1^2 random variables, which can then in turn be summed to obtain a χ^2 random variable with enhanced degrees of freedom. It is the task of §3.4 to describe how to set up standardized time series estimators for σ^2 so that they converge to these scaled χ^2 random variables as the sample size $n \to \infty$.

3.1 Itô Integral Representation of Area Functionals

The BM can be written as an Itô integral, $\mathscr{W}(t) = \int_0^t \mathrm{d}\mathscr{W}(s)$. The Brownian bridge is therefore equal to $\mathscr{B}(t) = t\mathscr{W}(1) - \mathscr{W}(t) = t\int_0^1 \mathrm{d}\mathscr{W}(s) - \int_0^t \mathrm{d}\mathscr{W}(s)$. It is also possible write the area functional as an Itô integral; in fact, after invoking integration by parts for Itô processes (see, e.g., Mikosch 1998, Equation (2.35)), it can be shown that

$$N(f) \;=\; \int_0^1 f(t)\mathscr{B}(t)\,\mathrm{d}t \;=\; \int_0^1 H(s)\,\mathrm{d}\mathscr{W}(s),$$

where $H(\cdot)$ is the deterministic function

$$H(s) \;\equiv\; \int_0^1 t f(t)\,\mathrm{d}t - \int_s^1 f(t)\,\mathrm{d}t,$$

and $\int_0^1 H(s)\mathrm{d}s = 0$. Moreover, the derivative of $H(\cdot)$ is equal to the weight function, i.e., $H'(s) = f(s)$. The starting and ending values are

$$H(0) \;=\; \int_0^1 (t-1)f(t)\,\mathrm{d}t \qquad \text{and} \qquad H(1) \;=\; \int_0^1 t f(t)\,\mathrm{d}t.$$

From Assumption A.3, we have that $f(t)$ is twice differentiable and bounded on $[0,1]$; and this implies that $\int_0^1 H^2(s)\,\mathrm{d}s < \infty$. This latter fact allows us to use the Itô isometry to obtain the variance of the area functional (Mikosch 1998, p. 108),

$$\mathrm{Var}\,[N(f)] \;=\; \mathrm{Var}\left[\int_0^1 H(s)\,\mathrm{d}\mathscr{W}(s)\right] \;=\; \int_0^1 H^2(s)\,\mathrm{d}s.$$

Therefore, in light of Assumption A.3's variance standardization, the function $H(\cdot)$ satisfies $\int_0^1 H^2(s)\,\mathrm{d}s = 1$.

For example, for constant weight function $f_0(t) = \sqrt{12}$, we have $H(s) = \sqrt{3}(2s-1)$; and hence $\int_0^1 H(s)\,\mathrm{d}s = 0$ and $\int_0^1 H^2(s)\,\mathrm{d}s = 1$.

The function $H(\cdot)$ can be seen as the weight function of observations X_i, whereas $f(\cdot)$ is the weight function of the standardized time series $T_n(\cdot)$. Foley and Goldsman (1999) showed that

$$N(f;n) \;=\; \frac{1}{n}\sum_{j=1}^{n} f\big(\tfrac{j}{n}\big)\sigma T_n\big(\tfrac{j}{n}\big) \;=\; \sum_{i=1}^{n}\left[\frac{h_i(f)}{n^{3/2}}\right]X_i. \tag{3}$$

For instance, for the constant weight f_0, we have $h_i(f) = \sqrt{3}(2i-n-1)$ and the weighted sum of observations with weights $H((i-0.5)/n)/\sqrt{n}$ is equal to the expression that we get from Foley and Goldsman (1999) with that choice of $h_i(f)$,

$$N(f_0;n) \;=\; \sum_{i=1}^{n}\left[\frac{H((i-0.5)/n)}{\sqrt{n}}\right]X_i \;=\; \sum_{i=1}^{n}\left[\frac{\sqrt{3}(2i-n-1)}{n^{3/2}}\right]X_i.$$

3.2 Covariance of Reflected Area Functionals

For a single reflection at $c \in [0,1]$, the RBM is given by $\mathscr{W}_c(t) = \int_0^t \delta_c(s)\,\mathrm{d}\mathscr{W}(s)$, where

$$\delta_c(s) = \begin{cases} 1, & \text{if } 0 \le s < c \\ -1, & \text{if } c \le s \le 1. \end{cases} \tag{4}$$

The reflected functional is

$$N_c(f) \;=\; \int_0^1 f(t)\mathscr{B}_c(t)\,\mathrm{d}t \;=\; \int_0^1 H(t)\,\mathrm{d}\mathscr{W}_c(t) \;=\; \int_0^1 \delta_c(t)H(t)\,\mathrm{d}\mathscr{W}(t). \tag{5}$$

By the Itô isometry, the covariance (and so the correlation due to the unit variances) between the original and the reflected functionals is

$$\mathrm{Cov}[N(f),N_c(f)] \;=\; \int_0^1 \delta_c(t)H^2(t)\,\mathrm{d}t \;=\; \int_0^c H^2(t)\,\mathrm{d}t - \int_c^1 H^2(t)\,\mathrm{d}t.$$

Let $\rho(c)$ denote the correlation between the original and the reflected functionals, $\rho(c) \equiv \mathrm{Cov}[N(f),N_c(f)]$. Note that $\rho'(c) = 2H^2(c) \geq 0$, $\rho(0) = -1$, and $\rho(1) = 1$. Therefore, by the Intermediate Value Theorem, for any weight function $f(\cdot)$, there exists a reflection point that makes the original and the reflected area functionals uncorrelated.

Are there sets of reflection points resulting in reflected area functionals that are uncorrelated with each other as well as with the original functional? We will answer in the affirmative. To this end, we define a generic multiply reflected BM process by

$$\mathscr{W}_{\mathscr{T}}(t) \;=\; \int_0^t \delta_{\mathscr{T}}(s)\,\mathrm{d}\mathscr{W}(s),$$

where

$$\delta_{\mathscr{T}}(s) \;=\; \begin{cases} 1, & \text{if } s \in \mathscr{T} \\ -1, & \text{if } s \in [0,1] \setminus \mathscr{T} \end{cases}$$

and where $\mathscr{T} \subseteq [0,1]$ can be regarded as the domain of "$+1$" implied by reflection points so that $[0,1] \setminus \mathscr{T}$ is the domain of "-1". For instance,

- for no reflection, $\mathscr{T} = [0,1]$;
- for one reflection at $0 < c < 1$, $\mathscr{T} = [0,c)$, as in Equation (4);
- for two reflections at $0 < a < c < 1$, $\mathscr{T} = [0,a) \cup [c,1]$; and
- for three reflections at $0 < a < b < c < 1$, $\mathscr{T} = [0,a) \cup [b,c)$.

Then, similar to (5), the multiply reflected area functional is

$$N_{\mathscr{T}}(f) \;=\; \int_0^1 \delta_{\mathscr{T}}(t)H(t)\,\mathrm{d}\mathscr{W}(t).$$

Now, consider the monotone increasing function $\Lambda(t) \equiv \int_0^t H^2(s)\,\mathrm{d}s$, $0 \leq t \leq 1$, as well as the set

$$\widehat{\mathscr{T}} \;\equiv\; \Lambda(\mathscr{T}) \;=\; \big\{\Lambda(t)\,|\,t \in \mathscr{T}\big\}.$$

It can be shown that the covariance between two multiply reflected functionals with respective "$+1$" domains $\mathscr{T}_i$ and $\mathscr{T}_j$ is given by

$$\mathrm{Cov}[N_{\mathscr{T}_i}(f),N_{\mathscr{T}_j}(f)] \;=\; \int_0^1 \delta_{\mathscr{T}_i}(t)\delta_{\mathscr{T}_j}(t)H^2(t)\,\mathrm{d}t \;=\; \int_0^1 \delta_{\widehat{\mathscr{T}_i}}(y)\delta_{\widehat{\mathscr{T}_j}}(y)\,\mathrm{d}y. \tag{6}$$

By taking $\delta_{\widehat{\mathscr{T}_j}}(y)$, $j = 0,1,2,\ldots$, to be, e.g., *Walsh functions* (Wang 2012), it immediately follows that the covariance given in Equation (6) is zero. The inverse transformation of the sign switching points, which are dyadic fractions, gives the locations of the reflection points, i.e., $c = \Lambda^{-1}(\frac{a}{2^b})$ where a is a positive integer less than 2^b and b is a natural number.

To recapitulate: In the spirit of Foley and Goldsman (1999), we have used an orthonormal system of weighting functions, $H_j(t) \equiv \delta_{\mathscr{T}_j}(t)H(t)$, $j = 0,1,2,\ldots$, to obtain a set of uncorrelated standard normal area functionals, $N_{\mathscr{T}_j}(f)$, $j = 0,1,2,\ldots$. Since these area functionals are (jointly) normal, they are in fact i.i.d. standard normal.

3.3 Examples

We present several examples to illustrate how one can obtain reflection points that guarantee the independence of the four functionals $N_{\mathscr{T}_j}(f)$, $j=0,1,2,3$, obtained by using the first four Walsh functions.

For the constant weight function $f_0(t)=\sqrt{12}$, we have $H(s)=\sqrt{3}(2s-1)$ and $\Lambda(t)=\int_0^t H^2(s)\,\mathrm{d}s = 3t-6t^2+4t^3$. Hence,

- for one reflection, $c=\Lambda^{-1}(1/2)=1/2$;
- for two reflections, $c_1=\Lambda^{-1}(1/4)=0.10315$ and $c_2=\Lambda^{-1}(3/4)=0.89685$; and
- for three reflections, $c_1=\Lambda^{-1}(1/4)=0.10315$, $c_2=\Lambda^{-1}(1/2)=0.5$, and $c_3=\Lambda^{-1}(3/4)=0.89685$.

For the quadratic weight function $f_2(t)=\sqrt{840}(3t^2-3t+1/2)$, we have $H(s)=\sqrt{210}(s-3s^2+2s^3)$ and $\Lambda(t)=70t^3-315t^4+546t^5-420t^6+120t^7$. It follows that

- for one reflection, $c=\Lambda^{-1}(1/2)=1/2$;
- for two reflections, $c_1=\Lambda^{-1}(1/4)=0.222986$ and $c_2=\Lambda^{-1}(3/4)=0.777014$; and
- for three reflections, $c_1=\Lambda^{-1}(1/4)=0.222986$, $c_2=\Lambda^{-1}(1/2)=0.5$, and $c_3=\Lambda^{-1}(3/4)=0.777014$.

For the cosine weight functions $f_{\cos,j}(t)=\sqrt{8}\pi j\cos(2\pi jt)$ for $j=1,2,\ldots$, we have $H(s)=\sqrt{2}\sin(2j\pi s)$ and $\Lambda(t)=t-\frac{\sin(4j\pi t)}{4j\pi}$. Hence, for $j=1,2,\ldots$,

- for one reflection, $c=\Lambda^{-1}(1/2)=1/2$;
- for two reflections, $c_1=\Lambda^{-1}(1/4)=1/4$ and $c_2=\Lambda^{-1}(3/4)=3/4$; and
- for three reflections, $c_1=\Lambda^{-1}(1/4)=1/4$, $c_2=\Lambda^{-1}(1/2)=1/2$, and $c_3=\Lambda^{-1}(3/4)=3/4$. $\lhd$

3.4 The New Estimators

Here we put all of the above results and notational fog together to give the explicit form of the new multiply reflected weighted area estimators for σ^2. To begin, suppose that $\mathscr{T}_i=\Lambda^{-1}(\widehat{\mathscr{T}_i})$, $i=0,1,2,\ldots$, where $\widehat{\mathscr{T}_i}$ is the domain of "+1" for the ith Walsh function as described in §3.2. Further, for $j=1,2,\ldots,n$, define

$$X_{\mathscr{T}_i,j} \;\equiv\; \begin{cases} X_j, & \text{if } \lfloor jn\rfloor/n\in\mathscr{T}_i \\ -X_j, & \text{otherwise,}\end{cases} \qquad j=1,2,\ldots,n,$$

and

$$T_{\mathscr{T}_i,n}(t) \;\equiv\; \frac{\lfloor nt\rfloor(\bar{X}_{\mathscr{T}_i,\lfloor nt\rfloor}-\bar{X}_{\mathscr{T}_i,n})}{\sigma\sqrt{n}}, \quad 0\le t\le 1,$$

where $\bar{X}_{\mathscr{T}_i,k}\equiv\sum_{j=1}^{k}X_{\mathscr{T}_i,j}/k$, for $k=1,2,\ldots,n$.

Next we generalize formula (2) for multiple reflections,

$$N_{\mathscr{T}_i}(f;n) \;\equiv\; \frac{1}{n}\sum_{j=1}^{n} f\big(\tfrac{j}{n}\big)\sigma T_{\mathscr{T}_i,n}\big(\tfrac{j}{n}\big) \;\xrightarrow[n\to\infty]{\mathscr{D}}\; N_{\mathscr{T}_i}(f) \;=\; \int_0^1 f(t)\sigma\mathscr{B}_{\mathscr{T}_i}(t)\,\mathrm{d}t \;=\; \sigma\int_0^1 \delta_{\mathscr{T}_i}(t)H(t)\,\mathrm{d}\mathscr{W}(t). \quad (7)$$

Thus, $N_{\mathscr{T}_i}(f;n)$ is the reflected standardized time series functional corresponding to the analogous STS functional from Equation (3). Our new overall multiply reflected area estimator $\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k)$ is simply the average of the squares of the k area functionals $N_{\mathscr{T}_i}(f;n)$,

$$\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k) \;\equiv\; \frac{1}{k}\sum_{i=0}^{k-1}\mathscr{A}_{\mathscr{T}_i}(f;n) \;=\; \frac{1}{k}\sum_{i=0}^{k-1}[N_{\mathscr{T}_i}(f;n)]^2.$$

We shall investigate its properties forthwith.

4 ESTIMATOR PERFORMANCE

In this section, we start with a bit of standard theory and then undertake some Monte Carlo simulation to evaluate the efficacy of the new estimators.

4.1 Some Easy Asymptotic Results

By the convergence result (7), the Continuous Mapping Theorem (Billingsley 1968, Corollary 1, p. 31), and the fact that the $N_{\mathscr{T}_i}(f)$'s, $i = 0, 1, 2, \ldots$, are i.i.d. standard normal, we have

$$\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k) \xrightarrow[n\to\infty]{\mathscr{D}} \frac{1}{k}\sum_{i=0}^{k-1}[N_{\mathscr{T}_i}(f)]^2 \sim \frac{\sigma^2\chi_k^2}{k}.$$

Thus, under suitable uniform integrability assumptions, we see that as the sample size $n \to \infty$,

$$\mathrm{E}\big[\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k)\big] \to \mathrm{E}\big[\sigma^2\chi_k^2/k\big] = \sigma^2$$

and

$$\mathrm{Var}\big[\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k)\big] \to \mathrm{Var}\big[\sigma^2\chi_k^2/k\big] = 2\sigma^4/k.$$

This indicates that the new estimator is asymptotically unbiased for σ^2 and that the variance decreases in the number of Walsh functions used.

4.2 A Small Monte Carlo Example

We take our new multiply reflected area out for a test drive on a stationary first-order autoregressive [AR(1)] process. Let $X_i = \phi X_{i-1} + \varepsilon_i$, $i = 1, 2, \ldots$, where $-1 < \phi < 1$, the ε_i's are i.i.d. $\mathrm{Nor}(0, 1-\phi^2)$, and X_0 is initialized as a standard normal random variate independent of the subsequent ε_i's. Then, as is well known, $X_i \sim \mathrm{Nor}(0,1)$ for all i, $R_j = \mathrm{Cov}(X_i, X_{i+j}) = \phi^{|j|}$ for all i, j, and $\sigma^2 = (1+\phi)/(1-\phi)$. Table 1 presents results on the performance of $\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k)$ when it is applied to an AR(1) process with $\phi = 0.9$, in which case it turns out that $\sigma^2 = 19$ and $2\sigma^4 = 722$. The $k = 2$ row roughly corresponds to the single-reflection case studied in Meterelliyoz et al. (2015). In any case, for various choices of the weighting function f, sample size n, and number of Walsh functions k, the table gives:

- Exact bias of $\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k)$ as an estimator of σ^2. These results are based on machinations similar to those employed in Foley and Goldsman (1999) and Meterelliyoz et al. (2015); and
- Estimated variance ($\widehat{\mathrm{Var}}$) of $\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k)$ based on 100,000 Monte Carlo replications.

The obvious takeaways from Table 1 are that:

- As n becomes large, the bias of $\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k)$ dissipates at about the rate $1/n$ for all of the estimators under study except for the case in which we use the weighting function f_2 with $k = 1$ or 2.
- In particular, the introduction of the reflected estimators often *adds* bias (except for the case of f_2 with one reflection).
- On the other hand, the variance of $\bar{\mathscr{A}}_{\mathrm{R}}(f;n,k)$ decreases more-or-less proportionately to the number of reflected estimators k.

These results are in line with previous research, including Meterelliyoz et al. (2015), and suggest the usual "bias–variance tradeoff." So even though the new reflected estimators exhibit significant bias, they can achieve smaller variances by re-using the same data over and over again — in particular, variances that are smaller than those obtained by the single-reflected estimators of Meterelliyoz et al. (2015).

Table 1: Exact bias and Monte Carlo estimates of the variance of reflected estimators for an AR(1) process with $\phi = 0.9$ ($\sigma^2 = 19$ and $2\sigma^4 = 722$). Each Monte Carlo estimate of the variance ($\widehat{\text{Var}}$) is based on 10,000 replications.

$n \rightarrow$	512		1024		2048		4096	
Estimator	Bias	$\widehat{\text{Var}}$	Bias	$\widehat{\text{Var}}$	Bias	$\widehat{\text{Var}}$	Bias	$\widehat{\text{Var}}$
$\mathscr{A}(f_0;n)$	−1.05	646	−0.53	667	−0.26	725	−0.13	745
$\bar{\mathscr{A}}_{\mathrm{R}}(f_0;n,2)$	−1.05	322	−0.53	333	−0.26	342	−0.13	353
$\bar{\mathscr{A}}_{\mathrm{R}}(f_0;n,4)$	−2.38	140	−1.19	158	−0.60	171	−0.30	172
$\bar{\mathscr{A}}_{\mathrm{R}}(f_0;n,8)$	−4.71	53	−2.47	71	−1.24	79	−0.62	84
$\bar{\mathscr{A}}_{\mathrm{R}}(f_0;n,16)$	−7.98	19	−4.77	28	−2.51	34	−1.26	39
$\mathscr{A}(f_2;n)$	−0.25	697	−0.07	704	−0.02	743	−0.004	712
$\bar{\mathscr{A}}_{\mathrm{R}}(f_2;n,2)$	−0.24	350	−0.06	356	−0.02	363	−0.004	366
$\bar{\mathscr{A}}_{\mathrm{R}}(f_2;n,4)$	−1.57	149	−0.74	173	−0.36	172	−0.17	174
$\bar{\mathscr{A}}_{\mathrm{R}}(f_2;n,8)$	−3.75	61	−1.88	73	−0.93	83	−0.46	85
$\bar{\mathscr{A}}_{\mathrm{R}}(f_2;n,16)$	−7.02	21	−3.94	29	−2.00	36	−1.00	42

5 CONCLUSIONS

This paper studied the efficacy of employing multiple reflection points on each realization of the standardized time series, and so generalized the basic reflected estimator originally investigated by Meterelliyoz et al. (2015). Generally speaking, our estimators have excellent variance, but some exhibit high bias that is not ready for prime time. The good news is that this paper has provided the basic theory to allow for a rich class of estimators that will be exploited in the future to overcome such issues.

For instance, augmentations that we did not include in the current article due to space limitations include:

- batching to reduce estimator variance;
- jackknifing to reduce bias;
- optimal batch size calculation to minimize mean squared error;
- overlapping versions of the reflected estimators to reduce variance;
- reflection point selection to minimize bias;
- use of cosine weights à la Foley and Goldsman (1999);
- enhanced Monte Carlo evaluation of the new estimators; and
- incorporation of the new estimators into sequential procedures.

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