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OPTIMAL COMPUTING BUDGET ALLOCATION FOR RANKING THE TOP DESIGNS WITH STOCHASTIC CONSTRAINTS

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ABSTRACT

Comparing with the well-studied unconstrained ranking and selecting problems in simulation, literatures on constrained ranking and selection problems are relatively fewer. In this paper, we consider the problem of ranking the top-m designs subjected to stochastic constraints, where the design performance of the main objective as well as the constraint measures can only be estimated from simulation. Using the optimal computing budget allocation framework, we derive an asymptotically optimal allocation rule. The effectiveness of the suggested rule is demonstrated via numerical experiments.

1 INTRODUCTION

We consider the problem of ranking the top-m feasible designs from a finite number of designs, assuming that a main objective and constraint measures of each design can only be obtained through simulation. Although simulation has been successfully applied to analyze and evaluate complex systems where no analytical solutions are available, it is computationally expensive since a large number of simulation replications are needed in order to have a steady mean performance value. As a result, it is practically useful and important to allocate the simulation replications efficiently. Since the number of designs for comparison is finite, this problem is closely related with the ranking and selection (R&S) in statistics (Bechhofer, Santner, and Goldsman 1995). In recent years, R&S procedures have been successfully applied in simulation (Andradóttir et al. 2005; Chen and Lee 2010).

In the literature, most of the works deal with unconstrained R&S problems, which are well studied from the indifference-zone (IZ) formulation and the optimal computing budget allocation (OCBA) framework. The IZ formulation first established by Bechhofer (1954) focuses on finding a feasible way to guarantee the pre-specified probability of correct selection is achieved. The optimal computing budget allocation (OCBA) focuses on the efficiency of simulation by intelligently allocating further replications based on the means and variances (Chen et al. 2000). Depending on the objective of the study, these R&S procedures have been further developed to select the best subset (Chen et al. 2008; Zhang et al. 2015), select the Pareto designs for multi-objective simulation optimization problems (Lee, Chew, and Teng 2010; Lee et al. 2010), select the best design based on opportunity cost (He, Chick, and Chen 2007; Gao and Chen 2015; Gao and Chen 2015) and rank all designs completely (Xiao, Lee, and Ng 2014).

Previous research on constrained R&S problems is relatively fewer compared with unconstrained problems. Among these works, some focus more on providing a guarantee on the probability of correct selection. For example, Andradóttir and Kim (2010) proposed a two-stage procedure to select the best in the presence of one constraint. The first stage aims to screen out all infeasible designs, while the best design is selected in the second stage. Morrice and Bulter (2006) used the utility functions to convert the constrained problem to an unconstrained one. The constrained R&S problems were also converted feasibility determination in Batur and Kim (2010) and Szechtman and Yücesan (2008). More recently, Lee et al. (2012), Hunter and Pasupathy (2013) and Pasupathy et al. (2014) used the optimal computing budget allocation framework and derived the simulation procedures for selecting the best design subjected stochastic constraints. These procedures focus on improving the efficiency of simulation rather than guaranteeing the probability of correct selection.

This paper aims to derive an efficient simulation budget allocation procedure for ranking the top feasible designs. In many multi-criteria decision making problems, identifying top designs are not enough. The relative ranking of the top designs is required since they have different importance in making the final decision. To the best of our knowledge, no previous work has studied the simulation budget allocation for ranking the top designs subjected to stochastic constraints. The next section formulates the R&S problem for ranking the top feasible designs. Section 3 derives the asymptotically optimal allocation rule. Numerical experiments are provided in Section 4, followed by the conclusion in Section 5.

2 PROBLEM FORMULATION

We consider the problem of ranking the top-m designs from a given set of k designs. Assume that the number of feasible designs is not less than m. Each design has $H+1$ performance measures. Let $X_{i,h,n}$ denote the nth simulation output of the main objective when $h=0$, and constraint measures if $h\in\{1,\cdots,H\}$ for the design i. N_i denotes the number of simulation replications allocated to design i. Let $J_{i,h}$, $\sigma_{i,h}^2$ and $\bar{J}_{i,h}$ denote the mean, the variance and the sample mean, i.e., $J_{i,h}=E(X_{i,h,n})$, $\sigma_{i,h}^2=Var(X_{i,h,\mathrm{n}})$ and $\bar{J}_{i,h}=(1/N_i)\sum_{n=1}^{N_i}X_{i,h,n}$. The main objective values, i.e., $J_{i,0},\ \forall i\in\{1,\cdots,k\}$ are used to determine the relative ranking of all designs and the constraint measures $J_{i,h}, h\in\{1,\cdots,H\}$ are used to check the feasibility. Without loss of generality, it can be assumed that design i is feasible if $J_{i,h}\le c_h, \forall h\in\{1,\cdots,H\}$. In this paper, the simulation outputs, i.e., $X_{i,h,n}$, are assumed to be normally distributed and independent from replication to replication, as well as independent across different designs. The normality assumption is typically satisfied and used in simulation because the outputs are generally obtained from batch means such that the Central Limit Theorem holds.

For any arbitrary set A, A^c denotes its complement. Under the assumption that $J_{1,0}<J_{2,0}<\cdots<J_{m,0}$, the probability of correctly ranking the top-m feasible designs can be written as follows:

$$\begin{aligned}PCR=P\Bigg\{&\left[\bigcap_{i=1}^{m}\bigcap_{h=1}^{H}\left(\bar{J}_{i,h}\le c_h\right)\right]\bigcap\left[\bigcap_{i=1}^{m-1}\left(\bar{J}_{i,0}<\bar{J}_{i+1,0}\right)\right]\\&\bigcap\left[\bigcap_{j=m+1}^{k}\left\{\left(\bigcap_{h=1}^{H}\left(\bar{J}_{i,h}\le c_h\right)\right)\bigcap\left(\bar{J}_{j,0}<\bar{J}_{m,0}\right)\right\}^{C}\right]\Bigg\}.\end{aligned}\tag{1}$$

Given a fixed simulation budget, the ranking of the top-m feasible designs cannot be determined with certainty. A common way to deal with this problem is to allocate the simulation budget efficiently such that the probability of correctly ranking the top-m feasible designs can be maximized. However, as shown in (1), evaluating the PCR is computationally intractable. To overcome this technical difficulty, we propose a lower bound on the PCR such that we can evaluate it in an fast and inexpensive way. Theorem 1 below provides the lower bound.

Theorem 1. A lower bound on the probability of correct ranking can be given as follows:

$$PCR \geq \sum_{i=1}^{m}\sum_{h=1}^{H} P\left(\bar{J}_{i,h} \leq c_h\right) + \sum_{i=1}^{m-1} P(\bar{J}_{i,0} < \bar{J}_{i+1,0})$$
$$- \sum_{j=m+1}^{k}\left[\min\left[\min_{h\neq 0} P\left(\bar{J}_{j,h} \leq c_h\right), P\left(\bar{J}_{j,0} < \bar{J}_{m,0}\right)\right]\right] + 2 - m\left(H+1\right) \tag{2}$$
$$= APCR$$

Theorem 1 can be proven using Bonferroni inequality. As the result of Theorem 1, we can convert our objective from maximizing the PCR to maximizing the APCR because the PCR goes to one when the APCR goes to one.

Therefore, we consider the following optimization problem:

$$\max_{N_1,\cdots,N_k} APCR$$
$$s.t. \ \sum_{i=1}^{k} N_i = T, N_i \geq 0, \forall i \in \{1,\cdots,k\}. \tag{3}$$

The optimal solution of (3) is the desired asymptotically optimal budget allocation rule, which can be obtained via maximizing the APCR.

3 OPTIMAL SIMULATION BUDGET ALLOCATION

Given the optimization model in (3), the objective is to derive the optimal values of N_i, $\forall i \in \{1,\cdots,k\}$ such that the APCR can be maximized. Theorem 2 below gives the asymptotically optimal solutions to model (3).

Let $\delta_{i,j,0} = J_{i,0} - J_{j,0}$ denote the mean difference of design i and j for their main objectives for any $i, j \in \{1,\cdots,k\}$, and $\sigma^2_{i,j,0} = \sigma^2_{i,0} / N_i + \sigma^2_{j,0} / N_j$ is the corresponding variance. Let $\rho_{i,h} = J_{i,h} - c_h$ denote the difference of the stochastic constraints with its corresponding performance measure for each design $i = 1,\cdots,k$. Let $q_i \equiv \arg\max_{h\in\{1,\cdots,H\}}\left(\rho_{i,h} / \sigma_{i,h}\right), \forall i = 1,\cdots,k$ denote the index of the dominating constraint measure for each design. Let $r_j \equiv \arg\min_{j\in\{m+1,\cdots,k\}}\left(-\delta_{m,j,0} / \sigma_{m,j,0}\right)$, and let $\alpha_i = N_i / T$ denote the proportion of the simulation budget allocated to each design.

Define the sets as follows.

$$\Theta_O = \left\{ j \mid j \in \left\{m+1,\cdots,k\right\}, \min_{h\in\{1,\cdots,H\}} P\left(\bar{J}_{j,h} \leq c_h\right) \geq P\left(\bar{J}_{j,0} \leq \bar{J}_{m,0}\right)\right\}$$
$$\Theta_F = \left\{ j \mid j \in \left\{m+1,\cdots,k\right\}, \min_{h\in\{1,\cdots,H\}} P\left(\bar{J}_{j,h} \leq c_h\right) < P\left(\bar{J}_{j,0} \leq \bar{J}_{m,0}\right)\right\}$$
$$\Theta_{DO} = \left\{ i \middle| i \in \left\{2,\cdots,m-1\right\}, \min_{h\in\{1,\cdots,H\}} P\left(\bar{J}_{i,h} \leq c_h\right) \geq \min\left\{P\left(\bar{J}_{i,0} \leq \bar{J}_{i-1,0}\right), P\left(\bar{J}_{i,0} \geq \bar{J}_{i+1,0}\right)\right\}\right\}$$
$$\Theta_{DF} = \left\{ i \middle| i \in \left\{2,\cdots,m-1\right\}, \min_{h\in\{1,\cdots,H\}} P\left(\bar{J}_{i,h} \leq c_h\right) < \min\left\{P\left(\bar{J}_{i,0} \leq \bar{J}_{i-1,0}\right), P\left(\bar{J}_{i,0} \geq \bar{J}_{i+1,0}\right)\right\}\right\}$$

The optimal simulation budget allocation is expressed using $\eta_i, \forall i \in \{1,\cdots,k\}$, which are defined as follows.

For the first design, η_1 is defined as follows：

$$\eta_1 = \min\left(\frac{\rho^2_{1,q_1}}{\sigma^2_{1,q_1} / \alpha_1 T}, \frac{\delta^2_{1,2,0}}{\sigma^2_{1,2,0}}\right). \tag{4}$$

For each design $j \in \left\{m+1,\cdots,k\right\}$, η_j is defined as follows:

$$\eta_j = \begin{cases} \dfrac{\rho^2_{j.q_j}}{\sigma^2_{j.q_j} / \alpha_j T}, & \text{if } j \in \Theta_F \\ \dfrac{\delta^2_{m,j,0}}{\sigma^2_{m,j,0}}, & \text{if } j \in \Theta_O. \end{cases} \tag{5}$$

For each design $i \in \{2,\cdots,m-1\}$, η_i is defined as follows:

$$\eta_i = \begin{cases} \dfrac{\delta^2_{m-1,m,0}}{\sigma^2_{m-1,m,0}}, & \text{if } i \in \Theta_{DO} \\ \dfrac{\rho^2_{m.q_m}}{\sigma^2_{m.q_m} / \alpha_m T}, & \text{if } i \in \Theta_{DF}. \end{cases} \tag{6}$$

For design m, η_m is defined as follows.

$$\eta_m = \min\left(\frac{\rho^2_{m.q_m}}{\sigma^2_{m.q_m} / \alpha_m T}, \frac{\delta^2_{m,r_j,0}}{\sigma^2_{m,r_j,0}}, \frac{\delta^2_{m-1,m,0}}{\sigma^2_{m-1,m,0}} \right). \tag{7}$$

Theorem 2. The asymptotically optimal allocation rule $\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_k)$ that maximizes the APCR is such that

$$\eta_1 = \eta_i = \eta_m = \eta_j, \forall i \in \{1,\cdots,m-1\}, \forall j \in \{m+1,\cdots,k\}. \tag{8}$$

4 NUMERICAL EXPERIMENTS

In this section, we conduct three sets of numerical experiments in order to investigate the performance of our proposed simulation budget allocation rule, which is named as CmR-OCBA. The proposed rule is compared with proportional to variance allocation (PTV) and equal allocation (EA). PTV allocates simulation budget proportionally to the variance of the each design. In this paper, the variance of each design refers to the performance variance of the design's main objective. EA allocates simulation equally to each design. Both PTV and EA can serve as benchmarks against which improvement can be measured.

The allocation rule CmR-OCBA is implemented sequentially. Initially, we allocate 20 replications to each design. Based on the simulation outputs, we can obtain the sample mean and sample variance of each design. They are used as the estimation of the population mean and population variance. Then, the sample means and sample variances are substituted into equations (4) – (8) to compute η_q, $\forall q \in \{1,\cdots,k\}$. Let $q^* = \arg\min_{q\in\{1,\cdots,k\}} \eta_q$ denote the design with the minimum value of η_q. In the next iteration, the 20 incremental replications are allocated to the design q^* such that the equality in (8) can be balanced. The simulation procedure repeats until the total simulation replications T are exhausted.

The constraint measures $c_h, h=1,2$ are set as $c_1 = 11$ and $c_2 = 9$. The experiment parameters are summarized in the Table 1 and Table 2. We can see that designs 4, 7, 8, 9 ,10, 11 ,12 and 13 are infeasible since their performance values of the first constraint are larger than $c_1 = 11$. Designs 5, 10, 15 and 20 are infeasible since their performance values of the first constraint are larger than $c_2 = 9$.

The simulation is run independently for 1000 times, and we count the number of times that we have made a correct ranking. The numerical results are summarized in Table 3. We can see that significant budget reduction is achieved via using our proposed allocation rule comparing with using PTV and EA. For example, our allocation rule requires only 4140 number of simulation replications in order to achieve a PCS of 95% for Scenario 2, but both PTV and EA require more than 8400 replications. If we fix the number of simulation replications to be 8400, we can see CmR-OCBA can achieve much higher PCS than PTV and EA in all three scenarios.

Table 1. Mean Performance Values of Each Design.

Design	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Main	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
Constraint 1	2	4	8	12	4	10	12	18	18	20	22	24	26	4	4	2	4	4	2	4
Constraint 2	2	4	6	8	10	2	4	6	8	10	2	4	6	8	10	2	4	6	8	10

Table 2. Numerical Experiments Parameters.

	Scenario 1	Scenario 2	Scenario 3
k	20	20	20
H	3	3	3
m	5	5	5
$\sigma_{i,0}$	15	5	10
$\sigma_{i,1}$	10	10	10
$\sigma_{i,2}$	5	10	10
constraints c_1	11	11	11
constraints c_2	9	9	9

Table 3. Numerical Comparison of CmR-OCBA, EA and PTV.

		Scenario 1	Scenario 2	Scenario 3
simulation budget T for reaching *PCS* of 95%	CmR-OCBA	6520	4140	5410
	EA	>8400	>8400	>8400
	PTV	>8400	>8400	>8400
simulation budget T for reaching *PCS* of 90%	CmR-OCBA	5080	3240	4450
	EA	>8400	7550	8020
	PTV	>8400	>8400	>8400
PCS when T=8400	CmR-OCBA	0.974	0.998	0.987
	EA	0.871	0.918	0.914
	PTV	0.477	0.649	0.535

5 CONCLUSIONS

The R&S procedures in simulation have been well studied and applied in many real world problems. The problem becomes more complex when the stochastic constraints are present in real industry. In this paper, the problem of ranking the top-m designs that are subjected to stochastic constraints is studied. Using the OCBA framework, we develop an efficient simulation budget allocation rule for ranking the top feasible designs. The numerical experiments have demonstrated the high efficiency of the proposed allocation rule.

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