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A MULTI-OBJECTIVE PERSPECTIVE ON ROBUST RANKING AND SELECTION

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ABSTRACT

In this study, we consider the robust Ranking and Selection problems with input uncertainty. Instead of adopting the minimax analysis in the classical robust optimization, we propose a novel method to approach this problem from the perspective of multi-objective optimization and Pareto optimality. More specifically, the performances of each design under various scenarios are reformulated as multiple objectives, and in this case, robust Ranking and Selection becomes a multi-objective Ranking and Selection. In order to determine the number of simulation replications to various scenarios of each design, a bi-level convex optimization is formulated by maximizing the surrogate of the large deviation rate function of the probability of false selection. Numerical results show the efficiency of the proposed procedure (PR-OCBA) compared with other methods.

1 INTRODUCTION

One successful and widely-used approach to dealing with the intractable complex optimization with randomness for decision-makers is the simulation optimization (SO) (see Hong and Nelson 2009; Pasupathy and Ghosh 2013; Fu et al. 2015; Xu, Huang, Chen, and Lee 2015 for a detailed review) which refers to the identification of the best solutions or designs by using efficient simulation techniques to evaluate their noisy performance. When the number of feasible designs is relatively small, the SO is reduced to the Ranking and Selection (R&S) problem (Branke, Chick, and Schmidt 2007; Kim and Nelson 2007) problem, which determines how to allocate the simulation budget to each design such that the "best" can be identified. There are three research streams in R&S (Chau, Fu, Qu, and Ryzhov 2014). The indifference-zone (IZ) approach (Kim and Nelson 2001; Nelson, Swann, Goldsman, and Song 2001; Andradóttir and Kim 2010; Teng, Lee, and Chew 2010; Frazier 2014; Healey, Andradóttir, and Kim 2014) aims at finding the budget allocation strategy which can guarantee a lower bound for the probability of correct selection ($P(CS)$) given that the performance mean of the best design is at least δ better than the others, where δ is the minimum difference to differentiate two designs. The expected value of information procedure (EVI) (Chick and Inoue 2001b; Chick and Inoue 2001a; Frazier and Kazachkov 2011; Chick and Frazier 2012; Xie, Frazier, and Chick

2016) selects the design with the largest expected value of information to simulate from the perspective of Bayesian statistics. Lastly, the optimal computing budget allocation approach (OCBA) (Chen, Lin, Yücesan, and Chick 2000; Lee et al. 2010; Lee, Chew, Teng, and Goldsman 2010; Lee et al. 2012; Xiao, Lee, and Ng 2014; Gao and Chen 2016a; Gao and Chen 2016b; Peng, Chen, Fu, and Hu 2016; Zhang et al. 2016) focuses on the maximization of $P(CS)$ given a computing budget constraint, and such procedure can further be applied to the case when simulation output follows correlated general distributions by adopting the large deviation principle (Glynn and Juneja 2004; Hunter and Pasupathy 2013; Hunter and Feldman 2015; Li et al. 2016). A general formulation of R&S can be illustrated by (1),

$$\arg\min_{i\in S} \quad E[\boldsymbol{H}(i,\omega|\theta)], \tag{1}$$

where the decision-maker wants to find the optimal design i from a finite set S with the smallest mean of the function $\boldsymbol{H}(i,\omega|\theta)$. Here, ω represents the intrinsic uncertainty within the simulation model, while θ and $\boldsymbol{H}(\cdot)$ are the input parameters and input models which are assumed to be given for most classical R&S work. However, it can be quite difficult to select the correct input model and the associated parameters in practice, which leads to the problem of R&S with input uncertainty (Lam 2016).

To tackle the R&S with input uncertainty, one typical way borrowed from the classical robust optimization is the Wald's maximin or minimax models (Wald 1945). This approach intends to find designs with the best worst-case performance, which can be mathematically formulated as (2) where Θ is the input parameter set, $\mathscr{H}$ is the input model set, and $\mathscr{U} = \Theta \times \mathscr{H}$ is the input uncertainty set. For example, Fan, Hong, and Zhang (2013) propose a two-layer R&S procedure under the IZ formulation. The first layer identifies the worst-case scenario and the best design is chosen in the second layer. Furthermore, an entropic descent algorithm by Ghosh and Lam (2015) is introduced to obtain a conservative bound considering the worst-case robust R&S. In addition, Gao, Xiao, Zhou, and Chen (2016) developed an asymptotically optimal budget allocation strategy under Gaussian assumption to select the best designs on the worst-case performance.

$$\arg\min_{i\in S} \max_{(\theta,\boldsymbol{H})\in\mathscr{U}} \quad E[\boldsymbol{H}(i,\omega|\theta)]. \tag{2}$$

Despite the fact that for a realized scenario $(\theta,\boldsymbol{H})$, the performance $E[\boldsymbol{H}(i^*,\omega|\theta)]$ of the optimal designs i^* given by (2) cannot be worse than their worst-case performance, such maximin or minimax models are criticized for their conservative decisions especially when the worst case is unlikely to happen in practice. Furthermore, this approach is from the perspective of the risk-averse decision-makers, while the risk-neutral and risk-seeking decision-makers might want to find the designs with optimal average-case performance and best-case performance, respectively. Finally, as pointed by Iancu and Trichakis (2013), such paradigm can produce suboptimal designs without the property of Pareto optimality in practice, and thus leads to inefficiency. To address these issues and provide more flexibility, we propose a novel approach from the perspective of multi-objective optimization and Pareto optimality to handle the robust R&S. More specifically, as presented in (3), the performances of each design under various scenarios are reformulated as multi-objectives where each scenario is treated as an objective measure, and we want to identify all the Pareto robust designs which are non-dominated by the others. It is noteworthy that the number of input uncertain scenarios can be positive infinite, but we can construct a finite uncertainty set by selecting some representative scenarios. Such scenarios might be selected as the quantiles of the performance of each design, or the status of the decision environment and the decisions by the adversarial player (e.g. sunny or rainy, stock prices go up or down, silent or betray).

$$\arg\min_{i\in S} \quad \{E[\boldsymbol{H}(i,\omega|\theta)], \forall(\theta,\boldsymbol{H})\in\mathscr{U}\}. \tag{3}$$

The organization of this paper is as follows. The robust R&S problem is formulated in section 2, followed by section 3 which illustrates the multi-objective optimization approach to solve the robust R&S. The performance of the proposed algorithm and the other two budget allocation strategies are compared in section 4, and we summarize this paper in section 5. Please refer to the journal version of this paper (Liu, Gao, and Lee 2017) for the proof of theorems and lemmas.

2 PROBLEM STATEMENT

In this study, we consider the robust R&S problem of identifying the best designs from a finite set S of r designs. For all the r designs, their input uncertainty set $\mathscr{U}$ is assumed to be identical and contains s scenarios which represent various input models and the relevant parameters. Such input uncertainty set can be a continuous set especially in the area of robust optimization, and thus the number of scenario for each design will be positive infinite. However, potential input models or parametric distribution families can be selected based on the historical data and the goodness-of-fit test, and then the quantiles of associated models can be used to construct the finite input uncertainty set $\mathscr{U}$. Therefore, the performances of each design under s scenarios can be treated as s objectives, and thus the robust R&S problem is reformulated as a multi-objective R&S problem.

Let $h_{ik}, \sigma_{ik}, H_{ik}, \bar{H}_{ik}$ denote the mean, standard deviation, random variable and sample mean for the performance of design i under scenario k, where $i \leq r$(the abbreviation for $i = 1,2,\ldots,r$ and $i \in S$) and $k \leq s$(the abbreviation for $k = 1,2,\ldots,s$ and $k \in \mathscr{U}$). In addition, let $\boldsymbol{h}_i = [h_{i1}, h_{i2}, \cdots, h_{is}]^T$ be the performance mean vector of design i, and $i \leq r$. Particularly, the values of h_{ik} and σ_{ik}^2 are unknown and can only be evaluated by the noise simulation as the sample mean and sample variance whose accuracy depends on the simulation budget allocation strategy. Hence, given the total simulation budget N, this study aims at determining the proportion of the simulation budget ($\alpha_{ik}N$) that should be allocated to evaluate the performance of design i under scenario k such that the Pareto robust set $S_p \subset S$ can be identified with high probability. Further, let $\boldsymbol{\alpha}_i = [\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{is}]^T, i \leq r$ denote the allocation proportion vector for design i. The Pareto robust set S_p, with cardinality $|S_P| = m$, is mathematically defined as below, and its complementary set is noted by $S_p^c = S \setminus S_p$.

Definition 1 (Pareto Dominance) Design i dominates design j, noted by $i \prec j$, if and only if $h_{ik} \leq h_{jk}, \forall k \leq s$ and $\exists k \leq s, h_{ik} < h_{jk}$. Otherwise we use $i \nprec j$ to represent that design i does not dominate design j. The notation $\hat{\prec}$ is used to represent the dominance relationship when each design's performance is evaluated by its sample mean.

Definition 2 (Pareto Robust Set) The Pareto robust set of S is defined by $S_p = \{i \in S : \nexists j \in S, j \prec i\}$ and the estimated Pareto robust set is $\hat{S}_p = \{i \in S : \nexists j \in S, j \hat{\prec} i\}$.

Throughout this paper, we make several common assumptions which can be found in the previous R&S papers (Glynn and Juneja 2004; Hunter and Pasupathy 2013; Li et al. 2016). Further, we assume the performances of a given design under any two scenarios are independent.

3 ASYMPTOTIC OPTIMAL ALLOCATION STRATEGY

3.1 Rate Function of Probability of False Selection

The false selection event will happen if there exists non-dominated design $i \in S_p$ which is estimated to be dominated by some other design $l \in S, l \neq i$; or there exists dominated design $j \in S_p^c$ which is estimated to be non-dominated by all the other design $l \in S, l \neq j$. Therefore, the probability of false selection ($P(FS)$) for the non-dominated designs can be formally expressed as (4),

$$P(FS) = P(\bigcup_{i \in S_p} \bigcup_{l \in S, l \neq i} l \hat{\prec} i \cup \bigcup_{j \in S_p^c} \bigcap_{l \in S, l \neq j} l \hat{\nprec} j), \tag{4}$$

which is normally intractable to derive the analytical closed-form expression. To handle this difficulty, we present the upper and lower bounds to approximate $P(FS)$ in Theorem 1, whic can be calculated by the Bonferroni inequality and the Lemma 1 of Lee, Chew, Teng, and Goldsman (2010).

Theorem 1 The $P(FS)$ is bounded from above by

$$P(FS) \leq (mr-2m+r)\max\{\max_{i\in S_p}\max_{l\in S,l\neq i} P(l\hat{\precsim} i), \max_{j\in S_p^c}\min_{l\in S,l\neq j} P(l\hat{\not\precsim} j)\},$$

and is bounded from below by

$$P(FS) \geq \max\{\max_{i\in S_p}\max_{l\in S,l\neq i} P(l\hat{\precsim} i), \max_{j\in S_p^c}\prod_{l\in S,l\neq j} P(l\hat{\not\precsim} j)\}.$$

Instead of minimizing the $P(FS)$, this study intends to derive the optimal allocation strategy by maximizing the rate function of $P(FS)$ from the perspective of large deviation principle. Relying on the Gärtner-Ellis theorem (please refer to Dembo and Zeitouni 2010, chap. 2), Theorem 2 demonstrates the bounds of the rate function of $P(FS)$, and let $I_{ik}(x) \triangleq \sup_{\theta_{ik}\in\mathbb{R}}\{\theta_{ik}x - \Lambda^{H_{ik}}(\theta_{ik})\}, \forall i\leq r, k\leq s, x\in\mathbb{R}$, be the Legendre-Fenchel transformation of the cumulant function $\Lambda^{H_{ik}}(\theta_{ik}) \triangleq \log E[\exp(\theta_{ik}H_{ik})]$. Note that $\Lambda^{H_{ik}}(\theta_{ik})$ is strictly convex, and thus $I_{ik}(x)$ as the supremum of infinitely many linear function is also strictly convex. Hence, $\lambda_{il}(\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_l)$ and $\eta_{jl}(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_l)$ below are strictly concave functions with respect to the associated α.

Theorem 2 The rate function of $P(FS)$ is bounded from below by

$$\lim_{n\to\infty} -\frac{1}{n}\log P(FS) \geq \min\{\min_{i\in S_p}\min_{l\in S,l\neq i}\lambda_{il}(\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_l), \min_{j\in S_p^c}\max_{l\in S,l\neq j}\eta_{jl}(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_l)\},$$

and is bounded from above by

$$\lim_{n\to\infty} -\frac{1}{n}\log P(FS) \leq \min\{\min_{i\in S_p}\min_{l\in S,l\neq i}\lambda_{il}(\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_l), \min_{j\in S_p^c}\sum_{l\in S,l\neq j}\eta_{jl}(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_l)\}.$$

where

$$\lambda_{il}(\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_l) = \sum_{k\leq s}\lambda_{ilk}(\alpha_{ik},\alpha_{lk}) = \sum_{k\leq s}[\inf_{x_{lk}\leq x_{ik}}(\alpha_{ik}I_{ik}(x_{ik}) + \alpha_{lk}I_{lk}(x_{lk}))],$$

$$\eta_{jl}(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_l) = \min_{k\leq s}\eta_{jlk}(\alpha_{jk},\alpha_{lk}) = \min_{k\leq s}\inf_{x_{lk}\geq x_{jk}}(\alpha_{jk}I_{jk}(x_{jk}) + \alpha_{lk}I_{lk}(x_{lk})).$$

Furthermore, the closed-form expressions of $\lambda_{ilk}(\alpha_{ik},\alpha_{lk})$ and $\eta_{jlk}(\alpha_{jk},\alpha_{lk})$ are presented in Lemma 1. In addition, it is well-known that $I_{ik}(x)$ is a convex function with the minimum value zero at $x = h_{ik}, \forall i\leq r, k\leq s$. Therefore, it can be shown that $\lambda_{ilk}(\alpha_{ik},\alpha_{lk}) > 0$ if $\min(\alpha_{ik},\alpha_{jk}) > 0$, othewise $\lambda_{ilk}(\alpha_{ik},\alpha_{lk}) = 0$. Similar results can be found for $\eta_{jlk}(\alpha_{jk},\alpha_{lk})$.

Lemma 1 The explicit expression of $\lambda_{ilk}(\alpha_{ik},\alpha_{lk}), \forall i\in S_p, l\in S, l\neq i, k\leq s$ can be represented as below

$$\begin{aligned}\lambda_{ilk}(\alpha_{ik},\alpha_{lk}) &\triangleq \inf_{x_{lk}\leq x_{ik}}(\alpha_{ik}I_{ik}(x_{ik}) + \alpha_{lk}I_{lk}(x_{lk})) \\ &= \mathbb{I}_{(h_{lk}\geq h_{ik})}(\alpha_{ik}I_{ik}(x(\alpha_{ik},\alpha_{lk})) + \alpha_{lk}I_{lk}(x(\alpha_{ik},\alpha_{lk}))),\end{aligned}$$

and the gradient of $\lambda_{ilk}(\alpha_{ik},\alpha_{lk})$ can be calculated by $\frac{\partial\lambda_{ilk}(\alpha_{ik},\alpha_{lk})}{\partial\alpha_{ik}} = I_{ik}(x(\alpha_{ik},\alpha_{lk}))$ and $\frac{\partial\lambda_{ilk}(\alpha_{ik},\alpha_{lk})}{\partial\alpha_{lk}} = I_{lk}(x(\alpha_{ik},\alpha_{lk}))$ if $h_{lk}\geq h_{ik}$, where $x(\alpha_{ik},\alpha_{lk})$ is the solution to the first-order condition $\alpha_{ik}\frac{\partial I_{ik}(x)}{\partial x} + \alpha_{lk}\frac{\partial I_{lk}(x)}{\partial x} = 0$.

The explicit expression of $\eta_{jlk}(\alpha_{jk},\alpha_{lk}), \forall j\in S_p^c, l\in S, l\neq j, k\leq s$ can be represented as below

$$\begin{aligned}\eta_{jlk}(\alpha_{jk},\alpha_{lk}) &\triangleq \inf_{x_{lk}\geq x_{jk}}(\alpha_{jk}I_{jk}(x_{jk}) + \alpha_{lk}I_{lk}(x_{lk})) \\ &= \mathbb{I}_{(h_{lk}\leq h_{jk})}(\alpha_{jk}I_{jk}(x(\alpha_{jk},\alpha_{lk})) + \alpha_{lk}I_{lk}(x(\alpha_{jk},\alpha_{lk}))),\end{aligned}$$

and the gradient of $\eta_{jlk}(\alpha_{jk},\alpha_{lk})$ can be calculated by $\frac{\partial\eta_{jlk}(\alpha_{jk},\alpha_{lk})}{\partial\alpha_{jk}} = I_{jk}(x(\alpha_{jk},\alpha_{lk}))$ and $\frac{\partial\eta_{jlk}(\alpha_{jk},\alpha_{lk})}{\partial\alpha_{lk}} = I_{lk}(x(\alpha_{jk},\alpha_{lk}))$ if $h_{lk}\leq h_{jk}$, where $x(\alpha_{jk},\alpha_{lk})$ is the solution to the first-order condition $\alpha_{jk}\frac{\partial I_{jk}(x)}{\partial x} + \alpha_{lk}\frac{\partial I_{lk}(x)}{\partial x} = 0$.

Particularly, if the performance samples of each design follow the multivariate Gaussian distribution, it is well-known that $I_{ik}(x) = \frac{(x-h_{ik})^2}{2\sigma_{ik}^2}, \forall i \le r, k \le s$. Therefore, the closed-form solution of the overall rate function under multivariate Gaussian assumption can be derived based on Lemma 1.

3.2 Optimal Allocation Problem

By maximizing the rate function of $P(FS)$, the true $P(FS)$ is guaranteed to converge to zero asymptotically. In this study, the upper bound of rate function of $P(FS)$ in Theorem 2 is selected as the surrogate of true rate to formulate the optimal allocation problem in (5) which leads to a bi-level convex optimization,

$$\begin{aligned}
\text{Problem P:} \quad \max \quad & z \\
\text{s.t.} \quad & z \le \min_{l \in S, l \ne i} \sum_{k \le s} \lambda_{ilk}(\alpha_{ik}, \alpha_{lk}), \quad \forall i \in S_p, \\
& z \le \sum_{l \in S, l \ne j} \min_{k \le s} \eta_{jlk}(\alpha_{jk}, \alpha_{lk}), \quad \forall j \in S_p^c, \\
& \sum_{i \le r} \sum_{k \le s} \alpha_{ik} = 1, \\
& z \in \mathbb{R}, \alpha_{ik} \ge 0, \forall i \le r, k \le s.
\end{aligned} \tag{5}$$

where $\lambda_{ilk}(\alpha_{ik}, \alpha_{lk})$ and $\eta_{jlk}(\alpha_{jk}, \alpha_{lk})$ can be solved via the inner convex optimization,

$$\begin{aligned}
\text{Problem P}_{ilk}: \quad & \inf_{x_{lk} \le x_{ik}} (\alpha_{ik} I_{ik}(x_{ik}) + \alpha_{lk} I_{lk}(x_{lk})), \\
\text{Problem P}_{jlk}: \quad & \inf_{x_{lk} \ge x_{jk}} (\alpha_{jk} I_{jk}(x_{jk}) + \alpha_{lk} I_{lk}(x_{lk})).
\end{aligned}$$

or utilizing the results in Lemma 1. Note that $\lambda_{ilk}(\alpha_{ik}, \alpha_{lk})$ and $\eta_{jlk}(\alpha_{jk}, \alpha_{lk})$ are strictly concave with respect to associated α, therefore there exists unique optimal solution $\boldsymbol{\alpha}^*$ for Problem P. In addition, by letting $z = 0$ and $\alpha_{ik} = \frac{1}{r \cdot s}, \forall i \le r, k \le s$, it can be easily checked that the Slater's condition hold for the convex optimization Problem P. Hence, the Karush-Kuhn-Tucker conditions (Karush 1939, Kuhn and Tucker 1951) are necessary and sufficient conditions for the optimality based on Boyd and Vandenberghe (2004). Problem P* in (6) is an equivalent formulation for the Problem P.

$$\begin{aligned}
\text{Problem P}^*: \quad \max \quad & z \\
\text{s.t.} \quad & z \le \sum_{k \le s} \lambda_{ilk}(\alpha_{ik}, \alpha_{lk}), \quad \forall i \in S_p, l \in S, l \ne i, \\
& z \le \sum_{l \in S, l \ne j} t_{jl}, \quad \forall j \in S_p^c, \\
& t_{jl} \le \eta_{jlk}(\alpha_{jk}, \alpha_{lk}), \quad \forall j \in S_p^c, l \in S, l \ne j, k \le s, \\
& \sum_{i \le r} \sum_{k \le s} \alpha_{ik} = 1, \\
& z \in \mathbb{R}, \alpha_{ik} \ge 0, \forall i \le r, k \le s, t_{jl} \in \mathbb{R}, \forall j \in S_p^c, l \in S, l \ne j.
\end{aligned} \tag{6}$$

Let $p_{il} \ge 0, \forall i \in S_p, l \in S, l \ne i; q_j \ge 0, \forall j \in S_p^c; r_{jlk} \ge 0, \forall j \in S_p^c, l \in S, l \ne j, k \le s;$ and $v \in \mathbb{R}$ be the dual variables for the four types of constraints in Problem P*. Then Problem P* is equivalent to the optimization problem $\min_{z \in \mathbb{R}, \boldsymbol{\alpha}_i \succeq 0, t_{jl} \in \mathbb{R}} \max_{\boldsymbol{p} \succeq 0, \boldsymbol{q} \succeq 0, \boldsymbol{r} \succeq 0, v \in \mathbb{R}} L(z, \boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, v) \triangleq -z + \sum_{i \in S_p} \sum_{l \in S, l \ne i} p_{il}(z - \sum_{k \le s} \lambda_{ilk}(\alpha_{ik}, \alpha_{lk})) + \sum_{j \in S_p^c} q_j(z - \sum_{l \in S, l \ne j} t_{jl}) + \sum_{j \in S_p^c} \sum_{l \in S, l \ne j} \sum_{k \le s} r_{jlk}(t_{jl} - \eta_{jlk}(\alpha_{jk}, \alpha_{lk})) + v(\sum_{i \le r} \sum_{k \le s} \alpha_{ik} - 1)$ by introducing the Lagrangian function, and its optimality conditions are presented in Theorem 3.

Theorem 3 The optimal allocation solution to Problem P* satisfies the stationarity conditions below,

$$\sum_{i\in S_p}\sum_{l\in S,l\neq i} p_{il} + \sum_{j\in S_p^c} q_j = 1,$$

$$\sum_{l\in S,l\neq a} \mathbb{I}_{(h_{lk}\geq h_{ak})} p_{al} I_{ak}(x(\alpha_{ak},\alpha_{lk})) + \sum_{i\in S_p,i\neq a} \mathbb{I}_{(h_{ak}\geq h_{ik})} p_{ia} I_{ak}(x(\alpha_{ik},\alpha_{ak}))$$

$$+ \sum_{j\in S_p^c} \mathbb{I}_{(h_{ak}\leq h_{jk})} r_{jak} I_{ak}(x(\alpha_{jk},\alpha_{ak})) = v, \forall a\in S_p, k\leq s,$$

$$\sum_{i\in S_p} \mathbb{I}_{(h_{bk}\geq h_{ik})} p_{ib} I_{bk}(x(\alpha_{Ik},\alpha_{bk})) + \sum_{l\in S,l\neq b} \mathbb{I}_{(h_{lk}\leq h_{bk})} r_{blk} I_{bk}(x(\alpha_{bk},\alpha_{lk}))$$

$$+ \sum_{j\in S_p^c,j\neq b} \mathbb{I}_{(h_{bk}\leq h_{jk})} r_{jbk} I_{bk}(x(\alpha_{jk},\alpha_{bk})) = v, \forall b\in S_p^c, k\leq s,$$

$$\sum_{k\leq s} r_{jlk} = q_j, \forall j\in S_p^c, l\in S, l\neq j,$$

and the complementary slackness conditions below,

$$p_{il}(z - \sum_{k\leq s} \mathbb{I}_{(h_{lk}\geq h_{ik})}(\alpha_{ik} I_{ik}(x(\alpha_{ik},\alpha_{lk})) + \alpha_{lk} I_{lk}(x(\alpha_{ik},\alpha_{lk})))) = 0, \forall i\in S_p, l\in S, l\neq i,$$

$$q_j(z - \sum_{l\in S,l\neq j} t_{jl}) = 0, \forall j\in S_p^c,$$

$$r_{jlk}(t_{jl} - \mathbb{I}_{(h_{lk}\leq h_{jk})}(\alpha_{jk} I_{jk}(x(\alpha_{jk},\alpha_{lk})) + \alpha_{lk} I_{lk}(x(\alpha_{jk},\alpha_{lk})))) = 0, \forall j\in S_p^c, l\in S, l\neq j, k\leq s,$$

and the primal and dual variables should be feasible.

4 NUMERICAL EXPERIMENTS

In this study, we compare the performance of the proposed allocation strategy Pareto Robust OCBA (PR-OCBA) derived by Problem P* with another two strategies, namely equal allocation (EA) and the proportional to variance allocation (PTV). PR-OCBA tries to solve Problem P* based on convex optimization solver CVXPY (http://www.cvxpy.org/) to obtain the asymptotic optimal allocation proportion $\boldsymbol{\alpha}^*$. EA allocates the simulation budget evenly to different designs' scenarios. PTV is introduced by Rinott (1978) which allocates simulation budget to each design's scenario proportional to its variance. Furthermore, the performance of each budget allocation strategy is evaluated by the $P(CS)$ and the speed-up ratio of required budget N_ε such that $P(FS)\leq\varepsilon\leq 1$. Let I be the value of rate function of $P(FS)$, and its lower and upper bounds are denoted by I_{ub} and I_{lb}. For sufficiently large total simulation budget N, the true $P(CS)$ can be approximated by $1-\exp(-NI)$ which falls into the interval $[1-\exp(-NI_{lb}), 1-\exp(-NI_{ub})]$. Let $P(FS)\approx\exp(-NI)\leq\varepsilon$, and thus $N_\varepsilon = -\frac{\log\varepsilon}{I} \in [-\frac{\log\varepsilon}{I_{ub}}, -\frac{\log\varepsilon}{I_{lb}}]$. Lastly, the performance of each design under different scenarios follows Gaussian distribution for the test cases we used throughout this study. The mean and standard deviation of each design's performance under different scenarios are known in advance, and thus there is no need to calculate the sample means and sample standard deviations.

4.1 Heap Configuration

In the first test, we consider a special configuration structure to provide some intuitions behind PR-OCBA. This configuration named after heap is described as follows for three variance structures,

- Constant-variance configuration: $h_{ik} = i+k-1, \sigma_{ik}^2 = 25, \quad \forall i\leq r, k\leq s.$
- Increasing-variance configuration: $h_{ik} = i+k-1, \sigma_{ik}^2 = 20+k, \quad \forall i\leq r, k\leq s.$
- Decreasing-variance configuration: $h_{ik} = i+k-1, \sigma_{ik}^2 = 31-k, \quad \forall i\leq r, k\leq s.$

It is obvious that for the heap configurations, design i dominates design $i+1, \forall i \leq r-1$, and the Pareto robust set only includes the first design. Figure 1 illustrates the visualized intuitions behind PR-OCBA through three heap configurations when $r = s = 10$. The first column represents the configurations and results for the increasing-variance case, followed by the constant-variance case and the decreasing-variance case. The first and second rows describe the performance mean matrix and standard deviation matrix with each row represents the design and each column stands for the scenario. The last row in Figure 1 demonstrates the optimal solution $\boldsymbol{\alpha}^*$ given by PR-OCBA. In all the nine sub-figures, the cells in blue have smaller values, while the red cells mean larger values. Note that the first design is the only Pareto robust design, and each design dominates the next design one after another. Hence, it is necessary and sufficient to identify the dominance relationship between the first and second design to avoid the false selection event which leads to more simulation budget allocated to the first two designs (illustrated by the first two red rows in the last row of Figure 1). Furthermore, it is quite intuitive that more budget should be allocated to those scenarios with larger variance, and therefore we can find that the first two rows get redder and redder when the variance increases.

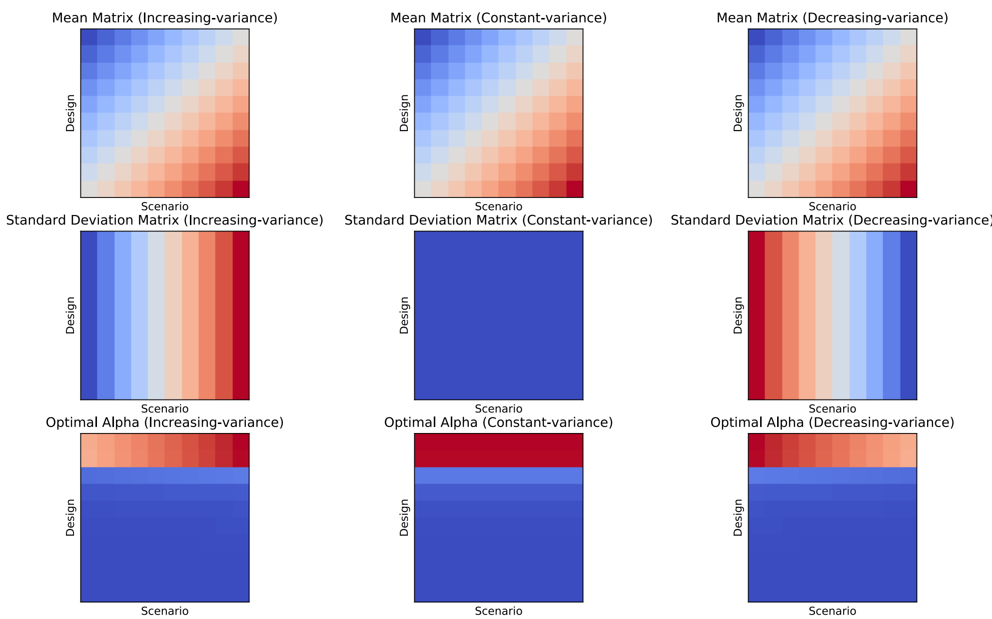

Figure 1: Demonstration of PR-OCBA Allocation.

Table 1 presents the $P(CS)$ comparison of PR-OCBA, PTV and EA given the total simulation budget $N = 2 \times 10^4$ under various heap configurations. Since the exact $P(CS)$ cannot be calculated directly, we provide the lower and upper bounds of the $P(CS)$ instead. It can be found that PR-OCBA performs the best among the three budget allocation strategies in all the cases conducted, followed by PTV and EA. Apparently, there is no need to consume too much simulation budget to the designs other than the first two given that intrinsic uncertainty is not considered. Therefore, PTV and EA demonstrate poor performance in all the cases tested, while PR-OCBA concentrates the simulation budget efficiently on the designs and scenarios that can lead to the false identification of Pareto robust set.

4.2 Random Configuration

In this test, the performance of each budget allocation strategy is evaluated via randomly generated design configurations. Specifically, h_{ik} and σ_{ik} are randomly generated from the Uniform(0, 5) and Uniform(1, 2), respectively $\forall i \leq r, k \leq s$. For each test case of a various number of designs and scenarios, 1000 random design configurations are generated. The numerical results are illustrated in Table 2 which shows the bounds

Table 1: P(CS) Comparison of PR-OCBA, PTV and EA ($N = 2 \times 10^4$).

Heap Configurations	# of design	# of scenario	PR-OCBA	PTV	EA
Constant-variance	r=5	s=3	[1.0000, 1.0000]	[1.0000, 1.0000]	[1.0000, 1.0000]
		s=5	[1.0000, 1.0000]	[0.9997, 0.9997]	[0.9997, 0.9997]
		s=10	[0.9936, 0.9999]	[0.9817, 0.9817]	[0.9817, 0.9817]
	r=10	s=3	[1.0000, 1.0000]	[0.9987, 0.9987]	[0.9987, 0.9987]
		s=5	[0.9955, 1.0000]	[0.9817, 0.9817]	[0.9817, 0.9817]
		s=10	[0.9329, 0.9999]	[0.8647, 0.8647]	[0.8647, 0.8647]
Increasing-variance	r=5	s=3	[1.0000, 1.0000]	[1.0000, 1.0000]	[1.0000, 1.0000]
		s=5	[1.0000, 1.0000]	[0.9998, 0.9998]	[0.9997, 0.9997]
		s=10	[0.9930, 0.9999]	[0.9802, 0.9802]	[0.9643, 0.9643]
	r=10	s=3	[1.0000, 1.0000]	[0.9995, 0.9995]	[0.9993, 0.9993]
		s=5	[0.9972, 1.0000]	[0.9871, 0.9871]	[0.9817, 0.9817]
		s=10	[0.9292, 0.9999]	[0.8593, 0.8593]	[0.8111, 0.8111]
Decreasing-variance	r=5	s=3	[1.0000, 1.0000]	[1.0000, 1.0000]	[1.0000, 1.0000]
		s=5	[0.9999, 1.0000]	[0.9992, 0.9992]	[0.9987, 0.9987]
		s=10	[0.9930, 0.9999]	[0.9802, 0.9802]	[0.9643, 0.9643]
	r=10	s=3	[0.9996, 1.0000]	[0.9968, 0.9968]	[0.9961, 0.9961]
		s=5	[0.9920, 1.0000]	[0.9719, 0.9719]	[0.9643, 0.9643]
		s=10	[0.9292, 0.9999]	[0.8593, 0.8593]	[0.8111, 0.8111]

of the speed-up ratio's median of the required budget N_ε such that $P(FS) \leq \varepsilon = 1 \times 10^{-6}$. This performance indicator measures how many times the required budget N_ε of PTV and EA is that of PR-OCBA. Similarly, PR-OCBA shows the most promising performance than PTV and EA. The result is not surprising given that simulation budget is not necessary to be allocated to the designs and scenarios that are not critical to avoid the false selection event, and PTV and EA just end up sampling inferior designs more often than PR-OCBA.

Table 2: Median of Speed-up Ratio of N_ε such that $P(FS) \leq 1 \times 10^{-6}$.

# of design	# of scenario	PTV	EA
r=3	s=3	[2.4973, 2.5529]	[2.5655, 2.6345]
	s=5	[2.5074, 2.5236]	[2.5024, 2.5149]
	s=10	[2.6001, 2.6001]	[2.5222, 2.5222]
r=5	s=3	[3.2046, 3.6718]	[3.1480, 3.5988]
	s=5	[2.4340, 2.5666]	[2.3789, 2.5574]
	s=10	[1.9020, 1.9020]	[1.8497, 1.8497]
r=10	s=3	[3.1867, 5.1068]	[3.1684, 5.0561]
	s=5	[5.8080, 6.8081]	[5.7543, 6.6370]
	s=10	[1.4494, 1.4494]	[1.4473, 1.4473]

5 CONCLUSIONS

Despite the intrinsic uncertainty within the simulation model, the selection and estimation of the input model family and associated input parameters are also fundamental for R&S problems due to the issues of input uncertainty. Therefore, it is important to develop a robust approach to select the best designs given the input uncertainty which can be seen in many real-world problems. Previous R&S with input uncertainty work majorly adopts the classic minimax model from the robust optimization work, while this study proposes a novel procedure which tackles the robust R&S from the perspective of multi-objective optimization and Pareto optimality. In addition, it is observed that the optimal allocation problem is a bi-level convex optimization which can be solved easily even for large scale problems. The numerical results show the promising performance of the proposed simulation budget allocation strategy (PR-OCBA) than the compared methods. Future research work can consider how to utilize the correlation information between performances of each design under different scenarios to propose more efficient budget allocation strategies. In addition, a closed-form budget allocation strategy should be developed instead of solving Problem P^* based on convex optimization solver. Furthermore, stochastic dominance and risk measure can be considered to compare the performance of each design when the probability distribution of scenarios in the input uncertainty set is known.

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