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PORTFOLIO RISK MEASUREMENT VIA STOCHASTIC MESH

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ABSTRACT

We propose a stochastic mesh approach to portfolio risk measurement under the nested setting in which revaluation of the portfolio value requires simulations. While stochastic mesh was originally proposed as a tool for American option pricing, we are interested in estimating via simulation the risk of the portfolio. We establish asymptotic properties of the stochastic mesh estimator for portfolio risk. In particular, we show that the estimator is asymptotically unbiased and consistent, and its mean squared error (MSE) converges to zero in a rate of Γ^{-1}, where Γ is the effort required to simulate the sample paths. This rate of convergence is the same as that under the non-nested setting. The proposed method allows for path dependence of financial instruments in the portfolio. Preliminary numerical experiments show that the proposed method works reasonably well.

1 INTRODUCTION

When measuring the risk of a portfolio that includes derivative contracts, reevaluation of the portfolio value for different scenarios of risk factors may require simulations, especially when risk managers use complex pricing models for which no closed-form formula of the portfolio value is available. This problem is usually referred to as portfolio risk measurement under the nested setting, and has received increasing attention in the simulation community in recent years.

A well known approach to portfolio risk measurement under the nested setting is via two-level simulation, also referred to as nested simulation. In particular, one simulates at outer level a number of possible scenarios of the risk factors over the time horizon of interest, and then simulates at inner level a number of sample paths for underlying assets of the derivative contracts until maturity for each of the outer-level scenarios. For more details on the two-level simulation approach, interested readers are referred to Lee (1998), Lee and Glynn (2003), and Gordy and Juneja (2010). One of the main issues for two-level simulaton is how to allocate computational budget to inner and outer levels. Let $\Gamma = n_1 n_2 c$ denote the total computational budget where n_1 and n_2 denote the outer- and inner-level sample sizes, respectively, and the constant c represents the computational effort required to simulate an inner-level sample while the effort to simulate an outer sample is often negligible. Lee (1998), Lee and Glynn (2003), and Gordy and Juneja (2010) analyzed asymptotic properties of the nested estimator, and showed that the optimal asymptotic mean squared error (MSE) of the two-level simulation estimator diminishes to zero at a rate of $\Gamma^{-2/3}$ if the underlying scenario space is continuous, and it was shown that the asymptotic MSE achieves the optimal rate when n_1 and n_2 are of orders $\Gamma^{2/3}$ and $\Gamma^{1/3}$, respectively. Along the same line of research, Broadie et al. (2011) proposed a method to sequentially allocate computational effort to inner-level simulations for estimating the probability of large portfolio losses, and showed that the resulting asymptotic MSE converges at a faster rate that can be arbitrary close to $\Gamma^{-4/5}$.

Another line of research for portfolio risk measurement under the nested setting is centered around smoothing ideas for estimating the portfolio loss as a conditional expectation given the risk factors. Broadie et al. (2015) proposed a least-squares regression method (LSM) to estimate the conditional expectation, and showed that the MSE of the regression method converges at a rate Γ^{-1} until reaching an asymptotic bias level. A major drawback of LSM is that it is biased in general, and sometimes choosing appropriate basis functions could be difficult. Unlike LSM, Hong et al. (2017) proposed a kernel smoothing approach to estimate the conditional expectation. While kernel smoothing suffers from curse of dimensionality, i.e., unsatisfactory performance when the dimension of the risk factors is high, Hong et al. (2017) argued that risk factors associated with individual derivative contracts are often low-dimensional. Based on this argument, they proposed a decomposition technique that decomposes a high-dimensional problem into a sequence of low-dimensional ones, making the kernel smoothing approach practically viable. Another smoothing approach is based on stochastic kriging that builds upon a spatial metamodel for the conditional expectation; see Liu and Staum (2010) for use of the stochastic kriging approach in estimating expected shortfalls.

In this paper, we propose a new approach to portfolio risk measurement that builds upon stochastic mesh, which was first proposed by Broadie and Glasserman (1997) (see also Broadie and Glasserman 2004) for pricing of American options. For more details of the stochastic mesh approach for American option pricing, interested readers are referred to Avramidis and Hyden (1999), Avramidis and Matzinger (2004), Broadie, Glasserman, and Ha (2000) and Liu and Hong (2009). In the context of American option pricing, the stochastic mesh method is attractive in that it provides an asymptotically unbiased estimator and achieves the fastest rate of convergence, while its major drawback is that its computation is relatively time consuming compared to its competitors, especially when the sample size is large. Motivated by the attractive theoretical properties of the stochastic mesh approach, we consider its application to portfolio risk measurement. It should be pointed out that there is a difference in time scale for acceptable computational times between option pricing and portfolio risk measurement. Compared to option price computation (or estimation) that is usually expected to be done within a small fraction of a second under a rapidly changing trading environment, measuring the risk of a portfolio, especially for large-scale portfolios, is often much more time consuming and computational time within a longer time frame such as minutes, or even a few hours, is often acceptable. In the context of measuring portfolio risk, a major part of the computational time is usually spent in simulating sample paths of the risk factors, especially when the portfolio involves a large amount of financial instruments and when complex pricing models are used, while the time spent in computing the estimator for given sample paths is usually negligible. A theoretical contribution of the paper is on analysis of asymptotic properties of the proposed stochastic mesh estimator. In particular, we show that the rate of convergence of its MSE is Γ^{-1}, which is the same as that under a non-nested setting where closed-form expression for the loss function is available and thus no inner simulations are required.

The rest of this paper is organized as follows. The problem is formulated in Section 2. We provide the general mathematical framework of the stochastic mesh method in Section 3. In Section 4, we establish the rate of convergence of the MSE of the stochastic mesh estimator. Preliminary numerical experiments are presented in Section 5, followed by conclusions in Section 6.

2 PROBLEM FORMULATION

Suppose that a risk manager is interested in measuring the risk of the portfolio up to a future time horizon t_τ, e.g., 2 days or 1 week. In the later presentation this time horizon is also referred to as *risk horizon*. The portfolio may consist of derivatives contracts that have maturities longer than t_τ, and its value at the risk horizon depends on a vector of risk factors that could be interest rates, commodity prices, stock prices, and/or underlying asset prices of the derivative contracts. We assume that the price dynamics of these risk factors are governed by a vector-value Markov process $\{S_t \in \mathbb{R}^d, t \geq 0\}$, which is defined on a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\mathscr{F}$ is a σ-algebra of subsets of Ω and σ-subalgebra $\mathscr{F}_t$ is generated by $\{S_u\}_{0\leq u\leq t}$, i.e., the set of information available up to t. Hence S_t is adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. In addition, suppose that the derivative contracts in the portfolio have finite maturities, and let T denote the maximum value of the maturity dates. Due to the discrete nature of computer simulation, a stochastic process is often simulated in discrete time points. Therefore, throughout the paper we work with a discretized version of S_t valued at a sequence of time points $0 = t_0 < t_1 < \cdots < t_N = T$; see Glasserman (2004) for greater detail on discretization methods of price dynamics. For notational simplicity, we write S_{t_i} and $\mathscr{F}_{t_i}$ as S_i and $\mathscr{F}_i$, respectively for $i = 0, 1, ..., N$. We further assume that the risk horizon t_τ is taking value in $\{t_1, \ldots, t_N\}$, and without loss of generality assume that τ takes value in $\{1, \ldots, N-1\}$.

By standard asset pricing theory (see, e.g., Chapter 6 in Duffie 1996), for $i = 0, 1, ..., N$, the value of the portfolio at time t_i can be represented as

$$V_i = \mathbb{E}[Y|\mathscr{F}_i],$$

where Y denotes the payoff of the portfolio (weighted with appropriate discounted factors), and the expectation is taken under a martingale pricing measure.

Then the portfolio loss at time t_τ can be written as

$$L = V_0 - V_\tau = V_0 - \mathbb{E}[Y|\mathscr{F}_\tau] = \mathbb{E}[V_0 - Y|\mathscr{F}_\tau],$$

where V_0 denotes the value of the portfolio at time 0, and is a known constant.

Typically, the portfolio loss at maturities $V_0 - Y$ can be viewed as a function of $(S_\tau, S_{\tau+1}, ..., S_N)$, and we define this function as $h(S_\tau, S_{\tau+1}, ..., S_N) \triangleq V_0 - Y$. Based on Markov property of S_i's, the portfolio loss at time t_τ is a function of S_τ and can be written as

$$L(S_\tau) = \mathbb{E}[h(S_\tau, S_{\tau+1}, ..., S_N)|S_\tau].$$

In portfolio risk measurement, risk manager is typically interested in a risk measure associated with the loss functional L. In this paper, we consider risk measures that are defined in the following form:

$$\alpha = \mathbb{E}g(L(S_\tau)), \tag{1}$$

where $g(\cdot)$ is a known function.

Different specifications of the function $g(\cdot)$ may lead to different risk measures. For instance, when g is an indicator function, i.e., $g(x) = 1_{\{x \geq y_0\}}$ for some threshold value y_0, α measures the probability that portfolio loss is larger than the given threshold y_0. When g is a hockey-stick function, e.g., $g(x) = (x - y_0)^+$, α measures the expected excess loss beyond y_0. When g is a quadratic function, e.g., $g(x) = (x - y_0)^2$, α measures the squared tracking error of the portfolio loss relative to a target y_0. Similar to the setting in Hong et al. (2017), in this paper we consider three types of g functions: smooth functions, a hockey-stick function, and an indicator function. The hockey-stick and indicator functions differ from a smooth function in terms of continuity and differentiability. The hockey-stick function is continuous everywhere but not differentiable at y_0, while the indicator function is discontinuous at y_0. In fact, the quantities of α defined for indicator and hockey-stick functions are closely related to value-at-risk and conditional value-at-risk, two widely used risk measures in practice.

To estimate the risk measure α as defined in (1), a major difficulty is that a closed-form formula of the functional form of L is usually not available. Therefore the function L has to be estimated. In the following section, we shall propose a stochastic mesh method for estimating this function L.

3 A STOCHASTIC MESH METHOD

We propose a stochastic mesh estimator for $L(s_\tau)$ for any $s_\tau \in \mathbb{R}^d$, where the function L is defined by

$$L(s_\tau) = \mathbb{E}[h(S_\tau, S_{\tau+1}, ..., S_N)|S_\tau = s_\tau]. \tag{2}$$

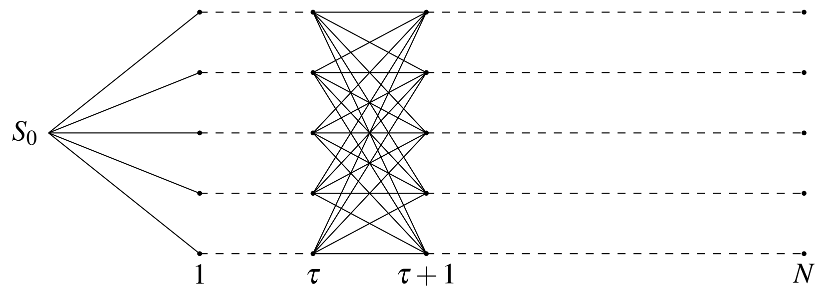

Figure 1: Illustration of sample paths for stochastic mesh.

To construct the estimator, we simulate m independent sample paths of $\{S_1,\ldots,S_N\}$, denoted by $\left\{S_k^{(j)}, k=1,\ldots,N, j=1,\ldots,m\right\}$. Throughout the paper, we assume that transition densities of the Markov chain $\{S_1,\ldots,S_N\}$ are known, and let $f_k(s_k,\cdot)$ denote the conditional density of S_{k+1} given $S_k=s_k$ for $k=0,\ldots,N-1$. We further let $f_k(\cdot)$ denote the marginal density of S_k for $k=1,\ldots,N$. An illustration of the sample paths is provided in Figure 1.

The main idea of the stochastic mesh method is that a sample path that does not satisfy $\{S_\tau=s_\tau\}$ can be used to estimate a conditional expectation given $S_\tau=s_\tau$, provided that this sample path is weighted with an appropriate likelihood ratio. For greater detail of stochastic mesh, interested readers are referred to Broadie and Glasserman (2004) and Section 8.5 of Glasserman (2004), and also Liu and Hong (2009) for further refinements. Specifically, let $\mathbf{S}\triangleq(S_{\tau+1},\ldots,S_N)$ and $\mathbf{s}\triangleq(s_{\tau+1},\ldots,s_N)$. Then we have,

$$\begin{aligned} L(s_\tau) &= \mathbb{E}[h(S_\tau,\mathbf{S})|S_\tau=s_\tau] \\ &= \int h(s_\tau,\mathbf{s}) f_\tau(s_\tau,s_{\tau+1}) f_{\tau+1}(s_{\tau+1},s_{\tau+2})\cdots f_{N-1}(s_{N-1},s_N)d\mathbf{s} \\ &= \int h(s_\tau,\mathbf{s}) \frac{f_\tau(s_\tau,s_{\tau+1})}{f_{\tau+1}(s_{\tau+1})} f_{\tau+1}(s_{\tau+1}) f_{\tau+1}(s_{\tau+1},s_{\tau+2})\cdots f_{N-1}(s_{N-1},s_N)d\mathbf{s} \\ &= \mathbb{E}\left[h(s_\tau,\mathbf{S})\frac{f_\tau(s_\tau,S_{\tau+1})}{f_{\tau+1}(S_{\tau+1})}\right]. \end{aligned} \tag{3}$$

In light of Equation (3), we may estimate $L(s_\tau)$ by

$$\bar{L}_m(s_\tau)=\frac{1}{m}\sum_{j=1}^m h\left(s_\tau,\mathbf{S}^{(j)}\right)\frac{f_\tau\left(s_\tau,S_{\tau+1}^{(j)}\right)}{f_{\tau+1}\left(S_{\tau+1}^{(j)}\right)}. \tag{4}$$

The estimator $\bar{L}_m$ is a weighted average, that makes use of all the sample paths, no matter whether they satisfy $\{S_\tau=s_\tau\}$ or not. It has nice properties that are summarized in the following proposition. Proof of the proposition is a straightforward application of the strong law of large numbers and is thus omitted.

Proposition 1 For any $s_\tau\in\mathbb{R}^d$, $\bar{L}_m(s_\tau)$ is an unbiased and strong consistent estimator of $L(s_\tau)$, i.e., $\mathbb{E}\bar{L}_m(s_\tau)=L(s_\tau)$ and $\bar{L}_m(s_\tau)\to L(s_\tau)$ almost surely as $m\to\infty$.

As mentioned in Section 8.5 of Glasserman (2004), in pricing American options, "multiplying weights along steps of a path through the mesh can produce exponentially growing variance" (see also Proposition 1 in Broadie and Glasserman 2004). To avoid this, Broadie and Glasserman (2004) propose an *average density method* to construct weights which have been dubbed *forward-looking weights* in Liu and Hong (2009). One of the advantages of the estimator in (4) is that its computational burden is $O(m)$, whereas that of the estimator constructed by forward-looking weights is $O(m^2)$.

We further simulate n samples for S_τ, denoted by $\left\{\tilde{S}_\tau^{(i)}, i=1,\ldots,n\right\}$, which are independent of the sample paths used in estimating L. Then straightforwardly an estimator of α is given by

$$\bar{\alpha}_{m,n} = \frac{1}{n}\sum_{i=1}^{n} g\left(\bar{L}_m(\tilde{S}_\tau^{(i)})\right).$$

Remark 1 When the portfolio consists of path-dependent derivative contracts such as Asian options, barrier options and lookback options, the loss function may depend on $\{S_1,\ldots,S_\tau\}$ and may not directly fit into the form in (2). The analysis can be extended to allow for such cases. More specifically, we define $\mathbf{S}_i = (S_1,...,S_i)$ for $i=1,\ldots,N$, and $\mathbf{S} = (S_{\tau+1},...,S_N)$. Then the portfolio loss L is in general a function of $\mathbf{S}_\tau$, i.e.,

$$L(s_1,\ldots,s_\tau) = \mathbb{E}\left[h(S_1,\ldots,S_\tau,\mathbf{S})|(S_1,\ldots,S_\tau)=(s_1,\ldots,s_\tau)\right]$$

for some function h. Then it can be easily verified that

$$L(s_1,\ldots,s_\tau) = \mathbb{E}\left[h(s_1,\ldots,s_\tau,\mathbf{S})\frac{f_\tau(s_\tau,S_{\tau+1})}{f_{\tau+1}(S_{\tau+1})}\right],$$

and thus an estimator for $L(\mathbf{s}_\tau)$ is given by

$$\bar{L}_m(s_1,\ldots,s_\tau) = \frac{1}{m}\sum_{j=1}^{m} h(s_1,\ldots,s_\tau,\mathbf{S}^{(j)})\frac{f_\tau\left(s_\tau,S_{\tau+1}^{(j)}\right)}{f_{\tau+1}\left(S_{\tau+1}^{(j)}\right)}.$$

4 ASYMPTOTIC ANALYSIS

This section is devoted to asymptotic analysis of the estimator $\bar{\alpha}_{m,n}$. Specifically, in Section 4.1, we establish the rate of convergence in $\mathscr{L}^{2p}$ norm of $\bar{L}_m(s_\tau)$ towards $L(s_\tau)$ under some mild assumptions, while the rest of the analysis is focused on study of the MSE of $\bar{\alpha}_{m,n}$ for the three types of functions of g: a smooth function, a hockey-stick function and an indicator function.

4.1 Analysis of $\bar{L}_m(s_\tau)$

To facilitate analysis, we provide two lemmas, whose proofs are provided in the appendix.

Lemma 1 Suppose U is a random variable with $\mathbb{E}U^{2p} < \infty$, where p is a positive integer. Then $\mathbb{E}(U - \mathbb{E}[U|\mathscr{G}])^{2p} \le 2^{2p}\mathbb{E}U^{2p}$, where $\mathscr{G}$ is an arbitrary σ-field. Furthermore, $\mathbb{E}(U-\mathbb{E}[U|\mathscr{G}])^2 \le \mathbb{E}U^2$.

Lemma 2 Let $\mathscr{G}$ denote an arbitrary σ-field and let $\{R_j\}_{j=1}^m$ be identically distributed random variables which conditional on $\mathscr{G}$, are independent of each other, such that $\mathbb{E}[R_j|\mathscr{G}] = 0$ for $1 \le j \le m$ and $\mathbb{E}R_1^{2p} < \infty$, where p is a fixed positive integer. Then,

$$\mathbb{E}\left(\frac{1}{m}\sum_{j=1}^{m} R_j\right)^{2p} = \frac{c_p \mathbb{E}R_1^{2p}}{m^p} + O\left(\frac{1}{m^{p+1}}\right),$$

as $m \to \infty$, where $c_p = \binom{2p}{2}\binom{2p-2}{2}\cdots\binom{2}{2}/p!$. Furthermore, $\mathbb{E}\left(\frac{1}{m}\sum_{j=1}^{m} R_j\right)^2 = \mathbb{E}R_1^2/m$.

Lemma 1 is a general result on moments for conditional expectations, and Lemma 2 provides a bound on the moments of a sample-mean type of estimator, which shall be useful for our asymptotic analysis.

For simplicity of notations, we define

$$w_{ij} = \frac{f_\tau(\tilde{S}_\tau^{(i)}, S_{\tau+1}^{(j)})}{f_{\tau+1}(S_{\tau+1}^{(j)})}, \quad \text{and} \quad w = \frac{f_\tau(\tilde{S}_\tau, S_{\tau+1})}{f_{\tau+1}(S_{\tau+1})}.$$

Then we can established an asymptotic bound for the moments of $\bar{L}_m(s_\tau)$, which is summarized in the following proposition, whose proof is provided in the appendix.

Proposition 2 Suppose that $\mathbb{E}\left(h(\tilde{S}_\tau,\mathbf{S})w\right)^{2p} < \infty$ with a positive integer p. Then,

$$\mathbb{E}\left(\bar{L}_m(\tilde{S}_\tau) - L(\tilde{S}_\tau)\right)^{2p} = \frac{C_p}{m^p} + O\left(\frac{1}{m^{p+1}}\right),$$

as $m \to \infty$, where $C_p = c_p 2^{2p}\mathbb{E}|h(\tilde{S}_\tau,\mathbf{S})w|^{2p} < \infty$ and $c_p = \binom{2p}{2}\binom{2p-2}{2}\cdots\binom{2}{2}/p!$.

Proposition 2 shows that the $2p$-moment of $\bar{L}_m(\tilde{S}_\tau)$ is of order $1/m^p$ provided a mild moment condition on $h(\tilde{S}_\tau,\mathbf{S})w$. It serves as a useful tool to analyze the MSE of $\bar{\alpha}_{m,n}$. In the following subsections, we analyze the MSE of $\bar{\alpha}_{m,n}$ for three different types of g functions individually.

4.2 Analysis for a Smooth Function

In this subsection, we consider the case when g is a smooth function. More specifically, we assume that g satisfies the following condition.

Assumption 1 The function $g(\cdot)$ is twice differentiable with bounded second derivative so that there exists a constant C_g such that

$$|g''(x)| \le C_g \quad \text{for any } x \in \mathbb{R},$$

and

$$\mathbb{E}|g'(L(S_\tau))|^2 < \infty, \quad \mathbb{E}|g(L(S_\tau))|^2 < \infty.$$

Asymptotic rate of convergence of the MSE of $\bar{\alpha}_{m,n}$ is summarized in the following proposition, whose proof is provided in the appendix.

Theorem 1 Suppose that Assumption 1 holds, $\mathbb{E}\left(h(\tilde{S}_\tau,\mathbf{S})w\right)^4 < \infty$, $\mathbb{E}|g'(L(\tilde{S}_\tau))h(\tilde{S}_\tau,\mathbf{S})w|^4 < \infty$ and $\mathbb{E}|g'(L(\tilde{S}_\tau))L(\tilde{S}_\tau)|^4 < \infty$. Then,

$$\begin{aligned} \mathrm{MSE}(\bar{\alpha}_{m,n}) &= \frac{2\mathbb{E}|g(L(S_\tau))|^2}{n} + 4\left(\mathbb{E}|g'(L(\tilde{S}_\tau))|^4\right)^{\frac{1}{2}}\left(\frac{C}{m^2}+O(m^{-3})\right)^{\frac{1}{2}} + C_g^2\left(\frac{C}{m^2}+O(m^{-3})\right) \\ &= O(\max\{m^{-1},n^{-1}\}), \end{aligned}$$

as $m,n \to \infty$, where $C = 48\mathbb{E}|h(\tilde{S}_\tau,\mathbf{S})w|^4$. In particular, if we let $m = n$, then $\mathrm{MSE}(\bar{\alpha}_{m,n}) = O(n^{-1})$.

Theorem 1 shows that the MSE of $\bar{\alpha}_{m,n}$ decays at a rate that is equal to $\max\{m^{-1},n^{-1}\}$. A particular attractive setting is $m = n$, in which the MSE decays at a rate of $1/n$. In this setting, the total computational budget $\Gamma = mc_1 + nc_2 = n(c_1 + c_2)$, where the constants c_1 and c_2 denote the computational efforts required to simulate a sample path of $(S_1,\ldots,S_N)$ and a sample path of $(\tilde{S}_1,\ldots,\tilde{S}_\tau)$, respectively. In this case, it can be seen that the MSE of the stochastic mesh estimator converges to zero at a rate of Γ^{-1}, which is the fastest rate of convergence that can be achieved for a Monte Carlo estimator. Theorem 1 also implies that $\bar{\alpha}_{m,n}$ is a consistent estimator of α when sample sizes m,n go to infinity.

It should also be pointed out that in the above stochastic mesh estimator, the reason we use independent sample paths $\{(\tilde{S}_1^{(i)},\ldots,\tilde{S}_\tau^{(i)}), i = 1,\ldots,n\}$ in the estimator is mainly for ease of analysis. During practical implementation, if we set $m = n$, it is reasonable to use $\{(S_1^{(i)},\ldots,S_\tau^{(i)}), i = 1,\ldots,n\}$ to replace $\{(\tilde{S}_1^{(i)},\ldots,\tilde{S}_\tau^{(i)}), i = 1,\ldots,n\}$, although the asymptotic analysis of its MSE may require more subtle analysis to handle the dependence structure within the estimator.

4.3 Analysis for Hockey-Stick and Indicator Functions

In this subsection, we study the MSE of $\bar{\alpha}_{m,n}$ when g is a hockey-stick or indicator functions.

When g is a hockey-stick function, it is Lipschitz continuous. It turns out that the MSE of $\bar{\alpha}_{m,n}$ for Lipschitz continuous functions g can be analyzed in a unified framework.

Assumption 2 The function $g(\cdot)$ is Lipschitz continuous, i.e., there exists a constant C_{Lip}, such that

$$|g(x_1) - g(x_2)| \leq C_{Lip}|x_1 - x_2| \quad \text{for any } x_1, x_2 \in \mathbb{R}.$$

Under Assumption 2, we carry out a similar analysis as in the previous subsection and establish an asymptotic result in the following theorem. The proof of the theorem is similar to that for Theorem 1, and is thus omitted.

Theorem 2 Suppose that Assumption 2 holds, and $\mathbb{E}\left(h(\tilde{S}_\tau, \mathbf{S})w\right)^2 < \infty$. Then,

$$\text{MSE}(\bar{\alpha}_{m,n}) = \frac{2C_{Lip}^2 C}{m} + O\left(\frac{1}{m^2}\right) + \frac{2\mathbb{E}|g(L(S_\tau))|^2}{n} = O(\max\{m^{-1}, n^{-1}\}),$$

as $m.n \to \infty$, where C is defined in Lemma 2. In particular, if we let $m = n$, then $\text{MSE}(\bar{\alpha}_{m,n}) = O(n^{-1})$.

Theorem 2 shows that the MSE of $\bar{\alpha}_{m,n}$ for a Lipschitz continuous g decays at the same order as that for a smooth g. When $m = n$, it achieves the fastest rate of convergence that is Γ^{-1}.

We further consider the case when g is an indicator function. Although it requires more elaborate analysis, it turns out the MSE of $\bar{\alpha}_{m,n}$ decays to zero at the same rate as that for smooth and Lipschitz continuous functions. This result is summarized in the following theorem, whose proof is omitted due to page limit.

Theorem 3 Let $g(x) = 1_{\{x \geq y_0\}}$ for some $y_0 \in \mathbb{R}$. Suppose that $\mathbb{E}\left(h(\tilde{S}_\tau, \mathbf{S})w\right)^2 < \infty$ and some regularity conditions hold. Then,

$$\text{MSE}(\bar{\alpha}_{m,n}) = O\left(\frac{1}{m}\right) + \frac{o(1)}{n} + \frac{1}{n} = O(\max\{m^{-1}, n^{-1}\}),$$

as $m, n \to \infty$. In particular, if we let $m = n$, then $\text{MSE}(\bar{\alpha}_{m,n}) = O(n^{-1})$.

5 NUMERICAL EXPERIMENTS

In this section, we construct an illustrative example to examine the performance of the proposed stochastic mesh estimator. More specifically, we consider a portfolio that consists of three European-style vanilla options with different strike prices, which are written on an underlying asset whose price dynamics is governed by the following geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where B_t is a standard Brownian motion process, μ is the rate of return of the underlying asset under the real-world probability measure. Under the risk-neutral pricing measure, the drift of the geometric Brownian motion is changed to r, the risk-free interest rate.

We assume that the options in the portfolio have the same maturity T, and we are interested in measuring the risk of the portfolio at a future time t_τ where $t_\tau < T$. During the implementation, we divided $[0, T]$ into N intervals evenly and let S_i denote the underlying asset valued at the ith time point. We first simulate S_τ under the real-world probability measure and then simulate $\mathbf{S} \triangleq (S_{\tau+1}, \ldots, S_N)$ under the risk-neutral probability measure. Denote the payoff of the portfolio at time T by $V_T(S_N)$. We let a constant V_0 denote the value of the portfolio at time 0. At time t_τ, the portfolio loss is $L(S_\tau) = \mathbb{E}[V_0 - V_T(S_N)|S_\tau]$.

We consider the estimation of the portfolio risk $\alpha = \mathbb{E}g(L(S_\tau))$ for three cases of g functions: a quadratic function $g(x) = x^2$, a hockey-stick function $g(x) = (x - y_0)^+$, and an indicator function $g(x) = 1_{\{t > y_0\}}$, where y_0 is a pre-specified threshold. In the numerical experiment, we let the initial underlying asset price $S_0 = 100$, $\mu = 5\%$, $r = 8\%$, $\sigma = 15\%$, $T = 1$, $t_\tau = 1/12$ and $\tau = 1$. Strike prices of the three options are set to be $K = 90, 100, 110$, and the loss threshold y_0 is set to be $y_0 = 5.8235$, the 90th percentile of L.

To measure the performance of the stochastic mesh estimator, we use the true value of α as a benchmark, which is approximated with high accuracy by applying a closed-form formula of $L(S_\tau)$ and taking average of 10^9 independent samples of $g(L(S_\tau))$. We then use this accurate estimate as a benchmark to measure the performance of the stochastic mesh estimator, including its bias, variance, MSE and relative root mean squared error (RRMSE), where RRMSE is defined as the percentage of the root MSE to the benchmark. All results reported are estimated based on 1000 independent replications, and we let $m = n$. Computational time of a single replication is about 10 seconds when the sample size is 10^4 using Matlab running on a PC with 3.40GHz Intel(R) Core(TM) i7-6700 CPU.

In Figure 2, we plot the estimated absolute biases, standard deviations, and the square roots of the MSEs of the stochastic mesh estimator with respect to different sample sizes. From the figure it can be seen that the MSEs of the estimator decrease as sample size increases, and a major part of the MSE comes from its variance while its bias is very small. Figure 3 shows that the convergence rate of MSE of the stochastic mesh estimators is of order Γ^{-1} (or n^{-1}), which is consistent with the theoretical result.

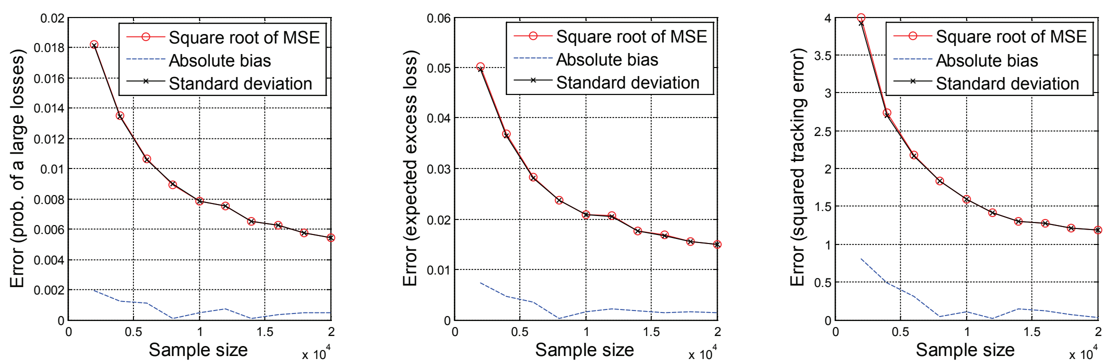

Figure 2: Estimated absolute bias, standard deviation, and square root MSEs of the stochastic mesh estimators

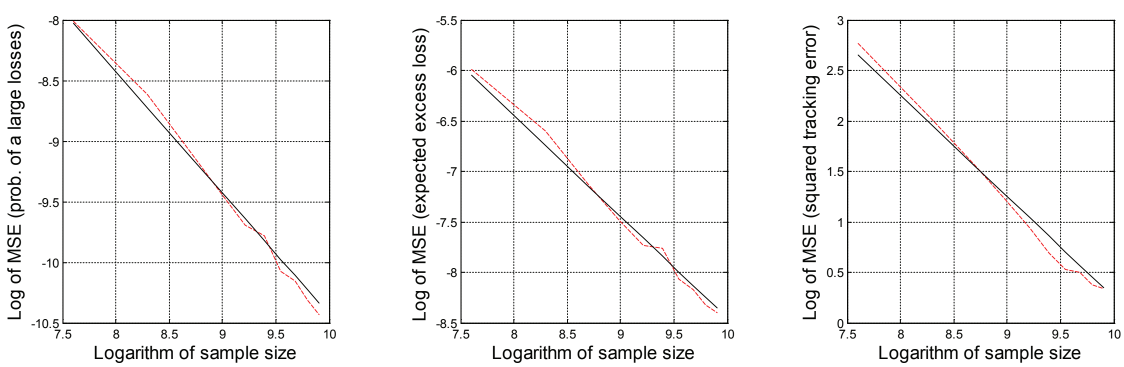

Figure 3: Illustration of convergence rates of MSEs of the stochastic mesh estimators.

Figure 4 illustrates the magnitude of RRMSEs of stochastic mesh estimators for different sample sizes. In particular, when estimating the probability of large losses, RRMSE is less than 6% when the sample size is 20,000. When estimating the expected excess loss, it is about 7% when the sample size is 20,000, and it is about 5% when the sample size is 12,000 in estimating squared tracking error. These numerical results show that the proposed stochastic mesh estimator works reasonably well.

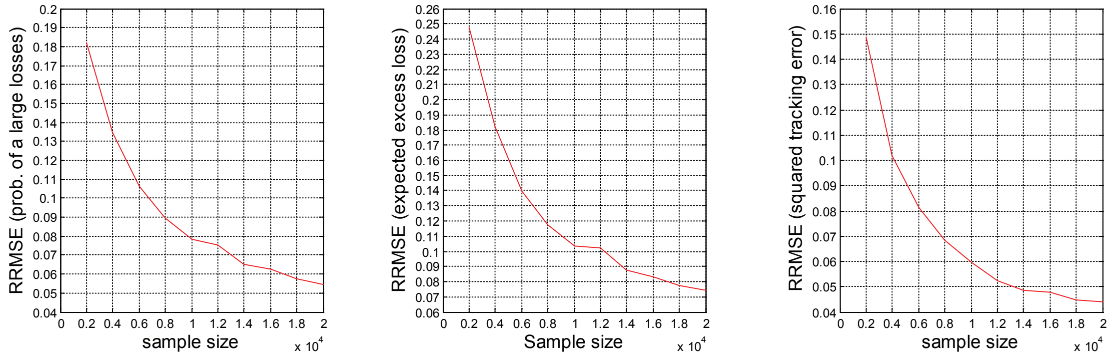

Figure 4: Performance of RRMSEs of stochastic mesh estimators.

6 CONCLUSIONS

In this paper, we have studied a stochastic mesh approach to portfolio risk measurement under the nested setting. We have analyzed the asymptotic MSE of the stochastic mesh estimator for various risk measures, and showed that the MSE decays to zero at a rate of Γ^{-1}, where Γ denotes the total computational budget. Preliminary numerical results have shown that the proposed approach may be a promising tool for measuring portfolio risk in practice.

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A PROOF OF LEMMA 1

Note that

$$
\begin{aligned}
&\mathbb{E}(U-\mathbb{E}[U|\mathscr{G}])^{2p} \le \mathbb{E}(|U|+|\mathbb{E}[U|\mathscr{G}]|)^{2p} = \mathbb{E}\left\{\sum_{l=0}^{2p}\binom{2p}{l}|U|^{2p-l}|\mathbb{E}[U|\mathscr{G}]|^{l}\right\} \\
&= \mathbb{E}U^{2p} + \mathbb{E}|\mathbb{E}[U|\mathscr{G}]|^{2p} + \sum_{l=1}^{2p-1}\binom{2p}{l}\mathbb{E}\left\{|U|^{2p-l}|\mathbb{E}[U|\mathscr{G}]|^{l}\right\} \\
&\overset{(*)}{\le} \mathbb{E}U^{2p} + \mathbb{E}|\mathbb{E}[U|\mathscr{G}]|^{2p} + \sum_{l=1}^{2p-1}\binom{2p}{l}(\mathbb{E}U^{2p})^{\frac{2p-l}{2p}}(\mathbb{E}|\mathbb{E}[U|\mathscr{G}]|^{2p})^{\frac{l}{2p}} \\
&\overset{(**)}{\le} \mathbb{E}U^{2p} + \mathbb{E}U^{2p} + \sum_{l=1}^{2p-1}\binom{2p}{l}(\mathbb{E}U^{2p})^{\frac{2p-l}{2p}}(\mathbb{E}U^{2p})^{\frac{l}{2p}} = \sum_{l=0}^{2p}\binom{2p}{l}\mathbb{E}U^{2p} = 2^{2p}\mathbb{E}U^{2p},
\end{aligned}
$$

where $(*)$ and $(**)$ follow from Hölder's and Jensen's inequalities, respectively, and the proof is completed.

B PROOF OF LEMMA 2

Note that

$$\mathbb{E}\left(\sum_{j=1}^{m} R_j\right)^{2p} = \sum_{j_1,\ldots,j_{2p}=1}^{m} \mathbb{E}(\mathbb{E}[R_{j_1}\cdots R_{j_{2p}}|\mathscr{G}]).$$

Because conditional independence of the R_j's implies that the summand is zero if there is one index different from the $2p-1$ others, the non-zero terms are thus summarized as follows:

terms	numbers
$\mathbb{E}(\mathbb{E}[R_{j_1}^{2p}\|\mathscr{G}])$	m
$\mathbb{E}(\mathbb{E}[R_{j_1}^{i_1}R_{j_2}^{i_2}\cdots R_{j_l}^{i_l}\|\mathscr{G}])$, $i_1,...,i_l \geq 2$ and $i_1+\cdots+i_l = 2p$ $(l<p)$	$O\left(\binom{2p}{i_1}\binom{2p-i_1}{i_2}\cdots\binom{i_l}{i_l}\binom{m}{l}\right)$
$\mathbb{E}(\mathbb{E}[R_{j_1}^2\cdots R_{j_p}^2\|\mathscr{G}])$	$\binom{2p}{2}\binom{2p-2}{2}\cdots\binom{2}{2}\binom{m}{p}$

For $l \leq p$, by the generalization of Hölder's inequality: assume that $r \in (0,\infty)$ and $q_1,...,q_n \in (0,\infty]$ such that $\sum_{k=1}^{n}\frac{1}{q_k} = \frac{1}{r}$, then for random variables $X_1,...,X_n$,

$$\left\|\prod_{k=1}^{n} X_k\right\|_r \leq \prod_{k=1}^{n} \|X_k\|_{q_k}, \tag{5}$$

where $\|\cdot\|_r = (\mathbb{E}|\cdot|^r)^{\frac{1}{r}}$.

Applying (5) by letting $n = l$, $q_k = 2p/i_k$, $k = 1,\ldots,l$, and $r = 1$, we have

$$\mathbb{E}\left|R_{j_1}^{i_1}R_{j_2}^{i_2}\cdots R_{j_l}^{i_l}\right| \leq (\mathbb{E}R_{j_1}^{2p})^{\frac{i_1}{2p}}\cdots(\mathbb{E}R_{j_l}^{2p})^{\frac{i_l}{2p}} = \mathbb{E}R_1^{2p} < \infty,$$

and similarly, $\mathbb{E}\left(R_{j_1}^2\cdots R_{j_p}^2\right) \leq \mathbb{E}R_1^{2p}$.

Thus, all terms are finite. Apparently, the number of the term $\mathbb{E}(\mathbb{E}[R_{j_1}^2\cdots R_{j_p}^2|\mathscr{G}])$ is strictly greater than that of terms $\mathbb{E}(\mathbb{E}[R_{j_1}^{i_1}R_{j_2}^{i_2}\cdots R_{j_l}^{i_l}|\mathscr{G}])$ $(l<p)$ when $m\to\infty$. Then we have,

$$\mathbb{E}\left(\frac{1}{m}\sum_{j=1}^{m}R_j\right)^{2p} = \frac{1}{m^{2p}}\left(m + c_p m^p\left(1+O\left(\frac{1}{m}\right)\right)\right)\mathbb{E}R_1^{2p} = \frac{c_p\mathbb{E}R_1^{2p}}{m^p} + O\left(\frac{1}{m^{p+1}}\right).$$

To prove the second half of the lemma, we note that

$$\mathbb{E}\left(\frac{1}{m}\sum_{j=1}^{m}R_j\right)^2 = \frac{1}{m^2}\left(\sum_{j=1}^{m}\mathbb{E}R_j^2 + \sum_{i\neq j}^{m}\mathbb{E}(\mathbb{E}[R_iR_j|\mathscr{G}])\right) = \mathbb{E}R_1^2/m,$$

because $\mathbb{E}[R_iR_j|\mathscr{G}] = \mathbb{E}[R_i|\mathscr{G}]\mathbb{E}[R_j|\mathscr{G}] = 0$ due to conditional independence of R_i and R_j, and $\mathbb{E}[R_i|\mathscr{G}] = 0$.

C PROOF OF PROPOSITION 2

Note that $\bar{L}_m(\tilde{S}_\tau) - L(\tilde{S}_\tau) = \frac{1}{m}\sum_{j=1}^{m}R_j$, where

$$R_j = h(\tilde{S}_\tau, \mathbf{S}^{(j)})w_{1j} - L(\tilde{S}_\tau),$$

for $j = 1,...,m$. Then it suffices to prove that the conditions of Lemma 2 hold for R_j and $\mathscr{G} = \sigma(\tilde{S}_\tau)$.

Note that

$$\mathbb{E}\left[h(\tilde{S}_\tau,\mathbf{S}^{(j)})w_{1j}\middle|\mathscr{G}\right] \overset{(*)}{=} \mathbb{E}\left[h(s_\tau,\mathbf{S}^{(j)})\frac{f_\tau(s_\tau,S_{\tau+1}^{(j)})}{f_{\tau+1}(S_{\tau+1}^{(j)})}\right]\Bigg|_{s_\tau=\tilde{S}_\tau} \overset{(**)}{=} \mathbb{E}h(s_\tau,\mathbf{S})|_{s_\tau=\tilde{S}_\tau} = L(\tilde{S}_\tau),$$

where $(*)$ follows from the Independence Lemma (see Lemma 2.3.4 in Shreve 2004) and $(**)$ is due to (3). Thus, $\mathbb{E}[R_j|\mathscr{G}]=0$. Moreover, conditional on $\mathscr{G}$, R_j is a function of $S_{\tau+1}^{(j)},\ldots,S_N^{(j)}$, so R_j's are conditionally independent of each other. To bound the second moment, we apply Lemma 1 with $U=h(\tilde{S}_\tau^{(1)},\mathbf{S}^{(j)})\tilde{w}_{1j}$ and $\mathscr{G}=\sigma(\tilde{S}_\tau^{(1)})$, leading to

$$\mathbb{E}|R_j|^{2p}\le 2^{2p}\mathbb{E}|h(\tilde{S}_\tau^{(1)},\mathbf{S}^{(j)})w_{1j}|^{2p}<\infty,$$

which completes the proof.

D PROOF OF THEOREM 1

It follows from the definition that

$$\begin{aligned}
\mathrm{MSE}(\bar{\alpha}_{m,n}) &= \mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n g(\bar{L}_m(\tilde{S}_\tau^{(i)}))-\mathbb{E}g(L(S_\tau))\right|^2\\
&= \mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n g(\bar{L}_m(\tilde{S}_\tau^{(i)}))-\frac{1}{n}\sum_{i=1}^n g(L(\tilde{S}_\tau^{(i)}))+\frac{1}{n}\sum_{i=1}^n g(L(\tilde{S}_\tau^{(i)}))-\mathbb{E}g(L(S_\tau))\right|^2\\
&\le 2\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n g(\bar{L}_m(\tilde{S}_\tau^{(i)}))-\frac{1}{n}\sum_{i=1}^n g(L(\tilde{S}_\tau^{(i)}))\right|^2+2\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n g(L(\tilde{S}_\tau^{(i)}))-\mathbb{E}g(L(S_\tau))\right|^2\\
&\le 2\mathbb{E}\left|g(\bar{L}_m(\tilde{S}_\tau^{(1)}))-g(L(\tilde{S}_\tau^{(1)}))\right|^2+2\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n g(L(\tilde{S}_\tau^{(i)}))-\mathbb{E}g(L(S_\tau))\right|^2, \qquad (6)
\end{aligned}$$

where the second inequality follows from Cauchy-Schwartz inequality.

Because $\mathbb{E}|g(L(S_\tau))|^2<\infty$, by Lemma 2 with $p=1$, we have

$$\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n g(L(\tilde{S}_\tau^{(i)}))-\mathbb{E}g(L(S_\tau))\right|^2=\frac{\mathbb{E}|g(L(S_\tau))-\mathbb{E}g(L(S_\tau))|^2}{n}\le\frac{\mathbb{E}|g(L(S_\tau))|^2}{n}, \qquad (7)$$

where the inequality follows from Lemma 1 with $p=1$.

For the first term of (6), by Taylor expansion, we have

$$\begin{aligned}
&\mathbb{E}\left|g(\bar{L}_m(\tilde{S}_\tau^{(1)}))-g(L(\tilde{S}_\tau^{(1)}))\right|^2\\
=\;& \mathbb{E}\left|g'(L(\tilde{S}_\tau^{(1)}))[\bar{L}_m(\tilde{S}_\tau^{(1)})-L(\tilde{S}_\tau^{(1)})]+\frac{1}{2}g''(\Xi)[\bar{L}_m(\tilde{S}_\tau^{(1)})-L(\tilde{S}_\tau^{(1)})]^2\right|^2\\
\le\;& 2\mathbb{E}\left(|g'(L(\tilde{S}_\tau^{(1)}))|^2|\bar{L}_m(\tilde{S}_\tau^{(1)})-L(\tilde{S}_\tau^{(1)})|^2\right)+2\mathbb{E}\left(\frac{1}{4}|g''(\Xi)|^2|\bar{L}_m(\tilde{S}_\tau^{(1)})-L(\tilde{S}_\tau^{(1)})|^4\right)\\
\overset{(*)}{\le}\;& 2\left(\mathbb{E}|g'(L(\tilde{S}_\tau^{(1)}))|^4\right)^{\frac{1}{2}}\left(\mathbb{E}|\bar{L}_m(\tilde{S}_\tau^{(1)})-L(\tilde{S}_\tau^{(1)})|^4\right)^{\frac{1}{2}}+\frac{C_g^2}{2}\mathbb{E}|\bar{L}_m(\tilde{S}_\tau^{(1)})-L(\tilde{S}_\tau^{(1)})|^4, \qquad (8)
\end{aligned}$$

where Ξ is a random variable that lies between $\bar{L}_m(\tilde{S}_\tau^{(1)})$ and $L(\tilde{S}_\tau^{(1)})$, and $(*)$ follows from Cauchy-Schwarz inequality.

Assembling the terms in Equations (6)-(8), we have the results in the theorem.

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