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BI-LEVEL STOCHASTIC APPROXIMATION FOR JOINT OPTIMIZATION OF HYDROELECTRIC DISPATCH AND SPOT-MARKET OPERATIONS

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ABSTRACT

We propose a bi-level formulation for the joint optimization of spot-market operations and optimal dispatch of hydroelectric power generation. The outer problem maximizes the net-profit from interacting with neighbours in the wider energy spot-market, where profit is the difference between the (non-convex) revenue generated from export and the cost for generating and transmitting the exported power. The latter is the optimal solution of the inner optimization problem, a stochastic linear program that determines the minimum-cost dispatch solution that meets existing local demand and planned export, subject to uncertainty in local demand and generation from run-of-the-river dams. The outer problem is solved using stochastic approximation, where the cost function gradient is obtained using parametric programming techniques on the inner problem. Experimental results establish the efficacy of this bi-level approach. We also provide some preliminary analysis of the convergence of this method.

1 INTRODUCTION

We consider the case of a power authority that controls a network of hydroelectric power generation stations, as illustrated in Figure 1, which it must use to fulfill electricity demand in its region. An integrated power system consisting of hydroelectric projects may have hydroelectric power plants attached to water storage dams, as well as run-of-river (ROR) hydroelectric power projects. Large storage dams store water during wet seasons and regulate water flow during dry seasons so that power from the storage project can follow a pre-planned schedule, and hence are considered as dispatchable. Energy production from ROR plants is uncertain because water flow in the river, and hence power production capacity, is largely determined by uncertain weather factors such as rainfall in the catchment area, spring temperatures that lead to snow-melt, and so on. The uncertainty factors have been observed to affect the hourly hydropower generation to the order of up to thirty percent of the maximum allowed capacity.

Large hydroelectric power generation stations are often far away from the consumption locations, requiring high-voltage transmission lines to transfer the power to the demand nodes. Electricity demand is diverse, including a large fraction of low-voltage domestic consumers and significant high-voltage level industrial consumers. While following well-understood trends, electricity usage at the hourly level also exhibits some residual uncertainty. In addition, demand centers are scattered over distant locations, and need multiple substations in order to step up or step down the voltage levels. As shown in Figure 1, an integrated power system network consists of multiple generation stations, transmission lines, substations, and distribution lines. Energy needs at each demand center can be supplied by a combination of local generators, transmission lines, substations, and distribution lines. Power losses on transmission lines and distribution lines also vary; as a result, the cost of energy transmission and distribution differ.

We study the optimal power dispatch determination problem in the order of the next hour. Power generation and transmission dynamics are typically non-linear in small time-scales due to Kirchoff's laws

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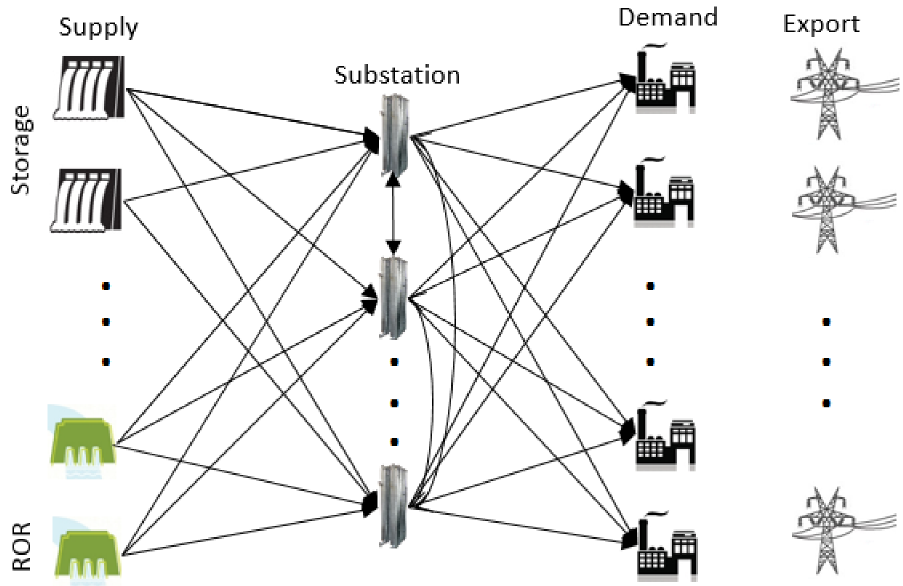


Figure 1: Hydroelectric Power Distribution Schematic Network Diagram

that govern power flow (Glavitsch and Bacher 1991); however, in the hourly time-scale of our model a linearized version called the DC power-flow model provides a good estimate (Shahidehpour, Yamin, and Li 2002). With linear dynamics of power flow, the formulation resembles an assignment problem (see Section 2), but the uncertainty in demand and ROR generation makes it a stochastic two-stage linear program with recourse. In the first stage, decisions on the dispatch of the more costly but dispatchable storage dams are taken. The second stage realizes the uncertain quantities and provides recourse in the form of selecting the optimal quantity of the cheaper but uncertain ROR generation to retain for meeting demand, spilling the rest of the runoff water if any. As is standard in the literature (Ruszczyński and Shapiro 2004), a sample-average approximation (SAA) scheme is used to obtain good candidate solutions to this two-stage linear stochastic program.

Governmental power authorities are usually mandated to satisfy local demand at cost. A key additional revenue generating activity is their operations in the spot-market for import/export of power with their regional neighbors. Spot market prices are determined by a variety of exogenous factors, and typically follow a non-linear, possibly non-convex, curve (Burger, Klar, and Schindlmayr 2004). The cost to produce any power that is exported from the region needs to be properly accounted for in optimizing the export. A disjoint optimization formulation may find it hard to appropriately approximate the true cost function emerging from the stochastic two-stage program, and so we seek to integrate planning for export with the dispatch decision making problem for the hydroelectric power production and distribution.

The joint-optimization problem is a non-linear two-stage stochastic program, where the objective function may even be non-convex; such problems are not computationally tractable. Our solution approach exploits an inherent structure to split the formulation into a bi-level optimization problem. From the dispatch perspective, the export (import) is an additional demand (supply) that is indistinguishable from other such demand (supply). Thus, we reformulate the problem so that all the revenue- and cost-function non-linearity is preserved in an outer level that maximizes the net profit from spot-market operations. The expected cost of producing the exports (or benefit from absorbing imports) is non-linear, and is modeled by sensitivity analysis on the inner optimization problem with respect to the export data.

We solve the outer profit-maximizing problem using a first-order-optimal method called Stochastic Approximation (SA). This method, first introduced by Robbins and Monro (1951), has a wide range of applications, such as in queueing systems (Dieker, Ghosh, and Squillante (2016), Chap. 9 in Kushner and Yin (2003)), signal processing (Chen 2003), inventory systems (Lim 2012), and resource allocation (Powell 2007). SA finds the roots of multidimensional functions of stochastic processes, and is applied to unconstrained stochastic optimization problem to find the roots of gradients, the first-order optimality condition, when estimates of gradients are available. The main challenge in SA is in obtaining a sufficiently high-quality estimate of the gradient of the objective function. Kiefer and Wolfowitz (1952) studied a gradient estimation scheme based on the method of finite-differences, a more sophisticated version of which, called simultaneous perturbation (Spall 1999), is considered the current best in terms of indirect estimation of gradients from function evaluations.

Direct methods of gradient estimation exploit properties of the function to estimate the gradient from sample paths of the function. Such methods include infinitesimal perturbation analysis (IPA) (Glasserman 1991a), score function methods, likelihood ratio, and weak derivatives (Sec 5.3 of Fu (2014)). In our application, we obtain the gradient of the optimal expected cost function of the inner optimization problem by exploiting a smoothness property of the sample paths, similar to the IPA approach, and using parametric programming techniques on the SAA linear program. However, unlike IPA which tends to produce unbiased gradient estimates, our gradient estimate has a bias that vanishes as the size of sampling in the SAA version the inner problem grows.

The contributions of this paper are:

- We solve a joint-optimization problem of simultaneously determining the profit-maximizing optimal spot-market export and power dispatch operations decisions that a hydroelectric generation authority can implement.
- The joint-optimization problem is a non-convex stochastic optimization formulation. This is solved by splitting the formulation into two parts, each of which can be solved efficiently using existing techniques.
- The outer optimization formulation is solved using a stochastic approximation approach, which is a stochastic first-order gradient-following procedure. The inner formulation is a two-stage stochastic linear program.
- The novelty of this approach lies in the extraction of gradient information from the inner stochastic linear program of sufficient quality that convergence of the outer stochastic approximation procedure can be guaranteed.
- The efficacy of this method is illustrated with numerical experiments.
- This bi-level SA approach is of potential interest to a wider class of joint-optimization problems where a similar decomposition can be imposed.

The remaining portion of the paper is organized as follows. The hydroelectric joint optimization problem is described in Section 2. The bi-level stochastic approximation algorithm to solve the problem is presented in Section 3. In Section 4, a preliminary analysis of the convergence of the algorithm is presented. Section 5 details the numerical experiments ran to test the algorithm, and also provides the results of these experiments.

2 PROBLEM FORMULATION

The hydroelectric network consists of $\mathcal{H}$ hydroelectric dams, where power generation follows a planned schedule. There are an additional $\mathcal{R}$ ROR hydroelectric generators, whose production is uncertain and depends on the amount of runoff water delivered to the river from its drainage footprint. The power generated from both these sources are consumed at $\mathcal{S}$ substations, all of which serve $\mathcal{K}$ energy consumers. Some demand locations also have power exchange capability with neighboring regions.

The overall objective is to maximize the net profit $p(\mathbf{e})$ generated from the spot-market export plan $\mathbf{e} = (e_1, \ldots, e_{\mathcal{K}}) \in \mathbb{R}^{\mathcal{K}}$ for the hydroelectric power authority. We let the e_k take negative values to allow for the cases when a) importing from neighbors is cheaper than generating electricity within the region to meet local demand, and b) the authority may wish to arbitrage energy prices between neighbors. The net profit is maximized as

$$\max_{\mathbf{e}\in\mathbf{R}^K} \quad p(\mathbf{e}) \quad \left(\sum_{k=1}^{\mathcal{K}} r_k(e_k)\right) - c(\mathbf{e}), \qquad \text{(OUTER)}$$

where $p(\mathbf{e})$ is the net-profit function, $r_k(e_k)$ is the revenue obtained from transacting energy with neighbor k, and $c(\mathbf{e})$ is the total cost incurred to produce export vector $\mathbf{e}$. We assume the revenue functions to be separable by neighbors. The functions r_k are deterministic non-convex functions with well-defined first derivatives with respect to e_k. We have $e_k = 0$ and $r_k(e_k) = 0$ if the k-th consumer does not have import/export capability.

The cost function $c(\mathbf{e})$ is obtained as the optimal solution to a minimal-cost hydroelectric power production stochastic program:

$$\begin{aligned} c(\mathbf{e}) \quad = \quad \min_{\mathbf{x}} & \quad f^t\mathbf{x} + h(\mathbf{x},\mathbf{e}) & \text{(INNER-I)} \\ \text{s.t.} & \quad A\mathbf{x} \le b,\ \mathbf{x} \ge 0. & (1) \end{aligned}$$

The (INNER-I) problem is the first-stage of the stochastic program, and finds the optimal production schedule $\mathbf{x}$ that minimizes the total production cost subject to certain linear constraints (detailed below). The objective function term $h(\mathbf{x},\mathbf{e})$ represents the expected cost of exercising the recourse available when the uncertainty in demand and ROR generation are realized:

$$h(\mathbf{x},\mathbf{e}) \quad = \quad \mathbb{E}_\omega[H(\mathbf{x},\mathbf{e},\omega)]$$

where

$$\begin{aligned} H(\mathbf{x},\mathbf{e},\omega) \quad = \quad \min_{\mathbf{y}_\omega} & \quad c^t\mathbf{y}_\omega & \text{(INNER-II)} \\ \text{s.t.} & \quad T_1\mathbf{x} + U_1\mathbf{y}_\omega = d_\omega + \mathbf{e} & (2) \\ & \quad \mathbf{y}_\omega \le g_\omega & (3) \\ & \quad T_2\mathbf{x} + U_2\mathbf{y}_\omega \le 0. & (4) \end{aligned}$$

We follow the standard prescription (Ruszczyński and Shapiro 2004) in replacing the expectation $h(\mathbf{x},\mathbf{e})$ of the second stage problem (INNER-II) of the first-stage (INNER-I) with its SAA $h_n(\mathbf{x},\mathbf{e},\Omega_n)$ over the set $\Omega_n = \{\omega_1,\ldots,\omega_n\}$ of n sampled scenarios

$$h_n(\mathbf{x},\mathbf{e},\Omega_n) = \frac{1}{n}\sum_{i=1}^{n} H(\mathbf{x},\mathbf{e},\omega_i),$$

and in approximating the (INNER-I) problem with the optimal solution of

$$c_n(\mathbf{e}) \quad = \quad \min_{\mathbf{x}\in\mathbf{R}^\ell} \quad f^t\mathbf{x} + h_n(\mathbf{x},\mathbf{e},\Omega_n) \quad \text{s.t. } (1) \qquad \text{(INNER-I SAA)}$$

2.1 Formulation Details

(INNER-I): The vector $\mathbf{x}$ consists of:

- the amount x_i^g of power generated from hydroelectric dam i,
- the power x_{ij}^f transmitted from dam i to substation j,
- the flow x_{pj}^s of power between substations p and j given that $p \neq j$, and

- the power x^d_{jk} distributed from the substations j to local demand and export regions k.

The amount of power produced by each hydroelectric dam x^g_i is limited by the design capacity $\bar{G}_i$ of the hydroelectric dams, i.e., $x^g_i \leq \bar{G}_i$. The power transmitted from hydroelectric dams to the substations should respect the capacity of the transmission lines $\bar{F}_{ij}$: $x^f_{ij} \leq \bar{F}_{ij}$. Further, the sum of the power transmitted from a hydroelectric dam to substations can not exceed the amount of power produced by that particular dam: $\sum_{j=1}^{\mathcal{S}} x^f_{ij} \leq x^g_i$.

(INNER-II): The vector $\mathbf{y}_\omega$, under the particular scenario ω, represents:

- the power $y^g_{i\omega}$ supplied from the ROR generator i,
- the power $y^f_{ij\omega}$ transmitted from the ROR hydroelectric generators i to substations j,
- the flow $y^s_{pj\omega}$ of the ROR generated power between substations p and j, $p \neq j$, and
- the distribution of power $y^d_{jk\omega}$ from the substations j to demand/export regions k.

We then have the following relationships:

$$\Big(\sum_{j=1}^{\mathcal{S}} x^d_{jk} + \sum_{j=1}^{\mathcal{S}} y^d_{jk\omega}\Big) = d_{k\omega} + e_k \tag{5}$$

$$y^g_{i\omega} \leq g_{i\omega} \tag{6}$$

$$\sum_{j=1}^{\mathcal{S}} y^f_{ij\omega} - y^g_{i\omega} \leq 0 \tag{7}$$

$$\Big(\sum_{i=1}^{\mathcal{R}} y^f_{ij\omega}\Big) + \Big(\sum_{i=1}^{\mathcal{H}} x^f_{ij}\Big) + \Big(\sum_{p=1}^{\mathcal{S}-1} x^s_{pj} + \sum_{p=1}^{\mathcal{S}-1} y^s_{pj\omega}\Big) \leq \bar{S}_j \tag{8}$$

$$\Big(\sum_{k=1}^{\mathcal{K}} x^f_{jk} + \sum_{k=1}^{\mathcal{K}} y^f_{jk\omega}\Big) - \Big(\sum_{p=1}^{\mathcal{S}-1} x^s_{jp} + \sum_{p=1}^{\mathcal{S}-1} y^s_{jp\omega}\Big) = 0 \tag{9}$$

$$(x^s_{pj} + y^s_{pj\omega}) \leq \bar{S}_{pj} \tag{10}$$

$$y^d_{jk\omega} \leq \bar{L}_{jk} \tag{11}$$

The power that is distributed from one or more substations to the demand regions should satisfy the scenario demand $d_{k\omega}$ and the export e_k, as in (5). Constraint (6) states that the power produced from the ROR generators $y^g_{i\omega}$ can not exceed the available generation capacity of the ROR generators from the water runoff for that specific scenario. The total power transmitted from an ROR generator to substations should also be equal to all the power produced by the ROR generator, as in (7). From (8), the aggregate power transmission from ROR generators to a substation, from the storage dams to the substation, and from other substations to this substation should respect the substation capacity $\bar{S}_j$. Per (9), the power that enters the substation is also equated to both the power that leaves the substation to local demand and export regions and to the power that leaves the same substation to other substations. The flow of power between the substations is limited by the substation-to-substation line capacity $\bar{S}_{pj}$, as in (10). Finally, from (11), the power distributed should not exceed the distribution line capacity $\bar{L}_{jk}$.

3 BI-LEVEL STOCHASTIC APPROXIMATION

The (OUTER) decision problem of determining the optimal spot-market policy is a general non-linear (possibly non-convex) stochastic optimization problem. This is solved using the Stochastic Approximation

(SA) method, which is a gradient-following method to search for local optima of stochastic optimization formulations. We apply the SA method to find a first-order optimal solution to the (OUTER) problem. The SA method is defined in terms of the iterative algorithm

$$\mathbf{e}^{(t+1)} = \mathbf{e}^{(t)} - \varepsilon_n \widehat{D_p}^{(t+1)}, \tag{12}$$

where the vector $\widehat{D_p}^{(t+1)}$ is an estimator of the gradient $\nabla_{\mathbf{e}} p(\mathbf{e})$ of $p(\mathbf{e})$ with respect to $\mathbf{e}$ at the current iterate $\mathbf{e}^{(t)}$. Construction of the gradient estimator $\widehat{D_p}$ is a crucial step in ensuring the SA iteration (12) converges to the desired set $\mathcal{E}^* = \{\mathbf{e} \mid \nabla_{\mathbf{e}} p(\mathbf{e}) = 0\}$. Since the r_k are deterministic and differentiable, the main step in constructing the gradient estimator $\widehat{D_p}$ is in estimating the gradient of the optimal objective value $c(\mathbf{e})$ of the inner optimization problem (INNER-I).

Let $\mathbf{x}^*(\mathbf{e})$ be an optimal solution to (INNER-I) for a given $\mathbf{e}$, namely

$$c(\mathbf{e}) = f^t \mathbf{x}^*(\mathbf{e}) + h(\mathbf{x}^*(\mathbf{e}), \mathbf{e}).$$

The gradient estimator $\widehat{D_p}$ is then constructed by observing

$$\nabla_{\mathbf{e}} c(\mathbf{e}) \quad = \quad \nabla_{\mathbf{e}} h(\mathbf{x}^*(\mathbf{e}), \mathbf{e}) \quad \approx \quad \nabla_{\mathbf{e}} h_n(\mathbf{x}_n^*(\mathbf{e}), \mathbf{e}, \Omega_n),$$

where $\mathbf{x}_n^*$ is the optimal solution to the SAA $h_n(\mathbf{x}, \mathbf{e}, \Omega_n)$ of the true second-stage cost $h(\mathbf{x}, \mathbf{e})$ of (INNER-I). Note that the first equality works because the first-stage problem (INNER-I) depends on $\mathbf{e}$ only through the second-stage's optimal objective value h. Let $\widehat{D_h}(\mathbf{e}) = \nabla_{\mathbf{e}} h_n(\mathbf{x}_n^*(\mathbf{e}), \mathbf{e}, \Omega_n)$. Hence, we set

$$\widehat{D_p}(\mathbf{e}) = \left(\sum_k \nabla_{\mathbf{e}} r_k(e_k) \right) - \widehat{D_h}(\mathbf{e}).$$

Each of the sample-path objectives $H(\mathbf{x}, \mathbf{e}, \omega)$ in (INNER-II) are linear programs, where the term $\mathbf{e}$ appears as a right-hand constant in the constraints (2). Results from parametric programming, detailed in Section 4, lead to the final form of the estimator $\widehat{D_p}(\mathbf{e})$:

$$\widehat{D_p}(\mathbf{e}) = \left(\sum_k \nabla_{\mathbf{e}} r_k(e_k) \right) - \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\pi}^*(\mathbf{x}_n^*(\mathbf{e}), \mathbf{e}, \omega_i). \tag{13}$$

Here, the gradient estimate $\widehat{D_h}(\mathbf{e})$ is set to the sample-average of the $\boldsymbol{\pi}^*(\mathbf{x}_n^*(\mathbf{e}), \mathbf{e}, \omega_i)$, which are the optimal dual vectors associated with the constraint (2) in (INNER-II) for the sampled realizations ω_i at the primal optimal $\mathbf{x}_n^*(\mathbf{e})$ of the (INNER-I SAA) approximation to the true (INNER-I) problem.

Algorithm 1 provides a description of the overall algorithm. The (OUTER) problem follows the SA method, and in each iteration the (INNER-I SAA) problem is solved with a sampled set of outcomes Ω_n using SAA techniques. The optimal primal solution x_n^* to (INNER-I SAA) and the associated duals $\boldsymbol{\pi}^*(\mathbf{x}_n^*(\mathbf{e}), \mathbf{e}, \omega_i)$ are obtained by solving the problem (INNER-I SAA) as a standard two-stage stochastic linear program. This is done by following an efficient method known in various forms as the Bender's decomposition approach, the L-shaped method or the cutting plane method (Ruszczyński 1997). This approach reformulates the (INNER-I SAA) program by replacing the function h_n with an auxiliary variable, say u. Then, starting from an initial point $\mathbf{x}$, the method solves the (INNER-II) program for each ω_i and gleans cuts to either the feasible region of $\mathbf{x}$ if (INNER-II) is infeasible for some ω_i at the given $\mathbf{x}$, or cuts to the auxiliary variable u to iteratively build up a piecewise-linear proxy for the h_n function. The cut-augmented program is solved to obtain the next candidate optimal solution $\mathbf{x}$, and the procedure is continued till a candidate solution $\mathbf{x}_n^*$ of sufficient accuracy is found. The function 2STAGESAA in Algorithm 1 provides a procedural sketch of this approach.

Algorithm 1 Bi-level Stochastic Approximation

Given:

- Maximum number of SA iterations T
- Constant τ in the definition of $\varepsilon_t = \frac{1}{t+\tau}$
- Constant n_0, η in the definition of $n_t = n_0 \eta^t$
- Termination criterion constants $\Delta_O, \Delta_I \ll 1$

Initialization

Set $t = 0$ and $\mathbf{e}^{(t)} = 0$ $\triangleright$ identify no-export as starting point

Estimation

while $\{\ t < T$ and $|p(\mathbf{e}^{(t)}) - p(\mathbf{e}^{(t+1)})| > \Delta_O\ \}$ **do**
    Compute step size $\varepsilon_t = 1/(t+\tau)$
    Sample new i.i.d. scenarios $\Omega^t_{n_t} = \{\omega^t_1, \dots, \omega^t_{n_t}\}$, where each scenario $\omega = \{d_\omega, g_\omega\}$
    Set $\widehat{D_h}^{(t)} = 2\textsc{StageSAA}\ (\mathbf{e}^{(t-1)}, \Omega^t_{n_t})$
    Set $\widehat{D_p}^{(t)} = \left(\sum_k \nabla_{\mathbf{e}} r_k(e_k^{(t-1)})\right) - \widehat{D_h}^{(t)}$
    Set $\mathbf{e}^{(t)} = \mathbf{e}^{(t-1)} - \varepsilon_t \widehat{D_p}^{(t)}$
    Increment $t \leftarrow t+1$
end while
Return $\mathbf{e}^{(t)}$

Two-stage Sample Average Approximation

function $2\textsc{StageSAA}(\mathbf{e}, \Omega_n)$
    Set inner iteration counter $u = 0$, and $\mathbf{x}^{(u)} = 0$
    while $\{u == 0\}$ or $\{|f^t(\mathbf{x}^{(u)} - \mathbf{x}^{(u-1)}) + h_n(\mathbf{x}^{(u)}, \mathbf{e}) - h_n(\mathbf{x}^{(u-1)}, \mathbf{e})| > \Delta_I\}$ **do**
        for $i = 1, \dots, n$ **do**
            Solve (INNER-II) with $\omega = \omega_i$
            if (INNER-II) is infeasible **then**
                Augment (INNER-I) with a feasibility cut from (INNER-II DUAL)
            else
                Augment (INNER-I) with an optimality cut from (INNER-II)
            end if
        end for
        Increment $u \leftarrow u+1$
        Solve the augmented (INNER-I) linear program to obtain $\mathbf{x}^{(u)}$
    end while
    $\mathbf{x}^*_n(\mathbf{e}) \leftarrow \mathbf{x}^{(u)}$
    $\widehat{D_h} \leftarrow \frac{1}{n} \sum_{i=1}^n \boldsymbol{\pi}^*(\mathbf{x}^*_n(\mathbf{e}), \mathbf{e}, \omega_i)$
    Return $\widehat{D_h}$
end function

The substitution of the n-sample SAA function h_n in place of h introduces a bias in the gradient estimator $\widehat{D_p}$ of gradient $\nabla_{\mathbf{e}} p$. Section 4 argues that this bias is of order $O_p(n^{-1/2})$. (The notation $O_p()$ is defined in the next section). This requires that the successive steps on the SA algorithm take increasing number of samples, i.e., $n_t \to \infty$ as $t \to \infty$, in order for the bias to die down. Theorem 1 provides that any rate of increase of n_t with respect to t is sufficient for the SA iterates to converge. Algorithm 1 picks the step-size sequence to be $\varepsilon_t = 1/(t+\tau)$ and the sample-size sequence to be $n_t = n_0\eta^t$, which are easily verified as satisfying the conditions of Theorem 1.

4 CONVERGENCE ANALYSIS

This section provides some preliminary results on the convergence of the SA iterations (12), starting with the quality of the estimator $\widehat{D_p}$ of the gradient $\nabla_{\mathbf{e}} p$ of the profit function p in the (OUTER) problem.

4.1 Estimating Gradient of h

Consider the dual of the (INNER-II) linear program:

$$\begin{array}{rclll} H_D(\mathbf{x},\mathbf{e},\omega) & = & \max\limits_{\boldsymbol{\pi},\boldsymbol{\nu}} & (d_\omega+\mathbf{e}-T_1\mathbf{x})^t\boldsymbol{\pi} \;+\; g_\omega^t\boldsymbol{\nu}_1 \;-\; T_2^t\boldsymbol{\nu}_2 & \text{(INNER-II DUAL)} \\ & & \text{s.t.} & U_1^t\boldsymbol{\pi} \;+\; \boldsymbol{\nu}_1 \;+\; U_2^t\boldsymbol{\nu}_2 \;\geq\; c^t & (14) \\ & & & \boldsymbol{\pi}\text{ unbounded, } \boldsymbol{\nu}_1,\boldsymbol{\nu}_2 \geq 0. & \end{array}$$

The dual variable $\boldsymbol{\pi}$ corresponds to the constraint containing $\mathbf{e}$, $\boldsymbol{\nu}_1$ the constraint with g_ω and the $\boldsymbol{\nu}_2$ represents the rest of the constraints. The feasible region of the (INNER-II DUAL) program, defined by the constraints (14), do not depend on the samples $\boldsymbol{\omega}$. The optimal objective solution $(\boldsymbol{\pi}^*,\boldsymbol{\nu}_1^*,\boldsymbol{\nu}_2^*)$, if finite, is one of the $\mathcal{V}$ finitely-many vertices of the corresponding polytope. If the optimal solution is unbounded for any given ω, $\mathbf{x}$ or $\mathbf{e}$, then the primal problem (INNER-II) is infeasible, implying that a feasibility cut has to be added in solving (INNER-I), as done in Step 17 of Algorithm 1.

Observe that $\nabla_{\mathbf{e}} H_D(\mathbf{x},\mathbf{e},\omega) = \boldsymbol{\pi}^*(\mathbf{x},\mathbf{e},\omega)$ almost surely (a.s.), except when there exists multiple optima. The independence of the feasible region also implies that the function $H_D(\mathbf{x},\mathbf{e},\omega)$ is piecewise-linear in $\mathbf{e}$, since the optimal solution $\boldsymbol{\pi}^*$ of (INNER-II DUAL), which is also the gradient of H_D, can only take a finite number of values, the $\mathcal{V}$ vertices of the polytope. For an optimal set of primal and dual variables $\mathbf{y}_\omega^*,\boldsymbol{\pi}^*,\boldsymbol{\nu}_1^*$ and $\boldsymbol{\nu}_2^*$, the objective values of the primal (INNER-II) and its dual (INNER-II DUAL) coincide. Thus, we have sketched the proof of the following lemma.

Lemma 4.1. *The sample-path function $H(\mathbf{x},\mathbf{e},\omega)$ is piecewise-linear continuous in $\mathbf{e}$, with at most a finite number of breakpoints.*

Lemma 4.1 shows that the sample-path function H is continuous in $\mathbf{e}$, and this is a sufficient condition (Glasserman 1991b) for interchanging the gradient $\nabla_{\mathbf{e}}$ and the expectation $\mathbb{E}$ operators.

Lemma 4.2. *The gradient $\nabla_{\mathbf{e}} h(\mathbf{x},\mathbf{e})$ of the expected second-stage cost $h(\mathbf{x},\mathbf{e})$ in (INNER-I) satisfies*

$$\nabla_{\mathbf{e}} h(\mathbf{x},\mathbf{e}) = \nabla_{\mathbf{e}}\mathbb{E}_\omega[H(\mathbf{x},\mathbf{e},\omega)] = \mathbb{E}_\omega[\nabla_{\mathbf{e}} H(\mathbf{x},\mathbf{e},\omega)].$$

Lemma 4.2 shows us an easy path to estimating the gradient $\nabla_{\mathbf{e}} h(\mathbf{x},\mathbf{e})$ consistently, i.e., with zero bias, at any $\mathbf{x}$ and $\mathbf{e}$ by using the sample-average analogue $\nabla_{\mathbf{e}} h_n(\mathbf{x},\mathbf{e},\Omega_n)$ for a sample set of n-scenarios Ω_n; that is, with the estimator

$$\nabla_{\mathbf{e}} h_n(\mathbf{x},\mathbf{e}) = \frac{1}{n}\sum_{i=1}^{n}\nabla_{\mathbf{e}} H(\mathbf{x},\mathbf{e},\omega_i) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\pi}^*(\mathbf{x},\mathbf{e},\omega_i),$$

where the last equality is due to the arguments preceding Lemma 4.1.

Note, however, that the gradient $\nabla_{\mathbf{e}} p$ desired for the application of the SA method on (OUTER) requires that we estimate $\nabla_{\mathbf{e}} h(\mathbf{x}^*(\mathbf{e}),\mathbf{e})$, i.e., the gradient is to be estimated at the true optimal solution $\mathbf{x}^*(\mathbf{e})$ to

(INNER-I) for a given $\mathbf{e}$ while (INNER-I SAA) only provides an estimate $\mathbf{x}_n^*$ of the true optimal $\mathbf{x}^*$. Hence, estimating $\nabla_{\mathbf{e}} h(\mathbf{x}^*(\mathbf{e}),\mathbf{e})$ with $\widehat{D_{h_n}} = \nabla_{\mathbf{e}} h_n(x_n^*(\mathbf{e}),\mathbf{e},\Omega_n)$ induces a bias.

How big is this bias? Shapiro (2003) tells us that $|\mathbf{x}_n^*(\mathbf{e}) - \mathbf{x}^*(\mathbf{e})| = O_p(n^{-1/2})$ for two-stage stochastic programs of form (INNER-I). (A sequence of random variables A_i is said to be $O_p(a_i)$ if for every $\varepsilon > 0$, there exists a $K_\varepsilon < \infty$ such that $P(|A_i| < K_\varepsilon a_i) \geq 1-\varepsilon$.) So, if the $\nabla_{\mathbf{e}} h$ function were sufficiently smooth in $\mathbf{x}$, then the following result holds by the continuous mapping theorem (Mann and Wald 1943).

Lemma 4.3. *If $\nabla_{\mathbf{e}} h(\mathbf{x},\mathbf{e})$ is continuous in $\mathbf{x}$ for a fixed $\mathbf{e}$, then the bias $\mathbf{b}$ satisfies*

$$\mathbf{b} = |\widehat{D_{h_n}}(\mathbf{e}) - \nabla_{\mathbf{e}} h(\mathbf{x}^*(\mathbf{e}),\mathbf{e})| = O_p(n^{-1/2}).$$

The $\nabla_{\mathbf{e}} h$ function is expected to satisfy the conditions of Lemma 4.3 given the continuity of H as discussed in the section leading up to Lemma 4.2.

4.2 Convergence of Algorithm 1

Lemma 4.3 establishes that the gradient estimator $\widehat{D_p}$ suffers a bias $\mathbf{b}$ in estimating $\nabla_{\mathbf{e}} p$ that drops with n, the total sampled scenarios, as $n^{-1/2}$. The SA method thus requires that the samples n_t per iteration t grow as t grows in order to eliminate the bias and converge to the true solution $\mathbf{e}^*$ of (OUTER).

Define $\boldsymbol{\delta}_t = \widehat{D_h}^{(t)} - \mathbf{b}_t$, where $\mathbf{b}_t$ is the bias in iteration t. Conditional on the sample path leading up to $\mathbf{e}^{(t)}$, the random variables $\boldsymbol{\delta}_t$ have zero mean, and $\mathbb{E}_{(t)}[\boldsymbol{\delta}_t^2] = \mathrm{Var}_{(t)}(\sum_i \boldsymbol{\pi}^*(\mathbf{x}_t^*(\mathbf{e}^{(t)}),\mathbf{e}^{(t)},\omega_i)/n_t)$. (The subscripts (t) on the mean and variances estimators indicate the conditioning on the sample path that has led up to the t-th iterate.) Then, the second-moment of $\boldsymbol{\delta}$ drops to zero as $n_t \to \infty$. To see this, note that the random quantity $\boldsymbol{\pi}^*(\mathbf{x}_t^*(\mathbf{e}^{(t)}),\mathbf{e}^{(t)},\omega)$ has its support on the $\mathcal{V}$ vertices of the feasible region of (INNER-II DUAL). Hence, the second-moment of δ is bounded above by the term given in the lemma below, where σ is an upper bound on the variance of this (unknown) finite-dimensional distribution.

Lemma 4.4. *The (conditional) second-moment of $\boldsymbol{\delta}$ is bounded above as*

$$\mathbb{E}_{(t)}[\boldsymbol{\delta}_t^2] \leq \frac{\sigma^2}{n_t}.$$

An upper bound on the variance of a probability mass function over a finite support set is straightforward to derive; for example, choose the two most distant points on the finite vertex set and assign them equal probability. We have all the tools needed to prove the main result of this section, which provides a sufficient condition that the n_t needs to satisfy in order for the SA method to converge a.s.

Theorem 1 Suppose the following conditions are satisfied:

- The step-size sequence ε_t satisfy $\sum_t \varepsilon_t^2 < \infty$, but $\sum_t \varepsilon_t$ diverges to ∞;
- The sample-size sequence $n_t \to \infty$ as $t \to \infty$, i.e., grows unboundedly; and
- The gradient $\nabla_{\mathbf{e}} h(\mathbf{x},\mathbf{e})$ is continuous in $\mathbf{x}$ for a fixed $\mathbf{e}$.

Then, the SA iterates $\mathbf{e}^{(t)}$ from (12) converge a.s. to the set $\mathcal{E}^* = \{\mathbf{e} \mid \nabla_{\mathbf{e}} p(\mathbf{e}) = 0\}$.

Proof sketch. The desired result follows from Theorem 5.3.1 in Kushner and Clark (1978), which proves convergence of the general SA iterations of the form (12) for the so-called martingale-noise case. A key assertion is condition A2.2.4" of Theorem 5.3.1 that requires the sequence $\{\varepsilon_t \boldsymbol{\delta}_t\}$ to satisfy condition (15) below, where, again, $\boldsymbol{\delta}_t$ represents the noise in estimating the gradient conditional on the history up to the current iterate $\mathbf{e}^{(t)}$. This, together with the condition that $\lim_{t\to\infty} \mathbf{b}_t = 0$ a.s., which follows from Lemma 4.3, enables us to satisfy the conditions of Theorem 5.3.1, and thus the iterates of the unconstrained problem converge. The (conditional) random variables δ_t have zero mean, and by Lemma 4.4, $\mathbb{E}_{(t)}[\delta_t^2] = O(n_t^{-1})$. Next, considering the sequence $\{M_j = \sum_{k=t}^{j} \varepsilon_k \boldsymbol{\delta}_k,\ j \geq t\}$, which is a martingale, we have from Doob's

Martingale Inequality (see, e.g., Jacod and Protter (2003), Theorem 26.1)):

$$\mathbb{P}\left[\sup_{t\le j\le T}\|\sum_{s=t}^{j}\varepsilon_s\boldsymbol{\delta}_s\|\ge\varepsilon\right]=\mathbb{P}\left[\sup_{s\le j\le T}\|M_j\|>\varepsilon\right] \le \frac{K}{\varepsilon^2}\,\mathbb{E}\|M_T\|^2$$

$$= \frac{K}{\varepsilon^2}\,\mathbb{E}\|\sum_{s=t}^{T}\varepsilon_s\boldsymbol{\delta}_s\|^2 \le \frac{K'}{\varepsilon^2}\sum_{s=t}^{T}O(\varepsilon_s^2 n_s^{-1}),$$

where $\|\cdot\|$ represents the l_2-norm and the last equality relies on the fact that the noise terms are i.i.d. From the choice of the parameter sequences ε_t and n_t, and letting $T\to\infty$, the right-hand side tends to 0 as $t\to\infty$, which renders the condition A2.2.4" in Kushner and Clark (1978) that for any $\varepsilon>0$

$$\lim_{t\to\infty}\mathbb{P}\left[\sup_{j\ge t}\|\sum_{s=t}^{j}\varepsilon_k\boldsymbol{\delta}_k\|\ge\varepsilon\right]=0. \tag{15}$$

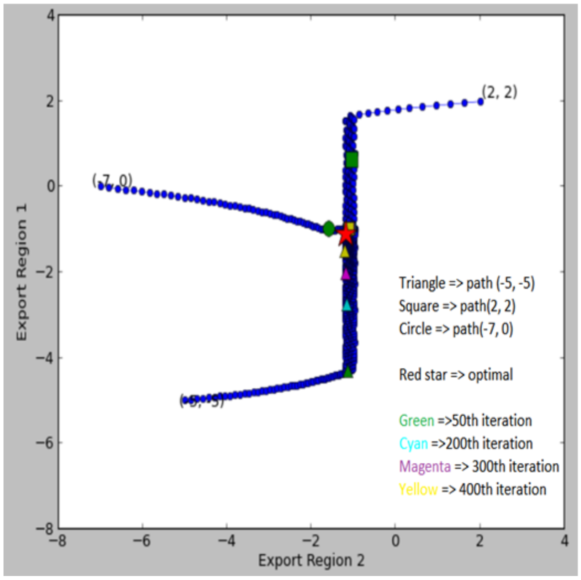
This completes the proof. □

5 NUMERICAL EXPERIMENTS

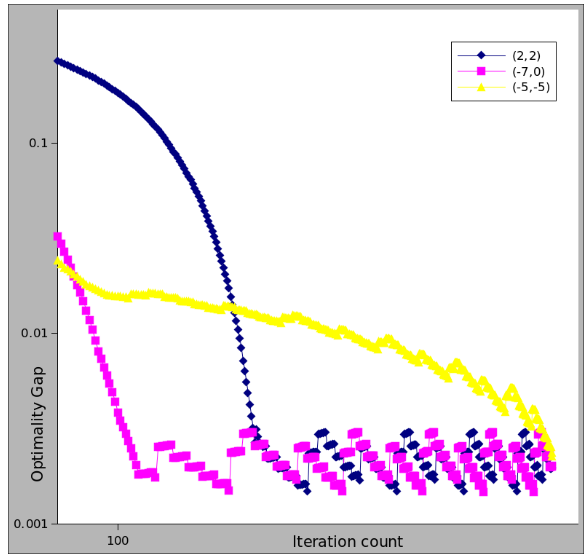
The problems (OUTER, INNER-I, and INNER-II) were solved using Python software version 2.7 and Gurobi software version 6.0.4 interface. The problems were run on Intel(R) Core(TM) i-7-5500U CPU 2.4 GHz machine. The parameters presented below were chosen based on historic hydroelectric power system operations data available in the public domain (US Energy Information Administration 2016).

- Number of storage dams ($\mathcal{H}$) = 7
- Number of ROR hydroelectric generators ($\mathcal{R}$) = 7
- Number of substations ($\mathcal{S}$) = 14
- Number of demand and export regions ($\mathcal{K}$) = 14
- Design capacity of hydroelectric dam ($\bar{G}_i$) in range of 10000 - 16000MW
- Cost of hydroelectric dam = 100 - 230 dollars per MW
- ROR generator design capacity ($g_{i\omega}$) = 1000 - 9000 MW
- Cost of ROR generator = 100 - 160 dollars per MW
- Transmission capacity from storage and ROR generators to substation ($\bar{F}_{ij}$) = 1 - 1.5 GW per day
- Transmission cost from storage and ROR generators to substation = 100 - 1500 dollars per MW
- Capacity of substation ($\bar{S}_j$) = 1000 - 50,000 MW
- Cost of substation = 10 - 250 dollars per MW
- Capacity of transmission line between substations ($\bar{S}_{pj}$)= 1000 - 15000 MW per day
- Cost of transmission line between substations = 100 - 1500 dollars per MW
- Capacity of distribution line ($\bar{L}_{jk}$) = 100 - 45000 MW
- Cost of distribution line = 10 - 50 dollars per MW
- Regional demand mean ($d_{k\omega}$) sampled uniformly from [50, 3000] MW per day
- ROR generation $g_{k\omega}$ sampled uniformly from [60, 100]% of ROR generator design capacity

Two of the demand regions were selected as export interconnects for the experimental analysis. We illustrate the effectiveness of the method using the graphical plot of three sample paths of the algorithm. The three starting export/import amounts of the two regions were selected arbitrarily, and the bi-level SA algorithm was run for $T=500$ iterations each. The results are presented in Figure 2, where the leftmost plot, Figure 2(a), shows the three resulting sample paths that the bi-level SA method follows. All three paths converge onto the same optimal solution (marked with a star). In addition, every 100-th iterate is marked on the sample paths. It is apparent that two of the sample paths converge very quickly to a neighborhood



(a) Three Sample Paths



(b) Convergence of Relative Optimality Gap

Figure 2: The performance of the Bi-level SA algorithm, with three distinct starting points. The figure on the right is in log-log axes.

close to the eventual optimal iterate, while the third is slower. This is also borne out by the rightmost plot, Figure 2(b), which shows a log-log plot of the relative optimality gap over the iterates. The plot indicates that there is a faster convergence from two of the three initial points, each of which "circles" the true optimal value for most of the later iterations, while the third sample path takes a while to converge.

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