Proceedings of the 2015 Winter Simulation Conference
L. Yilmaz, W. K. V. Chan, I. Moon, T. M. K. Roeder, C. Macal, and M. D. Rossetti, eds.

ON THE ROBUSTNESS OF FISHMAN'S BOUND-BASED METHOD FOR THE NETWORK RELIABILITY PROBLEM

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ABSTRACT

Static network unreliability computation is an NP-hard problem, leading to the use of Monte Carlo techniques to estimate it. The latter, in turn, suffer from the rare event problem, in the frequent situation where the system's unreliability is a very small value. As a consequence, specific rare event event simulation techniques are relevant tools to provide this estimation. We focus here on a method proposed by Fishman making use of bounds on the structure function of the model. The bounds are based on the computation of (disjoint) *mincuts* disconnecting the set of nodes and (disjoint) *minpaths* ensuring that they are connected. We analyze the robustness of the method when the unreliability of links goes to zero. We show that the conditions provided by Fishman, based on a bound, are only sufficient, and we provide more insight and examples on the behavior of the method.

1 INTRODUCTION

Static network unreliability evaluation is a problem with applications in a wide range of areas. It translates directly in telecommunications, where it is important to know the probability that a given set of nodes stay connected. As an extension of the model, by adding capacities, we could also wonder what are the capacities of the system to support communications between two given nodes. Finally, in electricity management, the probability of blackout has to be estimated to avoid major financial losses, and these static models appear as relevant tools in this family of problems too.

The type of model usually considered is an undirected graph where links can fail but with perfect nodes, even if failing nodes can also be considered without major loss of generality (Cancela, El Khadiri, and Rubino 2009). A state of the model is then given by the state of each link (failed or not) from which we can determine if the network works or not. But the state space actually increases exponentially with the number of links, and determining the unreliability is actually known to be an NP-hard problem (Ball 1986) in general. For this reason, Monte Carlo simulation (Asmussen and Glynn 2007) is a relevant approximation tool.

Another issue is the "stiffness" of the model: in most practical cases, failures are rare (even failure of links), and a failure of the system is (very) rarely observed. As a consequence, standard Monte Carlo simulation is inefficient because requiring a too large sample size to observe the event once in average, and even larger to obtain a confidence interval. So-called acceleration techniques have to be used to improve

Monte Carlo efficiency. Rare event simulation (Rubino and Tuffin 2009) has been extensively used and numerous methods have been developed for this specific network unreliability estimation. Our goal here is not to review all methods, the reader can have a look at many existing methods described in Cancela, El Khadiri, and Rubino (2009). More recent ones are Botev, L'Ecuyer, and Tuffin (2013), Botev et al. (2013), Cancela et al. (2014), L'Ecuyer et al. (2011), Murray, Cancela, and Rubino (2013), Tuffin, Saggadi, and L'Ecuyer (2014), sometimes working on extensions with dependent failures. Among those methods, some are known to perform very well in practice, even if we could always find examples in favor of one with respect to the others. Examples in this class are the sampling method of Fishman (1986) based on bounds on the structure function, the generalized splitting technique of Botev et al. (2013), the splitting method of Murray, Cancela, and Rubino (2013), the so-called turnip method in Gertsbakh and Shpungin (2010), the zero-variance IS in L'Ecuyer et al. (2011), and the recursive variance reduction technique in Cancela and El Khadiri (1995) and the combination of the last two techniques in Cancela et al. (2014). Those were compared in Cancela et al. (2014) on a simple benchmark, the dodecahedron topology.

We focus in this paper on the first above technique proposed by Fishman (1986). The idea of this method is to use bounds on the structure function describing, in terms of configurations of link states, if the considered nodes are connected. The bounds are based on the computation of (independent) *mincuts* disconnecting the set of nodes and (independent) *minpaths* ensuring that they are connected. The method is known to be among the most efficient ones in the literature. Our goal in this paper is to provide an analysis of the robustness of this method (see Glynn, Rubino, and Tuffin (2009) for a global view of the topic). In Fishman (1986), a necessary and sufficient condition on the set of mincuts and minpaths was provided, but it was done to ensure that a *bound* of the relative error is bounded whatever the reliabilities of the links. We are going first to illustrate that a Bounded Relative Error (BRE) property can be satisfied even when the condition is not fulfilled. We will describe then more closely what happens by analyzing the relative variance of the estimators. Finally, we will prove that even if in many cases we are able to determine an appropriate set of (independent) mincuts and minpaths in order to obtain a BRE, there exist topologies for which such a construction is not possible.

The remaining of the paper is organized as follows. In Section 2 we describe the main characteristics of the method proposed in Fishman (1986), focusing on whet interests us here, the robustness of the estimators. We start our robustness analysis also in that section, and we develop it fully in Section 3. Section 4 provides further discussions about BRE conditions. Section 5 concludes the paper.

2 FISHMAN'S BOUND-BASED METHOD AND ASSOCIATED RESULTS

2.1 General Model Description and Definitions

Consider an undirected graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{K})$ where $\mathscr{V}$ is the set of nodes, $\mathscr{E} = \{1, \ldots, E\}$ is the set of links connecting nodes, and $\mathscr{K}$ is a subset of the node-set, called the terminal-set. The graph is assumed to be $\mathscr{K}$-connected, which means that there is at least a path connecting any pair of nodes in the terminal-set $\mathscr{K}$.

We assume that links can fail. We denote by q_e the probability that link $e \in \mathscr{E}$ doesn't work, or stated otherwise, is down. Failure of links are assumed to be independent. Nodes on the other hand are assumed perfect in the sense that they do not fail. The goal is to determine the *unreliability* defined as the probability that the terminal-set $\mathscr{K}$ is not connected given the link failure randomness (note that in Fishman (1986) only the connection between two nodes is considered, but this can be easily generalized as we do here).

The random vector-state, or configuration, of the network is given by the vector $X = (X_1, \ldots, X_E)$ where for $1 \leq e \leq E$, X_e is a Bernoulli random variable whose value is 1 if link e is working, or up, and 0 if it is failed, or down. We aim at investigating the probability $r(\mathscr{G})$ that all nodes in the terminal-set $\mathscr{K}$ are connected. We define a structure function Φ of $\{0,1\}^E$ in $\{0,1\}$ such that $\Phi(x) = 1$ if all nodes in $\mathscr{K}$ are connected when the configuration is $x = (x_1, \ldots, x_E)$, and $\Phi(x) = 0$ otherwise. We then have that $\Phi(X)$ is a Bernoulli random variable such that $\mathbb{E}[\Phi(X)] = r(\mathscr{G})$. In this paper we consider the computation of $r(\mathscr{G})$ or equivalently, the computation of the *unreliability* parameter

$$q(\mathscr{G}) = 1 - r(\mathscr{G}) = \mathbb{E}\big(1 - \Phi(X)\big). \tag{1}$$

Let Ω be the set of all the configurations, partitioned into $\mathscr{U} = \Phi^{-1}(1)$, the configurations where the system is up, and $\mathscr{D} = \Phi^{-1}(0)$, the configurations where the system is down. The total number of configurations is 2^E, thus increasing exponentially with the number E of links. Actually, the computation of $q(\mathscr{G})$ is known to be NP-hard (Ball 1986). Exact combinatorial methods and bounding procedures, together with reduction techniques that allow to diminish the size of the models become quickly limited (Rubino 1998) as the size of the network increases. We can indeed observe that in communication networks, model sizes are often very large. Then Monte Carlo simulation methods (Asmussen and Glynn 2007) are alternatives leading to estimate $q(\mathscr{G})$ when $\mathscr{E}$ is of moderate to large size (Cancela, El Khadiri, and Rubino 2009).

Before describing Fishman's method, we recall a few definitions from graph theory. A *cut* (or $\mathscr{K}$-cut) in $\mathscr{G}$ is a subset $\mathscr{C}$ of links such that when all the links in $\mathscr{C}$ are failed, then the nodes of $\mathscr{K}$ are not all connected (and thus, the system is necessarily failed). A *mincut* $\mathscr{C}$ is a cut such that no strict subset of $\mathscr{C}$ is a cut. Similarly, a *path* (or $\mathscr{K}$-path) in the graph $\mathscr{G}$ is a subset P of links i such that when these links are up ($X_i = 1$ for all $i \in P$), the nodes in $\mathscr{K}$ are in the same connected component of the resulting graph, whatever the state of other links. A *minpath* of $\mathscr{G}$ is a path $P = P(\mathscr{G})$ for which no strict subset of $P(\mathscr{G})$ is also a path. So, when all links in a path are up, the system is also up, and when all links in a cut are down, so is the system.

2.2 Fishman's Method

The principle developed in Fishman (1986) consists in selecting two subsets Ω_1 and Ω_3 of configurations (which will be explicitly given below) such that $\Omega_1 \subseteq \mathscr{U}$ and $\Omega_3 \subseteq \mathscr{D}$. We also denote $\Omega_2 = \Omega \setminus \big(\Omega_1 \cup \Omega_3\big)$.

We can then remark that the reliability $r(\mathscr{G})$ of $\mathscr{G}$ can be written in the following way:

$$\begin{aligned} r(\mathscr{G}) = \mathbb{P}(X \in \mathscr{U}) &= \mathbb{P}(X \in \Omega_1) + \sum_{x \in \Omega_2} \Phi(x)\mathbb{P}(X = x) \\ &= \mathbb{P}(X \in \Omega_1) + \sum_{x \in \Omega_2} \Phi(x)\mathbb{P}(X = x \mid X \in \Omega_2)\mathbb{P}(X \in \Omega_2) \\ &= \mathbb{P}(X \in \Omega_1) + \mathbb{P}(X \in \Omega_2)\mathbb{E}(\Phi(X) \mid X \in \Omega_2). \end{aligned}$$

Denoting $\pi_i = \mathbb{P}(X \in \Omega_i)$, $i = 1,2,3$, we have

$$\pi_1 \le r(\mathscr{G}) \le \pi_1 + \pi_2, \quad \text{and} \quad r(\mathscr{G}) = \pi_1 + \pi_2 \mathbb{E}(\Phi(X) \mid X \in \Omega_2).$$

We can estimate the reliability of the system, *conditional on the fact that its configuration belongs to the subset* Ω_2, $\mathbb{E}(\Phi(X) \mid X \in \Omega_2)$, using

$$\widetilde{rc}_K = \frac{1}{K}\sum_{k=1}^{K} \Phi(Y^{(k)}),$$

where $Y^{(k)}, \ldots, Y^{(k)}$ are K independent copies of the random vector $Y \in \Omega$ whose distribution is

$$\mathbb{P}(Y = y) = \begin{cases} \dfrac{\mathbb{P}(X = y)}{\pi_2} & \text{if } y \in \Omega_2, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we sample the configurations conditionally to the fact that we are in Ω_2. It is an application of conditional Monte Carlo (Asmussen and Glynn 2007). This gives the unbiased estimator $\widetilde{r}_K$

of $r(\mathscr{G})$, $\widetilde{r}_K = \pi_1 + \pi_2 \widetilde{rc}_K$. For the sake of conciseness, we omit the details concerning the way this can be implemented, which needs algorithms to generate minpaths and mincuts, and a procedure for conditionally sampling the network configuration under the condition of being in subset Ω_2. We refer to Fishman (1986) for these issues.

To make the connection with the notation in Fishman (1986), let us rename $B = \pi_1$ and $A = \pi_1 + \pi_2$; then,

$$B \le r(\mathscr{G}) \le A \quad \text{ and } \mathbb{V}(\widetilde{r}_K) = \frac{\big[r(\mathscr{G}) - B\big]\big[A - r(\mathscr{G})\big]}{K}.$$

2.3 Bound Analysis

In order to choose Ω_1 and Ω_3, we start by building I disjoint minpaths, denoted $P_1, P_2, \dots, P_I$ (the graph is $\mathscr{K}$-connected, so, there is at least one such minpath). We denote by p_i the number of links in P_i, that is, $p_i = |P_i|$. We define Ω_1 as the set of configurations where at least one of the I minpaths is up. A path is said to be up iff all its links are up. Otherwise, the minpath is down.

As announced in the abstract, we focus here on the robustness of the discussed estimator. To make the discussion as clear as possible, we will consider only the *homogeneous* context where all links behave in the same way regarding its functioning conditions, that is, where the individual unreliability of each link e is independent of e, denoted $q_e = \varepsilon$ here. In the rest of the text, any expression such as $o(\varepsilon)$ or $o(\varepsilon^\ell)$ is to be understood when $\varepsilon \to 0$.

We now make more explicit the way the lower bound B on ε:

$$\begin{aligned}
B = \pi_1 = \mathbb{P}(X \in \Omega_1) &= 1 - \mathbb{P}(\text{all } P_i\text{s are down}) \\
&= 1 - \prod_{i=1}^{I} \mathbb{P}(P_i \text{ is down}) \qquad \text{(minpaths are disjoint)} \\
&= 1 - \prod_{i=1}^{I} \big(1 - \mathbb{P}(P_i \text{ is up})\big) = 1 - \prod_{i=1}^{I} \big(1 - \prod_{e \in P_i} (1-\varepsilon)\big) \\
&= 1 - \prod_{i=1}^{I} \big[1 - (1-\varepsilon)^{p_i}\big] = 1 - \prod_{i=1}^{I} \big[1 - \big(1 - p_i\varepsilon + o(\varepsilon)\big)\big] \\
&= 1 - \eta\varepsilon^I + o(\varepsilon^I),
\end{aligned}$$

where $\eta = p_1 p_2 \cdots p_I$.

Now, let us choose a set of disjoint mincuts $C_1, C_2, \dots, C_J$ with respective sizes $c_1, c_2, \dots, c_J$. We say that a mincut is down iff all its links are down. Otherwise, we say that it is up.

Define Ω_3 as the set of all configurations such that at least one C_j is down. We have

$$\begin{aligned}
\pi_3 = \mathbb{P}(X \in \Omega_3) = 1 - \mathbb{P}(\text{all } C_j\text{s are up}) &= 1 - \prod_{j=1}^{J} \mathbb{P}(C_j \text{ is up}) \qquad \text{(mincuts are disjoint)} \\
&= 1 - \prod_{j=1}^{J} \big(1 - \mathbb{P}(C_j \text{ is down})\big) = 1 - \prod_{j=1}^{J} \big(1 - \varepsilon^{c_j}\big).
\end{aligned}$$

Denoting by c the minimal size of the mincuts $C_1, C_2, \dots, C_J$, that is, $c = \min\{c_1, c_2, \dots, c_J\}$, and by ν the number of mincuts C_j having size c, we have

$$\pi_3 = 1 - \sum_{j=1}^{J} \varepsilon^{c_j} + \sum_{\substack{h,\ell:\ h,\ell=1,\dots,n \\ h<\ell}} \varepsilon^{c_h + c_\ell} + \cdots + (-1)^J \varepsilon^{c_1 + \cdots + c_J} = 1 - \nu\varepsilon^c + o(\varepsilon^c).$$

Since $A = \pi_1 + \pi_2$ and $\pi_1 + \pi_2 = 1 - \pi_3$, we have $A = 1 - \pi_3$, so,

$$A = 1 - \nu\varepsilon^c + o(\varepsilon^c).$$

2.4 Relative Error Bound Analysis (Fishman 1986)

In this paper, we will focus on the unreliability estimation, which is the interesting case in practice. Of course, computing $r(\mathscr{G})$ and $q(\mathscr{G}) = 1 - r(\mathscr{G})$, are equivalent procedures, mathematically speaking, but statistically speaking, it is obviously more useful to control the *relative error* when estimating $q(\mathscr{G})$ than when estimating $r(\mathscr{G})$. The reason is that, in almost all cases of interest, $q(\mathscr{G})$ is a (very) small value, while $r(\mathscr{G})$ is (very) close to 1.

The unbiased estimator of the unreliability can be written $\widetilde{q}_K = 1 - \widetilde{r}_K$, so, $\mathbb{V}(\widetilde{q}_K) = \mathbb{V}(\widetilde{r}_K)$. The relative error when estimating the unreliability $q(\mathscr{G})$, denoted here RE, is then

$$RE = \psi \frac{\sqrt{\mathbb{V}(\widetilde{r}_K)}}{q(\mathscr{G})},$$

so,

$$RE^2 = \psi^2 \frac{\mathbb{V}(\widetilde{r}_K)}{(q(\mathscr{G}))^2} = \psi^2 \frac{\left[r(\mathscr{G}) - B\right]\left[A - r(\mathscr{G})\right]}{K(1 - r(\mathscr{G}))^2},$$

where, for instance, $\psi = 1.96$ if we are interested in a confidence interval with confidence level 95% (Asmussen and Glynn 2007).

Now, looking at the last expression of RE^2, if one defines function f on the interval $[B, A]$ by

$$f(x) = \frac{(x - B)(A - x)}{(1 - x)^2},$$

we see that

$$\max_{x \in [B,A]} f(x) = \frac{(A - B)^2}{4(1 - A)(1 - B)}.$$

This means that

$$RE^2 \le \frac{\psi^2}{4K} \frac{(A - B)^2}{(1 - A)(1 - B)}. \tag{2}$$

This is proved in Fishman (1986), where it is the basis of the analysis of the method.

3 ROBUSTNESS ANALYSIS

Robustness refers to the behavior of the estimator of the unreliability $q(\mathscr{G})$ when the target becomes smaller and smaller. To formalize the idea, we first use a *rarity parameter* that controls how rare the wanted value is. In the case of the models we are considering here, the common unreliability of the links ε is the natural choice. Denoting by $\|x\|$ the norm $\|x\| = x_1 + \cdots + x_E$, the expression

$$q(\mathscr{G}) = \sum_{x:\, x \in \mathscr{D}} \mathbb{P}(\Phi(x) = x) = \sum_{x:\, x \in \mathscr{D}} (1 - \varepsilon)^{\|x\|} \varepsilon^{1 - \|x\|}$$

shows that $q(\mathscr{G})$ is a polynomial in ε. It is then easy to see that the $\mathscr{K}$-connectivity of $\mathscr{G}$ implies that

$$q(\mathscr{G}) = m\varepsilon^b + o(\varepsilon^b), \tag{3}$$

where $b, m \in \mathbb{N}$, and $b, m \ge 1$. Parameter b is called the graph's *breadth*. It is defined as the size of a mincut having minimal size (not to be confused with c, defined at the end of Subsection 2.3, which is the minimal size of a set of J user-chosen mincuts). This means, for instance, that $1 \le b \le c$. A mincut is said to be *optimal* if its size is b. Parameter m is simply the number of mincuts having size b. In Fig. 1, $b = 2$ and $m = 3$; in Fig. 2, $b = 2$ and $m = 2$; in Fig. 3, $b = 4$ and $m = 4$. Expression (3) implies that $q(\mathscr{G}) \to 0$ as $\varepsilon \to 0$.

Now, we want to know how $\widetilde{q}_K$ behaves as $\varepsilon \to 0$. The most important property of an estimator in this context is the Bounded Relative Error (BRE) one. This holds for any estimator E of the probability p of an event for which a rarity parameter θ is defined. If $p \to 0$ when $\theta \to \theta_0$, then, E has the BRE property with respect to θ iff the Relative Error of E remains bounded when $\theta \to \theta_0$. See Glynn, Rubino, and Tuffin (2009) and L'Ecuyer et al. (2010) for more informations on this topic.

Now, concerning the BRE property of the method of Fishman, a basic remark here is that Fishman develops his analysis of the Relative Error of his technique by analyzing how the upper bound given in Relation (2) depends on ε. More specifically, he finds conditions under which the upper bound remains bounded as $\varepsilon \to 0$. We will show that these conditions are not always satisfied, and then, that both BRE and not BRE, are possible. Actually, the analysis in Fishman (1986) considers the more general heterogeneous case where the links do not necessarily have the same individual reliabilities. For our purpose, it will be easier to discuss what happens in the simpler homogeneous case.

In this paper, we analyze the behavior of the relative error *RE* directly, as a function of ε. Observe first that the asymptotic development in (3) suggests that mincuts play the critical role when looking at the unreliability because the probability that none of the links in every optimal mincut (those having b links) works is of the same order of magnitude than $q(\mathscr{G})$ (nothing like this happens for the minpaths). In other words, this explains the different role in accuracy of the two bounds, due to the different role played by minpaths and mincuts in the rare case.

First, recall that the user starts by choosing $I \ge 1$ disjoint minpaths and $J \ge 1$ disjoint mincuts. We then have for the bounds A and B the representations

$$B = 1 - \eta\varepsilon^I + o(\varepsilon^I), \quad \text{and} \quad A = 1 - \nu\varepsilon^c + o(\varepsilon^c),$$

where $\eta = p_1 p_2 \cdots p_I$, $c = \min\{c_1, \dots, c_J\}$, and ν is the number of mincuts in the selection $\{C_1, \dots, C_J\}$ having size c. It is useful to recall that from the definitions of paths and cuts, we necessarily have $I \le c$. More precisely, we have, also by definition, $I \le b \le c$.

3.1 Analysis of the Bound of The *RE*

Let us rewrite the upper bound in (2) as $RE^2 \le \frac{\psi^2}{4K}\beta$, where

$$\beta = \frac{(A-B)^2}{(1-A)(1-B)}.$$

Replacing A and B by their developments in ε, we obtain

$$\beta = \frac{\big(\eta\varepsilon^I + o(\varepsilon^I) - \nu\varepsilon^c + o(\varepsilon^c)\big)^2}{\nu\eta\varepsilon^{c+I} + o(\varepsilon^{c+I})}.$$

This allows a precise discussion that what can happen with β, and the consequences on the BRE property.

- **Case $\boldsymbol{I = c}$.** In this case, $A = 1 - \nu\varepsilon^c + o(\varepsilon^c)$ and $B = 1 - \eta\varepsilon^c + o(\varepsilon^c)$. Writing

  $$A - B = (\eta - \nu)\varepsilon^c + o(\varepsilon^c),$$

  and since $A > B$, necessarily $\eta > \nu$. So,

  $$\beta = \frac{(\eta-\nu)^2\varepsilon^{2c} + o(\varepsilon^{2c})}{\nu\eta\varepsilon^{2c} + o(\varepsilon^{2c})} \to \frac{(\eta-\nu)^2}{\nu\eta} \quad \text{as } \varepsilon \to 0.$$

In this case, we have BRE since $RE < (\psi/2)\sqrt{\beta/K}$, as the upper bound remains bounded itself when $\varepsilon \to 0$.

- **Case $I < c$**. In this case,

$$\beta = \frac{\eta^2\varepsilon^{2I} + o(\varepsilon^I)}{\nu\eta\varepsilon^{c+I} + o(\varepsilon^{c+I})} = \frac{\eta^2 + o(1)}{\nu\eta\varepsilon^{c-I} + o(\varepsilon^{c-I})} \to \infty \quad \text{as } \varepsilon \downarrow 0.$$

  Here, the bound goes to ∞, but *we don't know* what happens with the RE itself (see below).

3.2 Analysis of The *RE*

Now, let's look directly at the *RE*. Proceeding as for β, we denote

$$RE^2 = \frac{\psi^2}{K}\gamma, \tag{4}$$

where

$$\gamma = \frac{\left[r(\mathscr{G}) - B\right]\left[A - r(\mathscr{G})\right]}{\left[1 - r(\mathscr{G})\right]^2}. \tag{5}$$

This leads to the following decomposition.

- **Case $I = b = c$**. Here, we have the maximal number of minpaths, and at least one optimal mincut (one having size b) in our selection. Then:
  - If we have selected *all* the mincuts with minimal size b (this implies, in particular, that they were all disjoint, as in Figure 1 below; see that this is not possible in the case of Figure 2), then $\nu = m$. We have

$$\gamma = \frac{o(\varepsilon^b)\left[(\eta - m)\varepsilon^b + o(\varepsilon^b)\right]}{m^2\varepsilon^{2b} + o(\varepsilon^{2b})}$$

    and $\gamma \to 0$ as $\varepsilon \downarrow 0$ (we can add that we always have $m > \eta$). We thus have BRE. We have even more, the Vanishing Relative Error property, which means that *RE* not only remains bounded but goes to 0 as $\varepsilon \to 0$ (see L'Ecuyer et al. (2010) for a precise definition, and for instance Cancela et al. (2014) for another example of estimators with this strong property).
  - If not, $\nu < m$ and

$$\gamma = \frac{\left[(\eta - m)\varepsilon^b + o(\varepsilon^b)\right]\left[(m - \nu)\varepsilon^b + o(\varepsilon^b)\right]}{m^2\varepsilon^{2b} + o(\varepsilon^{2b})} \to \frac{(\eta - m)(m - \nu)}{m^2} \text{ as } \varepsilon \downarrow 0.$$

    We conclude that BRE holds in this case.
- **Case $I < b < c$**. That is, not enough minpaths and no optimal mincut selected.

$$\gamma = \frac{\left[\eta\varepsilon^I + o(\varepsilon^I)\right]\left[m\varepsilon^b + o(\varepsilon^b)\right]}{m^2\varepsilon^{2b} + o(\varepsilon^{2b})} \to \infty \quad \text{as } \varepsilon \downarrow 0.$$

  In this case, we do not have BRE.
- **Case $I = b < c$**. That is, we have the maximal number of minpaths ($I = b$) but a bad set of mincuts.

$$\gamma = \frac{\left[(\eta - m)\varepsilon^b + o(\varepsilon^b)\right]\left[m\varepsilon^b + o(\varepsilon^b)\right]}{m^2\varepsilon^{2b} + o(\varepsilon^{2b})} \to \frac{\eta - m}{m} \quad \text{as } \varepsilon \downarrow 0.$$

  So, BRE holds here. Observe that in this case, the bound in (2) goes to infinity as $\varepsilon \to 0$.

- **Case $I < b = c$.** That is, not enough minpaths but at least one optimal mincut in $\{C_1,\ldots,C_J\}$.

$$\gamma = \frac{\left[\eta \varepsilon^I + o(\varepsilon^I)\right]\left[(m-\nu)\varepsilon^b + o(\varepsilon^b)\right]}{m^2\varepsilon^{2b} + o(\varepsilon^{2b})}.$$

Here, $\nu \le m$: if $\nu < m$, then $\gamma \to \infty$ as $\varepsilon \downarrow 0$. If $\nu = m$ (we selected all the mincuts with minimal size), then, we have

$$\gamma = \frac{\left[\eta \varepsilon^I + o(\varepsilon^I)\right] o(\varepsilon^b)}{m^2\varepsilon^{2b} + o(\varepsilon^{2b})}.$$

Here, more exploration is needed. This is done in the first two examples below.

This analysis shows that to have BRE, we need just one mincut of minimal size or *all* minpaths, that is, b minpaths, when possible.

First example, source-to-terminal case. The source-to-terminal problem is the simplest, and the richest in terms of literature, in particular for obtaining efficient algorithms, with the related procedures for computing minpaths, mincuts, number of minimal size mincuts, and so on.

If we look at the sufficient conditions from Fishman's paper, as discussed before, always in the homogeneous case, we must choose a number I of minpaths equal to the graph breadth b (that is, the largest possible value, since the minpaths being disjoint, necessarily $I \le b$).

As stated before, this is just a sufficient condition for BRE. Consider the example described in Fig. 1, where the goal is to determine the probability $q(\mathscr{G})$ that the two grey nodes are not connected.

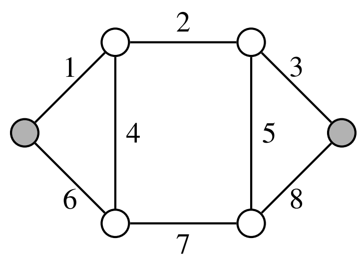

Figure 1: An homogeneous network with 8 links.

The target is here $q(\mathscr{G}) = 3\varepsilon^2 + 4\varepsilon^3 - 9\varepsilon^4 - 10\varepsilon^5 + 27\varepsilon^6 - 18\varepsilon^7 + 4\varepsilon^8 = 3\varepsilon^2 + o(\varepsilon^2)$. The breadth of this graph (obviously with respect to s and t) is $b = 2$. Suppose the user chooses just one minpath, say $\{1,2,3\}$, but *the* optimal set of mincuts, that is, all mincuts having minimal size $b = 2$: $\{1,6\}$, $\{2,7\}$ and $\{3,8\}$. This gives $A = (1-\varepsilon^2)^3 = 1 - 3\varepsilon^2 + 3\varepsilon^4 - \varepsilon^6$ and $B = 1 - 3\varepsilon + 3\varepsilon^2 - \varepsilon^3$. If one computes the squared relative error as a function of ε, when using the chosen minpath and the three mincuts ignoring the constant, that is, if one computes γ, we obtain

$$\gamma = \frac{\left[3\varepsilon + o(\varepsilon)\right]\left[4\varepsilon^3 + o(\varepsilon^3)\right]}{3^2\varepsilon^4 + o(\varepsilon^4)} \to \frac{4}{3} \quad \text{as} \quad \varepsilon \to 0.$$

This shows that we still have BRE in spite of the fact that the bound in (2) goes to ∞ as $\varepsilon \downarrow 0$.

In this example, the reader can check that if the user does not take the maximal number of minpaths (that is, two minpaths), the relative error goes to ∞ as ε goes to 0.

Second example, the all-terminal metric. Consider the bridge topology given in Fig. 2, where the considered metric is the all-terminal unreliability, that is, the probability that not all the nodes in the graph can communicate.

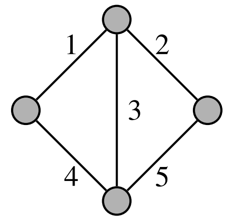

Figure 2: Bridge topology.

Here, each minpath has 3 edges and there are 8 such minpaths: $\{1,2,3\}$, $\{3,4,5\}$, $\{1,3,5\}$, $\{2,3,4\}$, $\{1,2,5\}$, $\{2,4,5\}$, $\{1,4,5\}$ and $\{1,2,4\}$. Since the total number of edges is 5, we cannot find two disjoint minpaths and the user is forced to choose only one (that is, the choice $I = 1$ is forced here).

The network has 6 mincuts: $\{1,4\}$, $\{2,5\}$, $\{1,2,3\}$, $\{1,3,5\}$, $\{3,4,5\}$ and $\{2,3,4\}$. The breadth of this network is thus $b = 2$ (and there are 2 mincuts with minimal size). Whatever the choice of mincuts (optimal or not), the user can not fulfill here the conditions proposed by Fishman.

Assume that the user choose any of the minpaths (so, $I = 1$) and the optimal set of mincuts with minimal size $b = 2$ (that is, the 2 mincuts with size 2). If you look at the bounds A and B as a function of ε, as with previous example, we again have that the bound used in Fishman's paper goes to ∞ but the estimator has the desired BRE. For the computations, we have:

$$q(\mathscr{G}) = 2\varepsilon^2 + 4\varepsilon^3 - 9\varepsilon^4 + 4\varepsilon^5,$$

$$B = (1-\varepsilon)^3 = 1 - 3\varepsilon + 3\varepsilon^2 - \varepsilon^3, \qquad A = (1-\varepsilon^2)^2 = 1 - 2\varepsilon^2 + \varepsilon^4.$$

For the analysis of the bound of the relative error,

$$\frac{(A-B)^2}{(1-A)(1-B)} = \frac{\left[3\varepsilon + o(\varepsilon)\right]^2}{\left[2\varepsilon^2 + o(\varepsilon^2)\right]\left[3\varepsilon + o(\varepsilon)\right]} \to \infty \text{ as } \varepsilon \to 0.$$

For the relative error itself,

$$\frac{\left[r(\mathscr{G}) - B\right]\left[A - r(\mathscr{G})\right]}{(1 - r(\mathscr{G}))^2} = \frac{\left[3\varepsilon + o(\varepsilon)\right]\left[4\varepsilon^3 + o(\varepsilon^3)\right]}{4\varepsilon^4 + o(\varepsilon^4)} \to 3 \quad \text{as} \quad \varepsilon \to 0.$$

Last example, all-terminal metric. Consider the complete graph with 4 nodes described in Fig. 3.

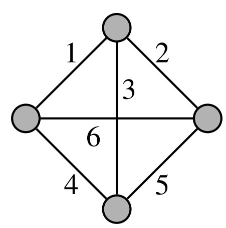

Figure 3: Complete graph with 4 nodes (K_4).

The unreliability of this system is $q(\mathscr{G}) = 4\varepsilon^3 + 3\varepsilon^4 - 12\varepsilon^5 + 6\varepsilon^6$.

In this case, there are 12 minpaths and 4 mincuts. The 12 minpaths are $\{2,5,4\}$, $\{5,4,1\}$, $\{4,1,2\}$, $\{1,2,5\}$, then $\{1,6,4\}$, $\{1,3,2\}$, $\{2,6,5\}$, $\{5,3,4\}$, then $\{1,3,5\}$, $\{2,3,4\}$, $\{1,6,5\}$, $\{2,3,4\}$. The 4 mincuts are $\{1,6,4\}$, $\{1,3,2\}$, $\{2,6,5\}$, $\{5,3,4\}$ (which are also minpaths). The 4 mincuts are all of size 3, which is thus the graph's breadth: $b = 3$.

The maximal number of disjoint minpaths we can find is $I = 2$, and they are either the pair $\{1,3,5\}$ with $\{2,6,4\}$, or $\{2,3,4\}$ with $\{1,6,5\}$. Any two mincuts are not disjoint, so, we can only choose $J = 1$ of them.

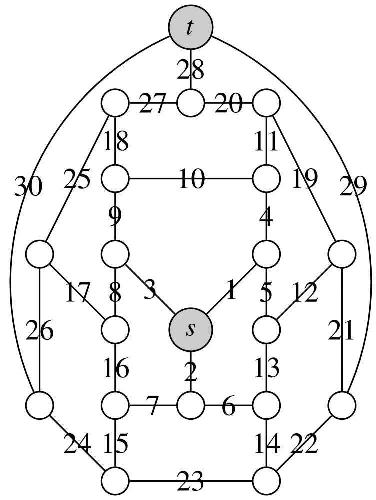

Figure 4: Dodecahedron topology.

For bound A, we have $A = 1-\varepsilon^3 + o(\varepsilon^3)$. For bound B, we have $B = 1-9\varepsilon^2 + o(\varepsilon^2)$.
The fraction to analyze for the asymptotic analysis of the relative error is

$$\gamma = \frac{\left[9\varepsilon^2 + o(\varepsilon^2)\right]\left[3\varepsilon^3 + o(\varepsilon^3)\right]}{16\varepsilon^6 + o(\varepsilon^6)} \sim \frac{27}{16\varepsilon},$$

and it goes to ∞ as $\varepsilon \to 0$.

In this example, we can not have BRE whatever our choice of minpaths and mincuts, using Fishman's approach.

4 MORE INSIGHT ON CONDITIONS FOR SATISFYING BRE

Fishman's bound provides a *sufficient* condition for getting BRE but not a *necessary* one. In order to have a necessary and sufficient condition we rather need to focus on the exact formulation of the relative error, (4) and (5). To explain the difference, the bound behavior just looks at the difference between the upper and lower bounds, while (5) looks at the product of differences between each bound and the target. From this, we can make two remarks:

- if A and B are of the order of $q(\mathscr{G})$ in terms of ε, then BRE is satisfied. The conditions provided by Fishman correspond to this case.
- But if one of the two values $1-A$ or $1-B$ is $q(\mathscr{G}) + o(q(\mathscr{G}))$, then the other one does not need to be as accurate. Indeed, assume that the bound based on cuts verifies $1-A = q(\mathscr{G}) + o(q(\mathscr{G}))$. If $q(\mathscr{G}) = \Theta(\varepsilon^b)$, then $q(\mathscr{G}) - (1-A) = \Theta(\varepsilon^d)$ with $d > b$. In order to get BRE, we only need to have $(1-B) - q(\mathscr{G}) = \Theta(\varepsilon^\ell)$ with $\ell \geq 2b-d$ because the squared relative error is of order $\Theta(\varepsilon^{\ell+d-2b})$. In other words, no need for B to be of the same order than $q(\mathscr{G})$ in terms of ε. This can be illustrated by applying the method on the dodecahedron topology described in Figure 4 where we want to compute the probability of connection of nodes s and t considering three disjoint mincuts. Table 1 illustrates that there is no need for B to be very accurate if A is, when all links have probability of failure ε and when ε decreases. Looking at the product, we actually satisfy BRE.
- If both A and B provide good estimations of $q(\mathscr{G})$, we can even satisfy Vanishing Relative Error, as we saw before.

Table 1: Empirical results of bounds for the dodecahedron topology, for several values of ε.

ε	$\big(1-B-q(\mathscr{G})\big)/\varepsilon^b$	$\big(q(\mathscr{G})-(1-A)\big)/\varepsilon^b$
0.5	1.5956E+00	3.5200E+00
0.1	6.5794E+01	8.7801E-01
0.05	9.0257E+01	3.5655E-01
0.01	1.1566E+02	6.1998E-02
0.005	1.1928E+02	3.0400E-02
0.001	1.2225E+02	6.0001E-03

5 CONCLUSIONS

This paper provides a more up-to-date view on Fishman's method efficiency, when the rare event situation is taken into account. We basically show the limits of the author's analysis, when we look for the fundamental Bounded Relative Error of the estimators. We illustrate how the modern way to address this analysis, by means of a "parameterization" of rarity and the corresponding asymptotic analysis, provides insight on why and when the procedure performs well. It appears that getting necessary and sufficient conditions in terms of minpaths and mincuts is difficult and extremely topology-dependent. To find effective conditions is still an open problem.

REFERENCES

Asmussen, S., and P. W. Glynn. 2007. *Stochastic Simulation*. New York: Springer-Verlag.

Ball, M. O. 1986, Aug.. "Computational Complexity of Network Reliability Analysis: An Overview". *IEEE Transactions on Reliability* 35 (3): 230–239.

Botev, Z. I., P. L'Ecuyer, G. Rubino, R. Simard, and B. Tuffin. 2013. "Static Network Reliability Estimation Via Generalized Splitting". *INFORMS Journal on Computing* 25 (1): 56–71.

Botev, Z. I., P. L'Ecuyer, and B. Tuffin. 2013. "Markov Chain Importance Sampling with Application to Rare Event Probability Estimation". *Statistics and Computing* 23 (3): 271–285.

Cancela, H., and M. El Khadiri. 1995. "A Recursive Variance-Reduction Algorithm for Estimating Communication-Network Reliability". *IEEE Transactions on Reliability* 44 (4): 595–602.

Cancela, H., M. El Khadiri, and G. Rubino. 2009. "Rare events analysis by Monte Carlo techniques in static models". In *Rare Event Simulation using Monte Carlo Methods*, edited by G. Rubino and B. Tuffin, 145–170. John Wiley & Sons.

Cancela, H., M. El Khadiri, G. Rubino, and B. Tuffin. 2014, December. "Balanced and Approximate Zero-Variance Recursive Estimators for the Static Communication Network Reliability Problem". *ACM Transactions on Modeling and Computer Simulation* 25 (1): 19.

Fishman, G. S. 1986. "A Monte Carlo Sampling Plan for Estimating Network Reliability". *Operations Research* 34 (4): 581–594.

Gertsbakh, I. B., and Y. Shpungin. 2010. *Models of Network Reliability*. Boca Raton, FL: CRC Press.

Glynn, P., G. Rubino, and B. Tuffin. 2009. "Robustness Properties and Confidence Interval Reliability Issues". In *Rare Event Simulation using Monte Carlo Methods*, edited by G. Rubino and B. Tuffin, 63–84. John Wiley & Sons.

L'Ecuyer, P., J. H. Blanchet, B. Tuffin, and P. W. Glynn. 2010. "Asymptotic Robustness of Estimators in Rare-Event Simulation". *ACM Transactions on Modeling and Computer Simulation* 20 (1): Article 6.

L'Ecuyer, P., G. Rubino, S. Saggadi, and B. Tuffin. 2011. "Approximate Zero-Variance Importance Sampling for Static Network Reliability Estimation". *IEEE Transactions on Reliability* 8 (4): 590–604.

Murray, L., H. Cancela, and G. Rubino. 2013. "A Splitting Algorithm for Network Reliability". *IIE Transactions* 45 (2): 177–189.

Rubino, G. 1998. "Network Reliability Evaluation". In *State-Of-The Art in Performance Modeling and Simulation*, edited by K. Bagchi and J. Walrand. Gordon and Breach Books.

Rubino, G., and B. Tuffin. (Eds.) 2009. *Rare Event Simulation using Monte Carlo Methods*. John Wiley & Sons.

Tuffin, B., S. Saggadi, and P. L'Ecuyer. 2014. "An Adaptive Zero-Variance Importance Sampling Approximation for Static Network Dependability Evaluation". *Computers and Operations Research* 45:51–59.

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