Proceedings of the 2012 Winter Simulation Conference
C. Laroque, J. Himmelspach, R. Pasupathy, O. Rose, and A. M. Uhrmacher, eds.

NEW CONTROL VARIATES FOR LÉVY PROCESS MODELS

Kemal Dinçer Dingeç
Wolfgang Hörmann

Department of Industrial Engineering
Boğaziçi University
34342 Bebek-İstanbul, TURKEY

ABSTRACT

We present a general control variate method for Monte Carlo estimation of the expectations of the functionals of Lévy processes. It is based on fast numerical inversion of the cumulative distribution functions and exploits the strong correlation between the increments of the original process and Brownian motion. In the suggested control variate framework, a similar functional of Brownian motion is used as a main control variate while some other characteristics of the paths are used as auxiliary control variates. The method is applicable for all types of Lévy processes for which the probability density function of the increments is available in closed form. We present the applications of our general approach for simulation of path dependent options. Numerical experiments confirm that our method achieves considerable variance reduction.

1 INTRODUCTION

A Lévy process is a general continuous time stochastic process with stationary and independent increments. Lévy processes gain an increasing importance in the mathematical finance literature due to the well known drawbacks of the classical Black Scholes geometric Brownian motion (GBM) model. Empirical evidence shows that asset returns follow clearly non normal distributions having higher kurtosis. Increments of a Lévy process can follow any type of distribution as long as it is infinitely divisible. For an overview of different types of Lévy processes and their application to option pricing, see Schoutens (2003) or Cont and Tankov (2003).

Let $\{L(t), t \geq 0\}$ be a Lévy process starting at zero $L(0) = 0$ and q be a function of the sample path of $L(t)$ on a discrete time grid $0 = t_0 < t_1 < t_2 < ... < t_d$ with equidistant time points $t_i = i\,\Delta t$. The unknown quantity that we are trying to estimate is the expectation

$$\mathrm{E}\left[q(L(t_1), \ldots, L(t_d))\right]. \tag{1}$$

For high dimensions and complicated functions, it is often not possible to find the exact value with a closed form solution. Such situations commonly arise e.g. when pricing options with path dependent payoffs.

Monte Carlo simulation is one of the widely used techniques for estimating the expectations of high dimensional problems. Compared to other methods, the availability of an error bound and the ease of implementation and parallelization make it attractive. However, an important disadvantage is that it is comparatively slow when precise results are required. Variance reduction techniques thus play an important role to increase the speed or precision of the simulation. Glasserman (2004) presents applications of different variance reduction methods for financial problems. As noted there, to design a successful variance reduction method, one has to exploit the specific nature of the problem. It is therefore quite difficult to design a general variance reduction method.

In the literature, there exist studies proposing variance reduction methods that work well for some special Lévy processes. However, there seem to exist only a limited number of studies proposing variance

978-1-4673-4782-2/12/$31.00 ©2012 IEEE

reduction methods, which are generally applicable to Lévy processes and different problem types. Dingeç and Hörmann (2012) developed a general control variate (CV) method for option pricing under Lévy processes. It is based on the idea of using the CV of the corresponding GBM option and reaches strong correlation by using common random numbers for the path simulation. In this paper, we extend the idea introduced by Dingeç and Hörmann (2012) to a larger class of problems and suggest to use some path characteristics as additional CVs together with a main CV. This results often in higher variance reduction due to the use of multiple CVs. We present the application of the new CV framework to path dependent options, which are contingent on the average and the maximum stock prices.

In Section 2, we formulate and explain the basic principles of the general CV framework. Section 3 presents our general CV candidates with their expectation formulas. We consider the application to path dependent options in Section 4, whereas Section 5 contains our conclusions.

2 GENERAL CV FRAMEWORK

When using multiple CVs the simulation output takes the form

$$Y = q(L) - c^T (V - \mathrm{E}[V]),$$

where $V = (V_1, \ldots, V_m)^T$ and $c = (c_1, \ldots, c_m)^T$ are the CV vector and the CV coefficient vector, respectively and m denotes the number of CVs used.

The optimal CV coefficients c^*, minimizing the variance of Y are the coefficients of the classical linear regression model with response variable $q(L)$ and covariates V. These least square estimates can be obtained by using a pilot run with a smaller sample size or by using the full sample of the simulation. The former approach leads to an unbiased estimate whereas the latter has a bias of order $O(1/n)$ with respect to the sample size n, which is negligible unless n is small. When the optimal coefficient c^* is used, the variance reduction factor of the control variate method with respect to the naive simulation is $VRF = 1/(1-R^2)$, a function sharply increasing with respect to the R^2 value of the regression model.

Let $\{W(t), t \geq 0\}$ be a Brownian motion (BM) with parameters μ and $\sigma > 0$,

$$W(t) = \mu t + \sigma B(t),$$

where $B(t)$ is a standard Brownian motion. Assume that increments of $\{W(t), t \geq 0\}$ are correlated with that of the original Lévy process $\{L(t), t \geq 0\}$ with a comonotonic copula. Also, assume that there exist a functional ζ of W, which is equal or similar to q and there exist a solution for $\mathrm{E}[\zeta(W)]$. The idea is to use $\zeta(W)$ as main CV for $q(L)$. The comonotonicity between the increments and the similarity between $\zeta()$ and $q()$ should imply high correlation and thus lead to large variance reduction. One technical difficulty is the simulation of the comonotonic copula. It requires inversion of the cumulative distribution functions (CDFs) of the increments of both processes for a common uniform random number. However, for the Lévy processes considered in the literature, at most the probability density function (PDF) of the increments is available in closed form while the CDF and the inverse CDF are not tractable. So, to use inversion we need the fast numerical inversion algorithm, see Derflinger et al. (2010), that requires as input only the PDF of the distribution.

When $\zeta(W)$ is used as single CV, the remaining variance is due to the difference between the two functions, $\zeta()$ and $q()$, and the two processes L and W. In practice, these functions are contingent on some characteristics of the paths such as average, maximum and terminal value. By using those characteristics as additional CVs we can further reduce the variance as they carry some information about the difference between $\zeta()$ and $q()$. The only requirement is the availability of their expectations. Moreover, in some cases, it is possible to obtain significant variance reduction by using only those characteristics as CVs without using $\zeta(W)$. Let $\gamma(W,L)$ be a function of the paths of W and L that evaluates the set of path characteristics. We call these additional CVs 'general CVs' since they are applicable to any $q()$, whereas $\zeta(W)$ is called 'special CV' as it is designed considering the special properties of $q()$.

Algorithm 1 presents the general CV method. In the algorithm, the CV coefficient vector is denoted $c = (c_1, c_2)$ where c_1 denotes the single coefficient of $\zeta(W)$ and c_2 the vector of coefficients necessary for $\gamma(W,L)$. The increments of W are generated by inversion of the standard normal CDF $\Phi()$:

$$F_{BM}^{-1}(U) = \mu\,\Delta t + \sigma\sqrt{\Delta t}\,\Phi^{-1}(U). \tag{2}$$

Here μ and σ are the unspecified parameters of the BM model. Our choice is to select them as $\mu = \mathrm{E}[L(1)]$ and $\sigma = \sqrt{\mathrm{Var}(L(1))}$. This selection leads to close to maximal correlation in our experiments.

Algorithm 1 The General CV Method

Require: Sample size n, simulation output function q, time interval Δt, number of time points d, quantile function of the Lévy process increments F_L^{-1}, special CV function ζ, general CV function γ, parameters of BM $\{\mu, \sigma\}$, CV coefficients c.

Ensure: An estimate of $\mathrm{E}[q(L)]$ and its $1-\alpha$ confidence interval.

for $i = 1$ to n **do**
  Generate independent uniform variates $U_j \sim U(0,1),\ j = 1, \ldots, d$.
  Set $X_j \leftarrow F_L^{-1}(U_j)$ and $Z_j \leftarrow F_{BM}^{-1}(U_j),\ j = 1, \ldots, d$, see (2).
  Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j,\quad j = 1, \ldots, d$.
  Set $Y_i \leftarrow q(L) - c_1\,(\zeta(W) - \mathrm{E}[\zeta(W)]) - c_2^T(\gamma(W,L) - E[\gamma(W,L)])$.
end for
Compute the sample mean $\bar{Y}$ and the sample standard deviation s of Y_i's.
return $\bar{Y}$ and the error bound $\Phi^{-1}(1-\alpha/2)\ s/\sqrt{n}$, where Φ^{-1} denotes the quantile of the standard normal distribution.

3 POSSIBLE CONTROL VARIATES

In Algorithm 1, the user has to provide the CV functions $\zeta()$ and $\gamma()$. The selection of $\zeta()$ of course depends on the problem type, as it is tailored to $q()$. However, the general CVs can be freely chosen from a large *pool* or *basket* of CV candidates. For the selection of the successful CVs from that basket, we can try to use our theoretical knowledge on the problem and guess which CV will be successful. In this paper, we propose an alternative approach, which is more automatic and universally applicable to all problems. We make a stepwise backward regression to detect the useful CVs in a pilot simulation run. First we start with a full regression model where all possible CVs in the basket are used. The CV with the smallest absolute t statistic is removed from the model if its value is smaller than 5. After removal, the t statistics of the other CVs are recomputed for the new regression model. These two steps are repeated till all absolute t values are above 5. The CV candidates, which remain in the regression model, are used for the main simulation. Instead of using only the significant CVs, one can prefer to use all CVs in the basket. However, simulation or evaluation of the expectation of some CVs can be computationally quite expensive. Therefore it is sensible to use backward regression to automatically eliminate the CVs that are not useful.

A single regression with k covariates requires $O(n_p k^2)$ operations, where n_p denotes the sample size of the pilot simulation. In the worst case, when all CVs in the basket are useless, the complexity of the backward regression becomes $O(n_p k^3)$. As we select n_p typically much smaller than n the regression round of the algorithm will not cause a substantial increase in the computational time unless k is larger.

Table 1 shows our basket of general CVs that contains the path characteristics of which the expectation is available in closed form. This list is not exhaustive and depends on our knowledge of the closed form solutions of the expectations. One can enlarge this basket by adding new CVs with new expectation formulas. We decided not to include CVs that require numerical methods to evaluate their expectations. The details of the expectation formulas of the CVs are given in Section 3.1. Some CVs have simpler expectation formulas if the processes $\exp(-rt+L(t))$ and $\exp(-rt+W(t))$ with some $r>0$ are martingales. We also give these simpler formulas in Table 1, as they are important for option pricing.

Table 1: A basket of general CVs. Expectation-M: simpler expectation formulas for the martingale case.

Label	CV	Expectation	Expectation-M
CVL1	$L(t_d)$	$d\,\mathrm{E}[X]$	
CVL2	$\exp(L(t_d))$	$M_{\Delta t}(1)^d$	e^{rT}
CVL3	$\frac{1}{d}\sum_{i=1}^{d} L(t_i)$	$\mathrm{E}[X](d+1)/2$	
CVL4	$\exp(\frac{1}{d}\sum_{i=1}^{d} L(t_i))$	$\prod_{i=1}^{d} M_{\Delta t}(i/d)$	
CVL5	$\frac{1}{d}\sum_{i=1}^{d}\exp(L(t_i))$	$\frac{1}{d}\sum_{i=1}^{d} M_{\Delta t}(1)^i$	$\frac{1}{d}\sum_{i=1}^{d} e^{r\,i\,\Delta t}$
CVW1	$W(t_d)$	$d\,\mu\,\Delta t$	
CVW2	$\exp(W(t_d))$	$e^{(d(\mu\Delta t+\sigma^2\Delta t/2))}$	e^{rT}
CVW3	$\frac{1}{d}\sum_{i=1}^{d} W(t_i)$	$\mu\,\Delta t(d+1)/2$	
CVW4	$\exp(\frac{1}{d}\sum_{i=1}^{d} W(t_i))$	$\exp(\tilde{\mu}+\tilde{\sigma}^2/2)$	
CVW5	$\frac{1}{d}\sum_{i=1}^{d}\exp(W(t_i))$	$\frac{1}{d}\sum_{i=1}^{d} e^{(i(\mu\Delta t+\sigma^2\Delta t/2))}$	$\frac{1}{d}\sum_{i=1}^{d} e^{r\,i\,\Delta t}$
CVW6	$\max_{0\le i\le d} W(t_i)$	see (5)	
CVW7	$\exp(\max_{0\le i\le d} W(t_i))$	x_d in (6)	
CVW8	$\sup_{0\le u\le t_d} W(u)$	see (8)	
CVW9	$\exp(\sup_{0\le u\le t_d} W(u))$	see (9)	

In Table 1, CVLs are based on the path characteristics of the original Levy process, while CVWs are based on the auxiliary BM. Indeed CVLs are internal CVs whereas CVWs are external. The internal CVs do not require any additional effort but often yield moderate variance reduction. However, when CVLs, CVWs and the special CV $\zeta(W)$ are used together, they often result in large variance reductions. CVL1-5 and CVW1-5 depend on the terminal value and arithmetic averages of the paths and their exponentials, while CVW6-9 depend on the maximum of the discrete path and the supremum of the continuous path of $W(t)$. The counterparts of CVW6-9 for $L(t)$ are not available in the CV basket as closed form solutions of their expectations only exist for the special case of Brownian motion.

Instead of the averages one could try to use all $L(t_i)$'s and $e^{L(t_i)}$'s as CVs. To keep the total number of CVs moderate we decided not to include them in our basket. It is also possible to add some other CVs depending on the averages of the squared values of $L(t_i)$ as their expectations are also available thanks to the variance formulas of the distribution of increments of the Lévy processes. However, their contribution is expected to be insignificant as the CVs depending on the exponentials already convey much of the information of the CVs depending on the squared values.

CVL1-5 and CVW1-7 are all easy to simulate as they are simple functions of the paths. However, CVW8 and CVW9 can not be calculated directly from the generated paths, as they are random functions of the discrete path. Note that to simulate $\sup_{0\le u\le t_d} W(u)$ we have to generate the maxima of d Brownian bridges. The maximum of these d local maxima gives us $\sup_{0\le u\le t_d} W(u)$. That is

$$\sup_{0\le u\le t_d} W(u) = \max_{1\le i\le d}\left(\sup_{t_{i-1}\le u\le t_i} W(u)\right).$$

To generate these local maxima, we need the cumulative distribution function of the maximum of a Brownian bridge. Fortunately, it has a simple structure, and random variate generation from that CDF can be done easily by using the inverse transformation method. The conditional CDF of the supremum of $W(t)$ is well known see, e.g., Shreve (2004), Glasserman (2004) and it is given by

$$P\left(\sup_{0\le u\le t} W(u)\le x \,\middle|\, W(t)=y\right) = 1-\exp\left(-\frac{2x(x-y)}{\sigma^2 t}\right), \tag{3}$$

where $x \geq \max(y,0)$. From that CDF, the maximum of the Brownian bridge can be generated by the inversion method as it was e.g. suggested in Beaglehole, Dybvig, and Zhou (1997) to decrease the simulation bias under a general diffusion process:

$$x = 0.5\left(y + \sqrt{y^2 - 2\sigma^2 t \log U}\right), \tag{4}$$

where $U \sim U(0,1)$ is a uniform random number.

Generating $\sup_{0\leq u\leq t_d} W(u)$ for a given discrete path $W(t_1),\ldots,W(t_d)$ introduces some extra randomness to the estimator, and as a direct consequence, increases the variance, since the sigma field has been expanded from $\sigma(W(t_1),\ldots,W(t_d))$ to $\sigma(W(t): 0 \leq t \leq t_d)$. As mentioned in Dingeç and Hörmann (2011), it is possible to use the conditional expectation of the maximum $\mathrm{E}[\sup_{0\leq u\leq t_d} W(u)|W(t_1),\ldots,W(t_d)]$ as CV to get a higher variance reduction. Our experiments with that idea showed that, as numerical integration is necessary to evaluate that expectation, the conditional expectation approach increases the computation time more than the variance reduction. We therefore do not suggest to use it.

Note that the CVs in the basket are strongly correlated with each other. This situation, which is called *multicollinearity* in the statistical literature, can be a problem for the accuracy of the estimates of the t statistics when the sample size is too small, as it inflates the standard errors of the estimates of the regression coefficients. So, for the pilot run, it is important to select a sample size which is not too small. Our numerical experience shows that the sample size of $n = 10^4$ is generally sufficient to get relatively stable estimates of the t values.

3.1 Expectation Formulas

3.1.1 Expectations of CVLs

The expectations of CVL1 and CVL3, $L(t_d)$ and $\frac{1}{d}\sum_{i=1}^{d} L(t_i)$, can be calculated by using the expectation formulas of the increment distribution of the corresponding Lévy process (see e.g., Table 2). For CVL2, CVL4 and CVL5, we need the moment generating function (MGF) of the increment distribution (see e.g., Table 2). Let X_i denote the increment of L over time interval $[t_{i-1},t_i]$. As the time lengths are equal to Δt, X_i's are iid copies of a random variate X. Expectations of CVL1 and CVL3 are given by $d\,\mathrm{E}[X]$ and $\mathrm{E}[X](d+1)/2$, respectively.

Let $M_{\Delta t}(u)$ denote the MGF of the increment for the time length Δt, that is $M_{\Delta t}(u) = \mathrm{E}[e^{uX}]$. Then

$$\mathrm{E}[\exp(L(t_i))] = \mathrm{E}\left[\exp\left(\sum_{j=1}^{i} X_j\right)\right] = M_{\Delta t}(1)^i.$$

The expectation of CVL2 is simply $M_{\Delta t}(1)^d$. Also, the expectation of CVL5 is $\frac{1}{d}\sum_{i=1}^{d} M_{\Delta t}(1)^i$. The expectation of CVL4 is given by

$$\mathrm{E}\left[\exp\left(\frac{1}{d}\sum_{i=1}^{d} L(t_i)\right)\right] = \prod_{i=1}^{d} M_{\Delta t}(i/d),$$

since

$$\frac{1}{d}\sum_{i=1}^{d} L(t_i) = \frac{1}{d}\sum_{i=1}^{d}(d-i+1)X_i.$$

3.1.2 Expectations of CVWs

Let Z_i denote the increment of W over time interval $[t_{i-1},t_i]$. Then Z_i's are iid copies of $Z \sim N(\mu\,\Delta t, \sigma^2\,\Delta t)$. The expectations of CVW1 and CVW3 are given by $\mathrm{E}[W(t_d)] = d\,\mu\,\Delta t$ and

$$\mathrm{E}\left[\frac{1}{d}\sum_{i=1}^{d} W(t_i)\right] = \frac{\mu\,\Delta t}{d}\sum_{i=1}^{d} i = \frac{\mu\,\Delta t(d+1)}{2},$$

respectively. By using the MGF of the normal distribution, we obtain the expectations of CVW2 and CVW5 as

$$\mathrm{E}\left[\exp(W(t_d))\right] = \exp(d(\mu\Delta t + \sigma^2\Delta t/2))$$

and

$$\mathrm{E}\left[\frac{1}{d}\sum_{i=1}^{d}\exp(W(t_i))\right] = \frac{1}{d}\sum_{i=1}^{d}\exp(i(\mu\Delta t + \sigma^2\Delta t/2)).$$

The expectation of CVW4 is obtained by using the fact that any linear combination of independent normal variates is again normal. Note that

$$\frac{1}{d}\sum_{i=1}^{d}W(t_i) = \frac{1}{d}\sum_{i=1}^{d}(d-i+1)Z_i$$

is equal in distribution to a normal variate with mean

$$\tilde{\mu} = \mathrm{E}\left[\frac{1}{d}\sum_{i=1}^{d} i Z_i\right] = \frac{\mu\,\Delta t(d+1)}{2}$$

and variance

$$\tilde{\sigma}^2 = \mathrm{Var}\left(\frac{1}{d}\sum_{i=1}^{d} i Z_i\right) = \frac{\sigma^2\,\Delta t}{d^2}\sum_{i=1}^{d} i^2 = \frac{\sigma^2\,\Delta t\,(d+1)\,(2d+1)}{6d}.$$

Hence we get

$$\mathrm{E}\left[\exp\left(\frac{1}{d}\sum_{i=1}^{d}W(t_i)\right)\right] = \exp(\tilde{\mu} + \tilde{\sigma}^2/2).$$

To find the expectation of CVW6, we use the fact that $\{W(t_i), i \geq 1\}$ is a random walk with iid increments. By Spitzer's identity (Spitzer 1956) we get

$$\mathrm{E}\left[\max_{0\leq i\leq d} W(t_i)\right] = \sum_{j=1}^{d}\frac{1}{j}\mathrm{E}\left[W(t_j)^+\right], \tag{5}$$

where

$$\mathrm{E}\left[W(t)^+\right] = \mu\, t\,\Phi(\mu\sqrt{t}/\sigma) + \sigma\sqrt{t}\,\phi(-\mu\sqrt{t}/\sigma)$$

and $\Phi()$ and $\phi()$ denote the CDF and the PDF of the standard normal distribution.

To find the expectation of CVW7, we use the recursion of Öhgren (2001). Let $\{X_j, j \geq 1\}$ be iid random variables and $X_0 = 0$. Define $Y_k = \sum_{j=0}^{k} X_j$ and $M_k = \max_{0\leq j\leq k} Y_j$. Also, let $x_k = \mathrm{E}\left[e^{M_k}\right]$ and $a_k = \mathrm{E}\left[e^{Y_k^+}\right]$. By using Spitzer's formula, Öhgren (2001) proves that

$$x_k = \frac{1}{k}\sum_{j=0}^{k-1} a_{k-j}\, x_j. \tag{6}$$

In our case, $X_j \sim N(\mu\,\Delta t, \sigma^2\,\Delta t)$, $Y_k \sim N(k\mu\Delta t, k\sigma^2\Delta t)$ and

$$a_k = \Phi\left(-\frac{\mu}{\sigma}\sqrt{k\Delta t}\right) + e^{(\mu+\sigma^2/2)k\Delta t}\,\Phi\left(\frac{\mu+\sigma^2}{\sigma}\sqrt{k\Delta t}\right). \tag{7}$$

Hence the expectation of CVW7, $\mathrm{E}\left[\exp(\max_{0\leq i\leq d} W(t_i))\right]$, is simply equal to x_d which can be calculated using the recursion (6) with the coefficients (7).

It should be noted that by using the formulas of Spitzer (1956) and Öhgren (2001) it is also possible to evaluate $\mathrm{E}[\max_{0\le i\le d} L(t_i)]$ and $\mathrm{E}[\exp(\max_{0\le i\le d} L(t_i))]$. However, in that case, we need numerical integration to evaluate each a_k. Furthermore, if the increment distribution is not closed under convolution, the evaluation of the a_k's requires the inversion of the characteristic functions.

The expectations of CVW8 and CVW9 are found by using the density of the maximum of Brownian motion with drift on a finite interval see e.g., Shreve (2004). By integration of the density, we obtain

$$\mathrm{E}\left[\sup_{0\le u\le t_d} W(u)\right] = \frac{\sigma^2}{2\mu}\left(2\,\Phi(b_1)-1\right)+\Phi(b_1)\,\mu\,t_d+\phi(b_1)\,\sigma\sqrt{t_d} \tag{8}$$

and

$$\mathrm{E}\left[\exp\left(\sup_{0\le u\le t_d} W(u)\right)\right] = 1+e^{zt_d}\Phi(b_1)-\Phi(b_2)+\left(\frac{\sigma^2}{2z}\right)\left[-\Phi(b_1-(2z/\sigma)\sqrt{t_d})+e^{zt_d}\Phi(b_1)\right], \tag{9}$$

where $b_1=\mu\sqrt{t_d}/\sigma$, $b_2=b_1-\sigma\sqrt{t_d}$ and $z=\mu+\sigma^2/2$.

3.2 A Simple Example

The easiest application of our general CV method uses only general CVs of the basket without a special CV. In this section, we present the use of this general framework for the simple example

$$q(L)=\exp\left(\max_{0\le i\le d} L(t_i)\right),$$

which demonstrates the effectiveness of the general method. More relevant examples can be found in Section 4 below. To produce the numerical results, Algorithm 1 was coded in R (R Development Core Team 2011). To find the approximate quantile function of the increments (F_L^{-1} in Algorithm 1), we used the numerical inversion algorithm of Derflinger et al. (2010) and the 'Runuran' package of Leydold and Hörmann (2012). The u-resolution of the numerical inversion algorithm was set to $\varepsilon_u = 10^{-10}$. For more details about the implementation, see Derflinger et al. (2010) and Dingeç and Hörmann (2012).

We assume that L is a generalized hyperbolic (GH) process. We fix the time step of the time grid to $\Delta t = 1/250$. The parameters of the distribution of the increments for the time length Δt are selected as $\lambda = 1.5$, $\alpha = 189.3$, $\beta = -5.71$, $\delta = 0.0062$, $\mu = 0.001$ (see Section 4.2). With these parameters, the increment distribution is close to normal but has a higher kurtosis. We try two cases $d=5$ and $d=50$. For $d=5$, the backward regression founds all CVs significant except for CVW8 and CVW9. The CV algorithm with those CVs yields a variance reduction factor (VRF) of 560. For $d=50$, the CVs, which are found to be significant by backward regression, are CVL1,2,4,5, CVW1,2,4,5 and CVW7. The obtained VRF is 395.

4 OPTION PRICING UNDER LÉVY PROCESSES

In this section, we present the application of our general CV algorithm to the simulation of path dependent options under Lévy processes. Suppose that we have an option on a stock with the price process $\{S(t), t\ge 0\}$ given by

$$S(t)=S(0)e^{L(t)},$$

where L is a Lévy process. Let ψ denote the payoff function of the option. For discretely monitored path-dependent options, ψ is a function from $\Re^d$ to $\Re$ where d denotes the number of control points in time. With time grid $0 = t_0 < t_1 < t_2 < ... < t_d = T$ and maturity T, the price of the option is given by the discounted risk neutral expectation of the payoff function

$$e^{-rT}\mathrm{E}\left[\psi(S(t_1),\ldots,S(t_d))\right],$$

where r is the deterministic risk free interest rate. Note that by setting $q(L) = \psi(S(0)e^L)$, the above expectation reduces to (1). So, Algorithm 1 can be applied.

In the application of our general CV approach, the first step is the selection of a special CV, ζ, which is a functional of $W(t)$. In the option pricing case, that functional corresponds to the payoff function of a similar option with analytically available price under GBM. Let ψ_{CV} denote the payoff function of this new option. Also, let $\left\{\tilde{S}(t), t \geq 0\right\}$ denote the stock price process which follows a GBM with parameters r and σ:

$$\tilde{S}(t) = \tilde{S}(0)e^{W(t)} = \tilde{S}(0)\exp((r-\sigma^2/2)t + \sigma B(t)).$$

where $B(t)$ is a standard BM. Then our special CV is $\zeta(W) = \psi_{CV}(\tilde{S})$. In this setting, the only unspecified parameter is the volatility σ of the GBM model, as the drift of BM is automatically set to $\mu = r - \sigma^2/2$ under the risk neutral measure. We set the volatility equal to the standard deviation of the increments of the original Lévy process $L(t)$, that is $\sigma = \sqrt{\mathrm{Var}(L(1))}$. We also use the same initial values for both processes $\widetilde{S}(0) = S(0)$.

For option pricing applications, some of the general CVs given in Table 1 have simpler expectation formulas due to the martingale property of the discounted stock price process under the risk neutral measure, $\mathrm{E}\left[e^{-rt}S(t)\right] = \mathrm{E}\left[e^{-rt}\tilde{S}(t)\right] = S(0)$. The expectations of CVL2 and CVW2 reduce to e^{rT}, the expectations of CVL5 and CVW5 are both equal to $\frac{1}{d}\sum_{i=1}^{d} e^{r\,i\,\Delta t}$.

4.1 Special CVs

We consider in the sequel the problem of pricing Asian options and fixed strike lookback options. As special CVs we use those suggested by Dingeç and Hörmann (2012).

4.1.1 Asian Options

We consider the arithmetic average Asian call option with payoff function

$$\psi_A(S) = \left(\frac{\sum_{i=1}^{d} S(t_i)}{d} - K\right)^+$$

where K is the strike price. We use the geometric average Asian call under GBM, suggested by Kemna and Vorst (1990), as special CV. It has the payoff function

$$\psi_G(\tilde{S}) = \left(\exp\left(\frac{\sum_{i=1}^{d}\log\tilde{S}(t_i)}{d}\right) - K\right)^+.$$

The formula of the geometric price is

$$e^{-rT}\mathrm{E}\left[\psi_G(\tilde{S})\right] = e^{-rT}\left(e^{\mu_{\tilde{s}}+\sigma_{\tilde{s}}^2/2}\,\Phi(-k+\sigma_{\tilde{s}}) - K\,\Phi(-k)\right), \tag{10}$$

where $\Phi()$ denotes the CDF of the standard normal distribution,

$$k = \frac{\log K - \mu_{\tilde{s}}}{\sigma_{\tilde{s}}}, \tag{11}$$

$$\mu_{\tilde{s}} = \log\tilde{S}(0) + (r-\sigma^2/2)\Delta t(d+1)/2, \tag{12}$$

and

$$\sigma_{\tilde{s}} = \frac{\sigma}{d}\sqrt{\Delta t\, d\,(d+1)\,(2d+1)/6}. \tag{13}$$

Here $\mu_{\tilde{s}}$ and $\sigma_{\tilde{s}}$ denote the mean and the standard deviation of the logarithm of the geometric average.

4.1.2 Lookback Options

We consider the fixed strike lookback call option with payoff function

$$\psi_L(S) = \left(\max_{0\le i\le d} S(t_i) - K\right)^+,$$

where K is the strike price. As special CV, we use the continuous lookback option under GBM (see Dingeç and Hörmann (2011)). It has the payoff function

$$\psi_{LC}(\tilde{S}) = \left(\sup_{0\le u\le t_d} \tilde{S}(u) - K\right)^+. \tag{14}$$

As shown in Section 3, it is possible to simulate $\sup_{0\le u\le t_d} \tilde{S}(u)$ from the discrete path. In Haug (2007), the formula of the price of the continuous fixed strike lookback call option is given for two cases. For $K > S(0)$,

$$\begin{aligned} e^{-rT}\mathrm{E}[\psi_{LC}(S)] &= S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2) \\ &\quad + S(0)e^{-rT}\left(\frac{\sigma^2}{2r}\right)\left[-(S(0)/K)^{-2r/\sigma^2}\Phi(d_1-(2r/\sigma)\sqrt{T}) + e^{rT}\Phi(d_1)\right], \end{aligned} \tag{15}$$

where

$$d_1 = \frac{\log(S(0)/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

For $K \le S(0)$,

$$\begin{aligned} e^{-rT}\mathrm{E}[\psi_{LC}(S)] &= e^{-rT}(S(0)-K) + S(0)\Phi(b_1) - S(0)e^{-rT}\Phi(b_2) \\ &\quad + S(0)e^{-rT}\left(\frac{\sigma^2}{2r}\right)\left[-\Phi(b_1-(2r/\sigma)\sqrt{T}) + e^{rT}\Phi(b_1)\right], \end{aligned} \tag{16}$$

where $b_1 = \frac{(r+\sigma^2/2)\sqrt{T}}{\sigma}$ and $b_2 = b_1 - \sigma\sqrt{T}$

4.2 Application of the General CV Method to Asian and Lookback Options

In this section the success of the general CV method suggested in Section 3 using the special CVs presented in Section 4.1 are reported for numerical examples.

Variance Gamma (VG), Normal Inverse Gaussian (NIG), Generalized Hyperbolic (GH) and Meixner (MXN) processes are among the most popular Lévy process models used for option pricing. Detailed information about these processes can be found in (Dingeç and Hörmann 2012) and in the references given there. In this paper, we report results only for the GH process as it is the most general process among the ones listed above. VG and NIG distributions are in fact subclasses of the GH distribution. Also, when the processes are calibrated by using the same data, the variance reductions obtained for the other three processes are observed to be very similar to the ones obtained for the GH process. The PDF, expectation and MGF formulas of the GH model are given in Table 2. Here, $\gamma = \sqrt{\alpha^2-\beta^2}$ and $K_\nu(x)$ denotes the modified Bessel function of the second kind of order ν.

The parameters of the GH process are estimated from the daily log return data of the Swiss stock "Swiss.Re" given in the R package 'ghyp' of Luethi and Breymann (2011). The sample consists of 500 daily log returns from 2005-01-19 to 2007-01-10. The parameters of the models were estimated by maximum likelihood estimation. The estimated daily parameters are $\lambda = 1.5$, $\alpha = 189.3$, $\beta = -5.71$, $\delta = 0.0062$,

Table 2: Formulas of the GH model.

Parameters	$\lambda, \alpha, \beta, \delta, \mu$
PDF	$f(x) = \frac{(\gamma/\delta)^\lambda}{\sqrt{2\pi} K_\lambda(\delta\gamma)} \, e^{\beta(x-\mu)} \frac{K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\sqrt{\delta^2+(x-\mu)^2}/\alpha\right)^{1/2-\lambda}}$
Expectation	$\mathrm{E}[X] = \mu + \frac{\beta\delta}{\gamma} \frac{K_{\lambda+1}(\delta\gamma)}{K_\lambda(\delta\gamma)}$
MGF	$M(u) = e^{u\mu} \left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda/2} \frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+u)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}$

$\mu = 0.001$. In our study, we have used the risk neutral Esscher measure proposed by Gerber and Shiu (1994) for option pricing. This change of measure implies the transformation of the real world skewness parameter β depending on the risk free interest rate r. For more details, see Gerber and Shiu (1994) and Dingeç and Hörmann (2012).

In the application of the general CV method, the first step is the selection of the CVs from the basket given in Table 1. For the Asian option example, the special CV (the geometric average option), CVL3,4,5, and CVW3,4,6,7 are found to be significant by the backward regression in the pilot simulation run. For the lookback option example, when $K > S(0)$, the special CV (the continuous option) and the general CVs are all found to be significant with large absolute t values. Note that the other case $K \leq S(0)$ is equivalent to the problem presented in Section 3.2. So, we will not consider it again.

We report as main result the variance reduction factors of our new CV method denoted by $VRF = \sigma_N^2/\sigma_{CV}^2$ where σ_N^2 and σ_{CV}^2 denote the variances of naive simulation and our new CV method, respectively. In our numerical results, we consider only the daily monitoring case $\Delta t = 1/250$. Table 3 shows the success of the new CV method for the two options and different strike prices. Substantial variance reductions are obtained in all cases. We see that the largest variance reductions are obtained for small strike prices. In Table 3, we also compare our final CV algorithm with the other CV algorithm variants using only general CVs (CVLs and CVWs) or only the single special CV. From the comparison, we see that using only the general CVs yields considerable VRFs, which are close to but often smaller than those using only the single special CV. We obtain the largest VRFs when the general CVs are used together with the special CV.

Table 3: Results for Asian and lookback options under GH process with $T = 1, \Delta t = 1/250, r = 0.05, S(0) = 100, n = 10^4$. Price: value of the simulation estimator of the option price, Error: 95% error bound. VRF-A: VRF obtained by using all significant CVs, VRF-G: VRF obtained by using only general CVs (CVLs and CVWs), VRF-S: VRF obtained by using only special CV.

Option	K	Price	Error	VRF-A	VRF-G	VRF-S
Asian	90	12.239	0.004	1,743	185	78
	100	4.912	0.005	530	51	64
	110	1.240	0.006	121	13	40
Lookback	110	7.534	0.012	294	57	57
	120	3.297	0.012	160	35	44
	130	1.266	0.011	79	17	32

Finally we compare the speed of the naive simulation and the new CV algorithm by reporting the time ratio t_N/t_{CV}, where t_N and t_{CV} are the CPU times of naive simulation and of the new multiple CV algorithm respectively. (Note that the corresponding variance reduction factor σ_N^2/σ_{CV}^2 is called VRF-A in Table 3.) The time ratio and the variance reduction factor are required to calculate the efficiency factor $EF = (\sigma_N^2 t_N)/(\sigma_{CV}^2 t_{CV})$ of the new CV algorithm. For the generation of GH variates in the naive simulation algorithm subordination was used (see Dagpunar (1989)), as it is the standard method in the literature. The time ratios t_N/t_{CV} were observed to be equal to 1.0 and 0.7 for Asian and lookback options, respectively. This shows that the speed up obtained by the use of the fast numeric inversion procedure is approximately as large as the slow down due to the extra computations required for the evaluation of the CVs. Note that in these time ratios the pilot simulation runs are excluded. We observed in our examples that the additional computational time necessary for the pilot simulation run with a sample size of one tenth of the main simulation is between 30% and 50% of the computational time of the main simulation.

5 CONCLUSIONS

In this study, we propose a general control variate framework for the functionals of Lévy processes. The new method exploits the strong correlation between the original Lévy process and an auxiliary Brownian motion. In the suggested framework, we use special control variates tailored to the functionals and general control variates selected from a large basket of control variate candidates. The application to path dependent options shows that the general framework results in successful control variate methods. In our examples, we observe moderate to large variance reductions.

ACKNOWLEDGMENTS

This work was supported by Boğaziçi Research Fund Project 5020 and TÜBİTAK (The Scientific and Technological Research Council of Turkey) Research Fund Project 111M108.

REFERENCES

Beaglehole, D. R., P. H. Dybvig, and G. Zhou. 1997. "Going to extremes: Correcting simulation bias in exotic option valuation". *Financial Analysts Journal* 53 (1): 62–68.

Cont, R., and P. Tankov. 2003. *Financial Modelling with Jump Processes*. Boca Raton, FL: Chapman & Hall/CRC Financial Mathematics Series.

Dagpunar, J. 1989. "An easily implemented generalised inverse Gaussian generator". *Communications in Statistics - Simulation and Computation* 18 (2): 703–710.

Derflinger, G., W. Hörmann, and J. Leydold. 2010, November. "Random variate generation by numerical inversion when only the density is known". *ACM Transactions on Modeling and Computer Simulation* 20 (4): 1–25.

Dingeç, K. D., and W. Hörmann. 2011. "Using the continuous price as control variate for discretely monitored options". *Mathematics and Computers in Simulation* 82 (4): 691–704.

Dingeç, K. D., and W. Hörmann. 2012. "A general control variate method for option pricing under Lévy processes". *European Journal of Operational Research* 221 (2): 368–377.

Gerber, H. U., and E. S. W. Shiu. 1994. "Option pricing by Esscher transforms". *Transactions of the Society of Actuaries* 46:99–191.

Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. New York: Springer-Verlag.

Haug, E. G. 2007. *The Complete Guide to Option Pricing Formulas*. New York: McGraw-Hill.

Kemna, A., and A. Vorst. 1990. "A pricing method for options based on average asset values". *Journal of Banking and Finance* 14:113–130.

Leydold, J., and W. Hörmann. 2012. *Runuran: R interface to the UNU.RAN random variate generators*. R package version 0.19.0.

Luethi, D., and W. Breymann. 2011. *ghyp: A package on the generalized hyperbolic distribution and its special cases*. R package version 1.5.5.

Öhgren, A. 2001. "A remark on the pricing of discrete lookback options". *Journal of Computational Finance* 4 (3): 141–147.

R Development Core Team 2011. *R: A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing. ISBN 3-900051-07-0.

Schoutens, W. 2003. *Lévy Processes in Finance: Pricing Financial Derivatives*. New York: John Wiley & Sons.

Shreve, S. 2004. *Stochastic Calculus for Finance II: Continuous Time Models*. New York: Springer.

Spitzer, F. 1956. "A combinatorial lemma and its application to probability theory". *Transactions of the American Mathematical Society* 82 (2): 323–339.

AUTHOR BIOGRAPHIES

KEMAL DİNÇER DİNGEÇ received his BS in Industrial Engineering in 2005 from Istanbul Technical University. He received his MS in Industrial Engineering in 2007 at Boğaziçi University. Currently, he is a PhD candidate at Industrial Engineering Department of the same institution. His research interests include stochastic models in manufacturing and finance, Monte Carlo simulation and variance reduction for financial simulations. His email addresses are kemal.dingec@boun.edu.tr and kdingec@yahoo.com.

WOLFGANG HÖRMANN is an associate professor in the Industrial Engineering Department at Boğaziçi University. His research centers on non-uniform random variate generation, variance reduction techniques and Monte Carlo methods in finance. His email address is hormannw@boun.edu.tr.