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A TWO-LEVEL LOAN PORTFOLIO OPTIMIZATION PROBLEM

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ABSTRACT

In this paper, we study a two-level loan portfolio optimization problem, a problem motivated by our work for some commercial banks in China. In this problem, there are two levels of decisions: at the higher level, the headquarter of the bank needs to decide how to allocate its overall capital among its branches based on its risk preference, and at the lower level, each branch of the bank needs to decide its loan portfolio based on its own risk preference and allocated capital budget. We formulate this problem as a two-level portfolio optimization problem and then propose a Monte Carlo based method to solve it. Numerical results are included to validate the method.

1 INTRODUCTION

Since the seminal work by Markowitz (Markowitz 1952), portfolio selection has become one of the pillars of today's finance research and practice. The main concern of an investor in portfolio selection is to balance the expected return and the risk of possible loss. In this paper, we study a loan portfolio selection problem motivated by the practice at some commercial banks in China. Different from traditional portfolio selection problem, in this problem, there are two levels of decisions. At the higher level, the headquarter of the bank needs to decide how to allocate its overall budget among its branches based on its risk preference, and at the lower level, each branch of the bank needs to decide its loan portfolio based on its own risk preference and allocated capital budget. We formulate this problem as a two-level portfolio optimization problem.

In this paper, we will use Monte Carlo methods to solve the portfolio selection problem. There are two different Monte Carlo methods that have been proposed to solve the portfolio selection problem: one is based on gradient method (Hong and Liu 2009) and the other is to convert the original problem into a sample-based linear programming program (Rockafellar and Uryasev 2000). Our method proposed in this paper is mainly based on the combination of (Hong and Liu 2009) and Lagrangian relaxation method.

The remainder of this paper is organized as follows. In Section 2, we present the formulation of the two-level loan portfolio optimization problem. A Monte Carlo based solution procedure is proposed in Section 3. In Section 4, we provide several numerical examples to validate our proposed method. Finally, a conclusion is provided in Section 5.

2 PROBLEM FORMULATION

In this section, we present the formulation for the two-level loan selection problem. Suppose that the bank has m branches and makes loans to n different types of customers (industries). Let x_{ij} be the amount of loan that branch i makes to customer j and $p_{ij}(t)$ be the unit value of the loan made to customer j by branch i at time t ($i=1,\ldots,m$, $j=1,\ldots,n$). We only consider two periods, i.e., $t=0,1$. Let $x_i=\{x_{i1},\ldots,x_{in}\}'$ and $p_i(t)=\{p_{i1}(t),\ldots,p_{in}(t)\}'$. The expected return of the loan portfolio for branch i is expressed as $E^i[(p_i(1)-p_i(0))'x_i]$, where we assume that each branch has its own distinct beliefs on the return of its loan portfolio. So, our model is heterogeneous for branches.

We use CVaR as a risk measure for the portfolio and let C_{α_i} be the upper limit loss for branch i, though our formulation can be extended to other risk measures as well, such as variance and VaR. Therefore, the loan portfolio selection problem for branch i can be formulated as follows:

$$
\begin{aligned}
(L_i) \qquad & \max \quad E^i[(p_i(1)-p_i(0))'x_i] \\
& \text{s.t.} \quad p_i(0)'x_i \le w_i \\
& \qquad CVaR_{\alpha_i}([p_i(0)-p_i(1)]'x_i) \le C_{\alpha_i} \\
& \qquad p_{ij}(0)x_{ij} \le c_j, \qquad j=1,\cdots,n \\
& \qquad x_{ij} \ge 0, \qquad j=1,\cdots,n,
\end{aligned}
$$

where w_i is the capital allocated to branch i by the bank and c_j is the upper limit set by the bank for the total loan lent to type j customers.

At the higher level, the headquarter has a total capital of w, and its objective is to maximize the total expected returns of all branches with an upper limit C_α on the overall risk. Therefore, the corresponding loan portfolio selection problem can be formulated as:

$$
\begin{aligned}
(H) \qquad & \max \quad \sum_{i=1}^{m} E^i[(p_i(1)-p_i(0))'x_i^*] \\
& \text{s.t.} \quad \sum_{i=1}^{m} p_{ij}(0)x_{ij}^* \le c_j, \qquad j=1,\cdots,n, \\
& \qquad CVaR_\alpha(\sum_{i=1}^{m}[p_i(0)-p_i(1)]'x_i^*) \le C_\alpha \\
& \qquad \sum_{i=1}^{m} w_i = w \\
& \qquad w_i \ge 0, \qquad i=1,\cdots,m,
\end{aligned}
$$

where x_i^* is the optimal solution of (L_i), which depends on w_i. Together, (L_i) and (H) form a two-level loan portfolio selection problem, which in general is quite difficult to solve. In the next section, we will propose a procedure to solve this two-level optimization problem.

We should point out that an alternative one-level formulation is also proposed in Hu et al. (2010) where numerical results are provided to compare it with the two-level formulation. In fact, their numerical results show that the one-level formulation is a good approximation for the two-level formulation.

3 A PROCEDURE FOR SOLVING THE TWO-LEVEL PROBLEM

In this section, we propose a numerical method to solve the two-level loan portfolio optimization problem presented in the previous section.

Let us first consider the lower level problem L_i. To solve L_i, we use the gradient-based simulation method proposed in (Hong and Liu 2009). For ease of exposition, let $f_i(x_i) = -E^i[(p_i(1)-p_i(0))'x_i]$. Define the Lagrange function

$$
\begin{aligned}
& L(x_i,\lambda_i,\mu_i,\tau_j,\gamma_j) \\
= \quad & f_i(x_i) + \lambda_i[p_i(0)'x_i - w_i] + \mu_i[CVaR_{\alpha_i}([p_i(0)-p_i(1)]'x_i) - C_{\alpha_i}] + \sum_{i=1}^{n}\tau_j(p_{ij}(0)x_{ij}-c_j) + \sum_{i=1}^{n}\gamma_j(-x_{ij})
\end{aligned}
$$

where λ_i, μ_i, τ_j, γ_j are Lagrange multipliers. Then, according to the duality theory, we have $\frac{\partial f_i(x_i^*)}{\partial w_i} = -\lambda_i^*$, where x_i^* is the optimal solution of $L_i(w_i)$ and λ_i is the corresponding Lagrange multiplier.

We notice that the objective function of the upper level problem (H) is the sum of all branches, i.e, $f(x) = \sum_{i=1}^{m} f_i(x_i(w_i))$, where $x \triangleq (x_1(w_1),\cdots,x_m(w_m))$. So the negative gradient of $f(x)$ can be expressed as $-\nabla_w f(x^*) \triangleq \lambda = (\lambda_1,\cdots,\lambda_m)'$, which is a descent direction of $f(x)$ in space R^m.

To solve the optimization problem (H), we use the following gradient-base stochastic approximation algorithm to update w: method can be generally formed as

$$
w^{(k+1)} = w^{(k)} + \alpha^{(k)} d^{(k)}
$$

where $d^{(k)}$ is a descent direction and $\alpha^{(k)} \geq 0$ is the step size. Since $\sum_{i=1}^{m} w_i = w$, we must have $\sum_{i=1}^{m} d_i^{(k)} = 0$. Hence, we need to project the descent direction λ on the hyperplane $\sum_{i=1}^{m} d_i^{(k)} = 0$ and obtain a new direction

$$d = \lambda - \frac{(\lambda, e)}{\|e\|_2^2} e$$

where $e = (1, \cdots, 1)'_{m\times 1}$.

Once the descent direction is determined, we need to choose an appropriate step-size $\alpha^{(k)}$. To do that, we start with an initial step-size $\alpha^{(k)}$, if the corresponding $w^{(k+1)}$ is a feasible solution, then we could increase the step size, otherwise, we would have to decrease it until the corresponding $w^{(k+1)}$ becomes feasible or we need to choose a different descent direction. We note that $0 \leq w_i^{(k+1)} = w_i^{(k)} + \alpha^{(k)} d^{(k)} \leq w \ (i = 1, \cdots, m)$, therefore

$$0 \leq \alpha^{(k)} \leq \frac{w - w_i^{(k)}}{d_i^{(k)}} \qquad \text{if } d_i^{(k)} > 0;$$

$$0 \leq \alpha^{(k)} \leq -\frac{w_i^{(k)}}{d_i^{(k)}} \qquad \text{if } d_i^{(k)} < 0.$$

Let $\alpha_t^{(k)}$ denote the t-th modification of step size $\alpha^{(k)}$ and $\beta_t^{(k)}$ denote the minimum value for the t-th modification. In order to insure that $w^{(k+1)}$ is feasible, we modify $\alpha_t^{(k)}$ as follows:

$$\alpha_t^{(k)} = \min\left\{\beta_t^{(k)}, \min_{d_i^{(k)}>0}\left\{\frac{w - w_i^{(k)}}{d_i^{(k)}}\right\}, \min_{d_j^{(k)}<0}\left\{-\frac{w_j^{(k)}}{d_j^{(k)}}\right\}\right\}.$$

Based on what we discussed above, we propose the following algorithm:

1. For $k = 0$ and a given initial feasible solution $w^{(k)} = (w_1^{(0)}, \cdots, w_m^{(0)})$, we solve each lower level problem $L_i(w_i^{(0)})$ and obtain the optimal solution $x_i^{(0)*}$, optimal value $f_i^{(0)*}$, and its corresponding Lagrange multiplier $\lambda_i^{(0)}$.

2. Let $\lambda^{(k)} = (\lambda_1^{(k)}, \cdots, \lambda_m^{(k)})$. In order to ensure $\sum_{i=1}^{m} w_i = w$ and we update the descent direction $d^{(k)}$ as $d^{(k)} = \lambda^{(k)} - \frac{(\lambda^{(k)},\ e^{(k)})}{\|e^{(k)}\|_2^2} e^{(k)}$.

3. In this step, we compute an appropriate step-size $\alpha^{(k)}$ along the descent direction $d^{(k)}$. The procedure works as follows:

a. For $t = 0$, let $w_t^{(k)} = w^{(k)}$, $\beta_t^{(k)} = 1$. Pre-specify $\eta > 1$, $\sigma < 1$, and $\varepsilon_1 << 1$.

b. Set

$$\alpha_t^{(k)} = \min\left\{\beta_t^{(k)}, \min_{d_i^{(k)}>0}\left\{\frac{w - w_i^{(k)}}{d_i^{(k)}}\right\}, \min_{d_j^{(k)}<0}\left\{-\frac{w_j^{(k)}}{d_j^{(k)}}\right\}\right\}.$$

If $\beta_t^{(k)} > \alpha_t^{(k)}$ or $\alpha_t^{(k)} < \varepsilon_1$, stop.

c. Let $w_t^{(k)} = w_{t-1}^{(k)} + \alpha_t^{(k)} d^{(k)}$ and solve each lower level problem $L_i(w_{t,i}^{(k)})$ to obtain its optimal solution $x_{t,i}^{(k)*}$ and the corresponding Lagrange multipliers $\lambda_{t,i}^{(k)}$.

d. Let $x_t^{(k)*} = (x_{t,1}^{(k)*}, \cdots, x_{t,m}^{(k)*})$, $\lambda_t^{(k)} = (\lambda_{t,1}^{(k)}, \cdots, \lambda_{t,m}^{(k)})$, and compute $d_{t+1}^{(k)}$ as in Step 2. If $(d_{t+1}^{(k)}, d^{(k)}) \leq 0$, stop and goto Step 4, otherwise, consider the following three cases:
If both $x_t^{(k)*}$ and $x_{t-1}^{(k)*}$ are feasible for (H), we increase the step-size by a factor of η (i.e., $\beta_{t+1}^{(k)} = \eta \beta_t^{(k)}$).
If $x_t^{(k)*}$ is feasible but $x_{t-1}^{(k)*}$ is not, stop and goto Step 4.
If $x_t^{(k)*}$ is infeasible for H, we set $\beta_{t+1}^{(k)} = \sigma \beta_t^{(k)}$.
Let $t \leftarrow t+1$ and goto 2.

4. Set $\alpha^{(k)} = \alpha_t^{(k)}$ and $w^{(k+1)} = w^{(k)} + \alpha^{(k)} d^{(k)}$. We solve each lower level problem $L_i(w_i^{(k+1)})$ and obtain optimal solution $x_i^{(k+1)*}$, optimal value $f_i^{(k+1)*}$ and the corresponding Lagrange multiplier $\lambda_i^{(k+1)}$.

5. If $|\sum_{i=1}^{m} f_i^{(k+1)*} - \sum_{i=1}^{m} f_i^{(k)*}| < \varepsilon_2$ (ε_2 is another pre-specified constant), stop; otherwise, set $k \leftarrow k+1$ and goto Step 2.

4 NUMERICAL EXAMPLES

To test the method we proposed in Section 3, in this section, we present three numerical examples.

4.1 Example 1

In this example, we have two branches ($m = 2$) and three types of customers ($n = 3$). The values of various parameters are given in the following table:

α	α_1	α_2	C_α	C_{α_1}	C_{α_2}	$p_i(0)$			c_1	c_2	c_3	w
99%	95%	97%	0.2	0.25	0.15	1	1	1	1	1	1	1

We assume that $\{p_i(0) - p_i(1), i = 1,2\}$ has a multivariate normal distribution whose mean and covariance matrix are given by (the numbers are generated randomly):

$$\mu = (\mu_1, \mu_2) = (-0.1730, -0.2714, -0.8757, -0.9797, -0.2523, -0.7373)$$

$$\Omega = \begin{pmatrix} 1.4846, & 0.3227, & 0.0456, & 0.1748, & 0.1060, & -0.8760 \\ 0.3227, & 0.8009, & 0.3184, & -0.2192, & -0.1618, & 0.0131 \\ 0.0456, & 0.3184, & 0.5031, & 0.0649, & -0.2677, & 0.0470 \\ 0.1748, & -0.2192, & 0.0649, & 0.6676, & -0.1124, & -0.0584 \\ 0.1060, & -0.1618, & -0.2677, & -0.1124, & 0.5729, & -0.1221 \\ -0.8760, & 0.0131, & 0.0470, & -0.0584, & -0.1221, & 0.6983 \end{pmatrix}$$

We first solve the two-level problem (L_i) + (H) using the enumerative method and then compare the results with our method proposed in Section 3. The results are provided in the following table:

	w_1^*	w_2^*	opt. value
Enumerative method	0.36404	0.63596	0.7975
Our method	0.36403	0.63597	0.7980

From the above table, it is clear that our method works very well and the results obtained based on it are very close to those obtained from the enumerative method.

4.2 Example 2

This example is similar to Example 1 and have the following parameter values:

α	α_1	α_2	C_α	C_{α_1}	C_{α_2}	$p_i(0)$			c_1	c_2	c_3	w
99%	96%	97%	0.3	0.25	0.2	1	1	1	1	1	1	1

$$\mu = (\mu_1, \mu_2) = (-0.3475, -0.4972, -0.3097, -0.3246, -0.2535, -0.3609)$$

$$\Omega = \begin{pmatrix} 0.7252, & 0.2385, & 0.1930, & -0.0334, & -0.1955, & -0.3863 \\ 0.2385, & 1.0934, & -0.1405, & -0.4391, & 0.1905, & 0.4216 \\ 0.1930, & -0.1405, & 0.3709, & -0.0822, & -0.1194, & 0.1824 \\ -0.0334, & -0.4391, & -0.0822, & 0.7109, & -0.0981, & -0.5308 \\ -0.1955, & 0.1905, & -0.1194, & -0.0981, & 1.2288, & 0.1146 \\ -0.3863, & 0.4216, & 0.1824, & -0.5308, & 0.1146, & 1.1934 \end{pmatrix}$$

The results are provided in the following table:

	w_1^*	w_2^*	opt. value
Enumerative method	0.12471	0.87529	0.174095
Our method	0.124924	0.875076	0.174217

The results are similar to those for Example 1.

4.3 Example 3

In this example, we have three branches ($m = 3$) and three types of customers ($n = 3$). The values of various parameters are given in the following table:

α	α_1	α_2	α_3	C_α	C_{α_1}	C_{α_2}	C_{α_3}	$p_i(0)$			c_1	c_2	c_3	w
99%	95%	97%	96%	0.4	0.25	0.15	0.2	1	1	1	1	1	1	1

Again, we assume that $\{p_i(0) - p_i(1), i = 1,2,3\}$ has a multivariate normal distribution whose mean and covariance matrix are given by (the numbers are generated randomly):

$$\mu = (\mu_1, \mu_2, \mu_3) = (-0.2981, -0.3639, -0.2428, -0.3765, -0.2080, -0.2692, -0.2115, -0.2243, -0.4059)$$

$$\Omega = \begin{pmatrix} 0.7012\ , & -0.1651\ , & -0.3435\ , & -0.5460\ , & 0.0630\ , & 0.5162\ , & 0.3282\ , & 0.2355\ , & 0.4400 \\ -0.1651\ , & 0.9495\ , & 0.0253\ , & -0.0931\ , & -0.2481\ , & -0.4407\ , & -0.4844\ , & 0.3813\ , & -0.7256 \\ -0.3435\ , & 0.0253\ , & 0.7095\ , & 0.5138\ , & -0.0368\ , & 0.1124\ , & -0.2512\ , & 0.2611\ , & -0.8672 \\ -0.5460\ , & -0.0931\ , & 0.5138\ , & 1.4692\ , & 0.2575\ , & -0.0575\ , & -0.4911\ , & -0.5253\ , & -0.5154 \\ 0.0630\ , & -0.2481\ , & -0.0368\ , & 0.2575\ , & 0.8545\ , & 0.2114\ , & 0.0087\ , & -0.6932\ , & 0.0146 \\ 0.5162\ , & -0.4407\ , & 0.1124\ , & -0.0575\ , & 0.2114\ , & 1.2193\ , & 0.0456\ , & 0.0686\ , & -0.2094 \\ 0.3282\ , & -0.4844\ , & -0.2512\ , & -0.4911\ , & 0.0087\ , & 0.0456\ , & 0.7636\ , & -0.1572\ , & 1.0078 \\ 0.2355\ , & 0.3813\ , & 0.2611\ , & -0.5253\ , & -0.6932\ , & 0.0686\ , & -0.1572\ , & 1.4252\ , & -0.6000 \\ 0.4400\ , & -0.7256\ , & -0.8672\ , & -0.5154\ , & 0.0146\ , & -0.2094\ , & 1.0078\ , & -0.6000\ , & 2.1854 \end{pmatrix}$$

The results are provided in the following table:

	w_1^*	w_2^*	w_3^*	opt. value
Enumerative method	0.562	0.23	0.208	0.241466
Our method	0.57989	0.22739	0.19271	0.24147

Similar to Examples 1 and 2, our method produces very good the solutions.

5 CONCLUSIONS

In this paper, we studied a two-level loan portfolio selection problem and proposed a numerical method to solve the problem. Numerical examples are provided to validate the method. We plan to investigate the convergence properties of the method in our future research.

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