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CONFIDENCE INTERVAL ESTIMATION USING LINEAR COMBINATIONS OF OVERLAPPING VARIANCE ESTIMATORS

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ABSTRACT

We develop new confidence-interval estimators for the mean and variance parameter of a steady-state simulation output process. These confidence intervals are based on optimal linear combinations of overlapping estimators for the variance parameter. We present analytical and simulation-based results exemplifying the potential of this technique for improvements in accuracy for confidence intervals.

1. INTRODUCTION

Steady-state simulations can be used to analyze a variety of complicated systems. To complete a successful steady-state simulation study, one has to carry out a careful statistical analysis of the simulation's output. If the output process $\{Y_i : i \geq 1\}$ is in steady state, then, usually, we are interested in estimating the unknown mean μ of the process. The sample mean $\bar{Y}_n \equiv \sum_{i=1}^{n} Y_i/n$ is the usual estimator for μ. But to complete the picture, the experimenter ought to estimate the sample mean's precision as well. The *variance parameter*,

$$\sigma^2 \equiv \lim_{n\to\infty} n\mathsf{Var}(\bar{Y}_n),$$

is one of the measures of the sample mean's variability.

The objective of our research is to develop new confidence interval estimators for μ and σ^2. These new confidence intervals are based on forming an optimal linear combination of overlapping variance estimators (OLCOVE) (Aktaran-Kalaycı 2006, Aktaran-Kalaycı, Goldsman, and Wilson 2007). It has been shown that an OLCOVE can have lower bias and lower variance than those of the constituent estimators used to construct the linear combination. Using the OLCOVEs, we expect the corresponding confidence intervals to have better performance than certain existing estimators in the literature.

The organization of the remainder of this article is as follows. We first review in §2 some background basics on OLCOVEs and standardized time series (STS). We approximate the distribution of an OLCOVE and then use this approximation to construct confidence intervals in §3. Section 3 also provides Monte Carlo results that support our findings. Finally, we offer a summary in §4.

2. BACKGROUND

2.1 Setup for Using STS Variance Estimators

We assume the stationary stochastic process $\{Y_i : i \geq 1\}$ has steady-state mean μ and variance parameter $\sigma^2 \in (0, \infty)$ such that the following Functional Central Limit Theorem (FCLT) holds.

Assumption FCLT. For $n = 1, 2, \ldots,$

$$X_n(t) \equiv \frac{\lfloor nt \rfloor \big(\bar{Y}_{\lfloor nt \rfloor} - \mu\big)}{\sigma\sqrt{n}} \quad \text{for} \quad t \in [0, 1] \qquad (1)$$

satisfies

$$X_n \underset{n\to\infty}{\Longrightarrow} \mathcal{W},$$

where: $\lfloor \cdot \rfloor$ denotes the greatest integer function; $\mathcal{W}$ is a standard Brownian motion process on [0, 1]; and $\underset{n\to\infty}{\Longrightarrow}$ denotes weak convergence in $D[0, 1]$, the associated probability space of functions on [0, 1] that are right-continuous with left-hand limits, as $n \to \infty$. See also Billingsley (1968).

Suppose we have n observations $Y_1, Y_2, \ldots, Y_n$ on hand, and we form $n - m + 1$ *overlapping batches*, each of size m. In particular, the observations $Y_i, Y_{i+1}, \ldots, Y_{i+m-1}$ constitute batch i, for $i = 1, 2, \ldots, n - m + 1$. Similar to the original definition in Schruben (1983), the standardized time series from overlapping batch i is

$$T^{\mathrm{O}}_{i,m}(t) \equiv \frac{\lfloor mt \rfloor \left(\bar{Y}^{\mathrm{O}}_{i,m} - \bar{Y}^{\mathrm{O}}_{i,\lfloor mt \rfloor} \right)}{\sigma \sqrt{m}} \quad \text{for } 0 \le t \le 1,$$

where $\bar{Y}^{\mathrm{O}}_{i,j} \equiv \sum_{\ell=0}^{j-1} Y_{i+\ell}/j$ for $i = 1, 2, \ldots, n - m + 1$ and $j = 1, 2, \ldots, m$. Alexopoulos et al. (2006) show that, under Assumption FCLT,

$$\sigma T^{\mathrm{O}}_{\lfloor sm \rfloor, m}(\cdot) \underset{m\to\infty}{\Longrightarrow} \sigma \mathcal{B}_{s,1}(\cdot) \quad \text{for fixed} \quad s \in [0, b-1],$$

where $b \equiv n/m > 1$ is a fixed ratio; and for $r \in [1, b)$ and $s \in [0, b - r]$, we let $\mathcal{B}_{s,r}(\cdot)$ denote a Brownian bridge process on the unit interval,

$$\mathcal{B}_{s,r}(t) = \frac{t[\mathcal{W}(s+r) - \mathcal{W}(s)] - [\mathcal{W}(s+tr) - \mathcal{W}(s)]}{\sqrt{r}}, \quad (2)$$

for $t \in [0, 1]$. Moreover, we also have the following useful asymptotic properties as $n \to \infty$:

$$\sqrt{n}\left(\bar{Y}_n - \mu\right) \sim \sigma \mathsf{Nor}(0, 1); \quad \text{and} \quad (3)$$

$$\sqrt{n}\left(\bar{Y}_n - \mu\right) \text{ and } \sigma T^{\mathrm{O}}_{1,n} \text{ are independent.} \quad (4)$$

2.2 Overlapping Area Variance Estimator

The idea is to form a separate estimator for σ^2 from each overlapping batch—even though the resulting estimators are clearly dependent—and then to average those estimators. In the context of this article, the fixed quantity b may in general take any real value exceeding one while we let the batch size $m \to \infty$; and thus the total sample size is always taken to be $\lfloor bm \rfloor$. The STS area estimator from the ith overlapping batch is

$$A^{\mathrm{O}}_i(f; m) \equiv \left[\frac{1}{m} \sum_{\ell=1}^{m} f(\ell/m) \sigma T^{\mathrm{O}}_{i,m}(\ell/m) \right]^2,$$

for $i = 1, 2, \ldots, \lfloor bm \rfloor - m + 1$, where $f(t)$ is a weighting function that has a continuous second derivative $\left(\mathrm{d}^2/\mathrm{d}t^2\right) f(t)$ on the unit interval [0, 1] and that satisfies the normalization condition $\mathsf{Var}(\int_0^1 f(t)\mathcal{B}_{0,1}(t)\,\mathrm{d}t) = 1$. Averaging, we have the *overlapping area estimator* (OAE) for σ^2,

$$\mathcal{A}^{\mathrm{O}}(f; b, m) \equiv \frac{1}{\lfloor bm \rfloor - m + 1} \sum_{i=1}^{\lfloor bm \rfloor - m + 1} A^{\mathrm{O}}_i(f; m).$$

Alexopoulos et al. (2006) show that

$$\mathcal{A}^{\mathrm{O}}(f; b, m) \underset{m\to\infty}{\Longrightarrow} \mathcal{A}^{\mathrm{O}}(f; b) \equiv \quad (5)$$
$$\frac{\sigma^2}{b-1} \int_0^{b-1} \left[\int_0^1 f(t) \mathcal{B}_{s,1}(t)\,\mathrm{d}t \right]^2 \mathrm{d}s.$$

The next theorem gives the expected value of the OAE. Before that we define the following quantities: $R_\ell \equiv \mathsf{Cov}(Y_1, Y_{1+\ell})$ for $\ell = 1, 2, \ldots$;

$$\gamma \equiv -2 \sum_{\ell=1}^{\infty} \ell R_\ell$$

(Song, and Schmeiser 1995);

$$F(t) \equiv \int_0^t f(s)\,\mathrm{d}s, \quad \bar{F}(t) \equiv \int_0^t F(s)\,\mathrm{d}s$$

for $0 \le t \le 1$; and

$$F^\star \equiv \left[\left(F(1) - \bar{F}(1) \right)^2 + \bar{F}^2(1) \right] \Big/ 2.$$

Theorem 1 (See, e.g., Foley and Goldsman 1999.) Suppose that $\{Y_i : i \ge 1\}$ is a stationary process for which Assumption FCLT holds and $\sum_{\ell=1}^{\infty} \ell^2 |R_\ell| < \infty$. Further, suppose that the family of random variables $\left\{ \left[\mathcal{A}^{\mathrm{O}}(f; b, m) \right]^2 : m = 1, 2, \ldots \right\}$ is uniformly integrable (cf. Billingsley 1968). If $f(\cdot)$ satisfies the above-mentioned smoothness and normalization requirements, then

$$\mathsf{E}[\mathcal{A}^{\mathrm{O}}(f; b, m)] = \sigma^2 + \frac{F^\star \gamma}{m} + o\!\left(\frac{1}{m}\right) \quad \text{as } m \to \infty. \quad (6)$$

Calculation of the asymptotic variance of the OAE can be performed using Eq. (5) as described in Alexopoulos et al. (2006), who show that as $m \to \infty$ the variance of the OAE depends on the choice of the function $f(\cdot)$. In fact, all the OAEs discussed subsequently have smaller limiting variances than those of the following:

(*a*) the benchmark nonoverlapping batch means variance estimator, whose variance is $2\sigma^4/(b-1)$ (see Chien, Goldsman, and Melamed 1997);

(*b*) the overlapping batch means variance estimator, whose variance is approximately $\frac{4}{3}\sigma^4/b$ (see Meketon and Schmeiser 1984); and

(*c*) the nonoverlapping version of the area estimator with any legal weighting function, whose variance is $2\sigma^4/b$ (see Goldsman, Meketon, and Schruben 1990).

Example 2 Schruben (1983) studied the nonoverlapping version of the area estimator with $f_0(t) \equiv \sqrt{12}$ for $0 \le t \le 1$. For the overlapping version, Theorem 1 yields $\mathsf{E}[\mathcal{A}^{\mathrm{O}}(f_0; b, m)] = \sigma^2 + 3\gamma/m + o(1/m)$. Further, as $m \to \infty$, some algebra involving Eq. (5) gives the asymptotic variance

$$\mathsf{Var}(\mathcal{A}^{\mathrm{O}}(f_0; b)) = \frac{24b - 31}{35(b-1)^2}\sigma^4 \quad \text{for } b \ge 2.$$

If one chooses weights having $F^\star = 0$, then the resulting OAE is *first-order unbiased* for σ^2, i.e., its bias is $o(1/m)$. Such an example is $f_2(t) \equiv \sqrt{840}(3t^2 - 3t + 1/2)$ for $0 \le t \le 1$ (see Goldsman, Meketon, and Schruben 1990, Goldsman and Schruben 1990); and for this weighting function, the asymptotic variance as $m \to \infty$ is

$$\mathsf{Var}(\mathcal{A}^{\mathrm{O}}(f_2; b)) = \frac{3514b - 4359}{4290(b-1)^2}\sigma^4 \quad \text{for } b \ge 2.$$

2.3 Optimal Linear Combination of Overlapping Area Estimators

For the OLCOVEs, we calculate k OAEs, $\mathcal{A}^{\mathrm{O}}(f; B_1, M_1)$, $\mathcal{A}^{\mathrm{O}}(f; B_2, M_2)$, ..., $\mathcal{A}^{\mathrm{O}}(f; B_k, M_k)$, where we use the same data over a variety of different integer batch sizes, M_1, M_2, ..., M_k. Specifically, for $j = 1, 2, \ldots, k$, we let $r_j \in [1, \infty)$ and take $M_j \equiv \lfloor r_j m \rfloor$ as the jth batch size, with the corresponding quantity $B_j \equiv \lfloor bm \rfloor / M_j$ so that the total sample size is $B_j M_j = \lfloor bm \rfloor$, and B_j is the associated sample-size-to-batch-size ratio. Then we form a linear combination of these k estimators and scale appropriately. We use standard regression techniques underlying the method of control variates (Lavenberg and Welch 1981) to determine scaling factors that preserve low bias and minimize variance.

Let $\boldsymbol{M} \equiv [M_1, M_2, \ldots, M_k]$ and $\boldsymbol{B} \equiv [B_1, B_2, \ldots, B_k]$. Let $\boldsymbol{\alpha} \equiv [\alpha_1, \alpha_2, \ldots, \alpha_{k-1}]$ denote the vector of weights used to form the linear combination

$$\mathcal{A}^{\mathrm{LO}}(f; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}) \equiv \sum_{i=1}^{k-1} \alpha_i \mathcal{A}^{\mathrm{O}}(f; B_i, M_i) + \left(1 - \sum_{i=1}^{k-1} \alpha_i\right) \mathcal{A}^{\mathrm{O}}(f; B_k, M_k). \tag{7}$$

Theorem 3 (Aktaran-Kalaycı 2006) Let $\boldsymbol{r} \equiv [r_1, r_2, \ldots, r_k]$ denote the vector of batch-size multipliers. If $b > \max\{r_j : j = 1, \ldots, k\}$, then let $\boldsymbol{\mathcal{A}}^{\mathrm{O}}(f; \boldsymbol{B}, \boldsymbol{r}, \boldsymbol{M})$ denote the $k \times 1$ vector whose jth component is $\mathcal{A}^{\mathrm{O}}(f; B_j, M_j)$ for $j = 1, \ldots, k$. Similarly, let $\boldsymbol{\mathcal{A}}^{\mathrm{O}}(f; b, \boldsymbol{r})$ denote the $k \times 1$ vector whose jth component is

$$\mathcal{A}_j^{\mathrm{O}}(f; b, r_j) \equiv (b - r_j)^{-1} \int_0^{b-r_j} \left[\sigma \int_0^1 f(t)\mathcal{B}_{s,r_j}(t)\,\mathrm{d}t\right]^2 \mathrm{d}s$$

for $j = 1, \ldots, k$. If Assumption FCLT holds and $f(t)$ is normalized as discussed in Section 2.2, then

$$\boldsymbol{\mathcal{A}}^{\mathrm{O}}(f; \boldsymbol{B}, \boldsymbol{r}, \boldsymbol{M}) \underset{m\to\infty}{\Longrightarrow} \boldsymbol{\mathcal{A}}^{\mathrm{O}}(f; b, \boldsymbol{r}). \tag{8}$$

We calculate the asymptotic covariances of the OAEs based on different batch sizes using Eq. (8) as described in Aktaran-Kalaycı, Goldsman, and Wilson (2007). Then we compute the variance-optimizing coefficients in the linear combination, $\boldsymbol{\alpha}^\star$, based on these covariances. Table 1 presents properties of various OLCOVEs based on different batch size sets $\boldsymbol{M}$. We see that the estimator has bias of the form $\gamma c/m$, where the constant c decreases from 3.00 to 2.59 as we add more and more terms (up to 20) to the linear combination; the estimator $\mathcal{A}^{\mathrm{LO}}(f_2; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$ only has $o(1/m)$ bias. Further, in this example, the standardized variance of $\mathcal{A}^{\mathrm{LO}}(f_0; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$ decreases from 0.686 to 0.610 (about a 12% reduction) as we add more terms, while that of $\mathcal{A}^{\mathrm{LO}}(f_2; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$ decreases from 0.819 to 0.688 (about a 16% reduction).

Table 1: Approximate asymptotic bias and variance of various OLCOVEs $\mathcal{A}^{\mathrm{LO}}(f; \boldsymbol{B}, \boldsymbol{M}_k, \boldsymbol{\alpha}^\star)$ (where $\boldsymbol{M}_k \equiv [m, 2m, \ldots, km]$) as $m \to \infty$.

	f_0		f_2	
k	$\frac{m}{\gamma}$Bias	$\frac{b}{\sigma^4}$Var	$\frac{m}{\gamma}$Bias	$\frac{b}{\sigma^4}$Var
1	3.00	0.686	$o(1)$	0.819
2	2.77	0.663	$o(1)$	0.782
3	2.71	0.638	$o(1)$	0.731
4	2.67	0.630	$o(1)$	0.722
5	2.64	0.625	$o(1)$	0.710
10	2.61	0.615	$o(1)$	0.695
20	2.59	0.610	$o(1)$	0.688

3. DENSITY AND CONFIDENCE-INTERVAL ESTIMATION

To estimate the probability density functions (p.d.f.'s) of the estimators, we generated 1,000,000 independent sample paths from an i.i.d. standard normal process, where each path consists of 10,240 observations with $m = 512$ and $b = 20$. Note that these settings are large enough for approximate convergence of the estimators to their limiting distributions. From these sample paths we computed the nonoverlapping, overlapping ($\mathcal{A}^{\mathrm{O}}(f; b, m)$), and OLCOVE

($\mathcal{A}^{\text{LO}}(f; \boldsymbol{B}, \boldsymbol{M}_5, \boldsymbol{\alpha}^\star)$) versions of area estimators, where $\boldsymbol{M}_5 = [m, \ldots, 5m]$.

From the computed estimates, we obtained the empirical p.d.f.'s. Superimposing the empirical p.d.f.'s of estimators gives Figure 1. The figure shows that the OLCOVEs have lighter tails (i.e., lower variance) than the underlying OAEs, and the OAEs have lighter tails than do their nonoverlapping counterparts.

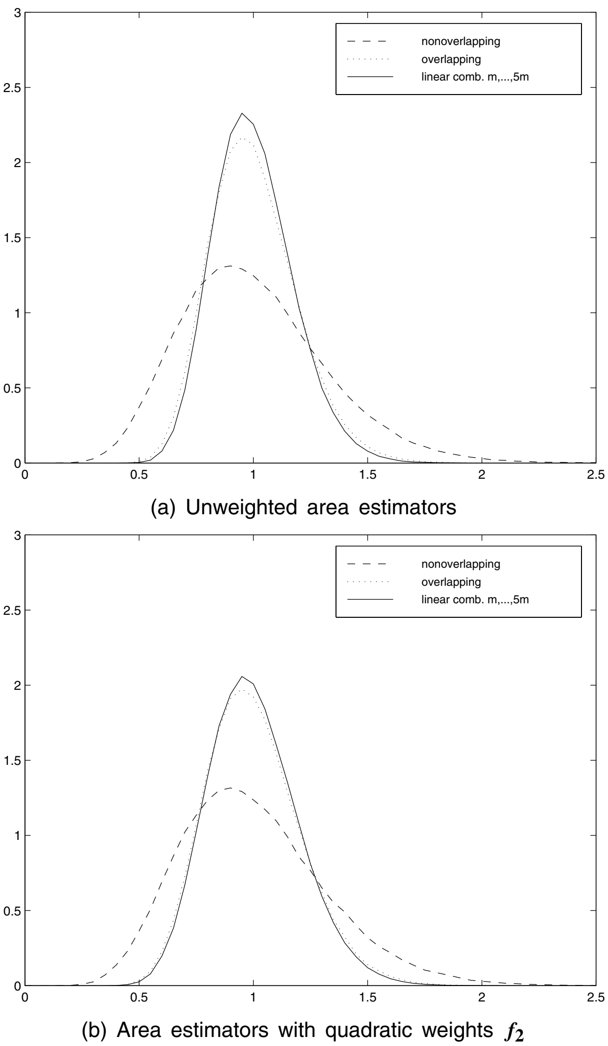


Figure 1: Estimated p.d.f.'s for STS area variance estimators based on 1,000,000 sample paths of the i.i.d. standard normal process with $m = 512$ and $b = 20$.

Letting $\dot\sim$ denote the phrase "is approximately distributed as," we adopt the approximation technique of Satterthwaite (1941) to obtain

$$\mathcal{V}(b, m) \;\dot\sim\; \mathsf{E}[\mathcal{V}(b, m)]\chi^2_{\nu_{\text{eff}}}\big/\nu_{\text{eff}}, \tag{9}$$

where $\mathcal{V}(b, m)$ is a generic variance estimator and the quantity ν_{eff} is called the effective degrees of freedom (d.o.f.) and is given by

$$\nu_{\text{eff}} = \left[\!\left[2\mathsf{E}^2[\mathcal{V}(b, m)]/\mathsf{Var}(\mathcal{V}(b, m)) \right]\!\right]. \tag{10}$$

Using (9) and (10), we calculate the effective d.o.f. for various OLCOVEs, with $\boldsymbol{M}_k = [m, \ldots, km]$ and the jth element of $\boldsymbol{B}$ defined as $B_j = \lfloor bm \rfloor/(jm)$ for $j = 1, \ldots, k$. The results are given in Table 2 for the various OLCOVEs under consideration here. From (5) we see that σ^2 simply serves as a scale parameter in the limiting distribution of $\mathcal{A}^{\text{O}}(f; b, m)$ as $m \to \infty$; and thus the formula (10) for the effective d.o.f. applies asymptotically to *all* stochastic processes satisfying Assumption FCLT as $m \to \infty$.

Table 2: Effective degrees of freedom ν_{eff} for various OLCOVEs with $b = 20$ for all processes satisfying Assumption FCLT as $m \to \infty$.

	Effective Degrees of Freedom				
Estimator	$\boldsymbol{M}_1$	$\boldsymbol{M}_2$	$\boldsymbol{M}_3$	$\boldsymbol{M}_4$	$\boldsymbol{M}_5$
$\mathcal{A}^{\text{LO}}(f_0; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$	56	58	60	60	61
$\mathcal{A}^{\text{LO}}(f_2; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$	47	49	52	53	53

To illustrate the accuracy of our approximations to the limiting p.d.f.'s of the variance estimators, Figure 2 superimposes the empirical p.d.f.'s of the OLCOVEs and the corresponding fitted p.d.f.'s based on the approximation (9) and (10) (using ν_{eff} from Table 2),

$$\mathcal{A}^{\text{LO}}(f; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star) \;\dot\sim\; \sigma^2\chi^2_{\nu_{\text{eff}}}\big/\nu_{\text{eff}}. \tag{11}$$

Figure 2 shows that we have very good approximations to the target p.d.f.'s. This finding immediately suggests that we can use OLCOVEs to construct approximately valid confidence intervals for the parameters μ and σ^2.

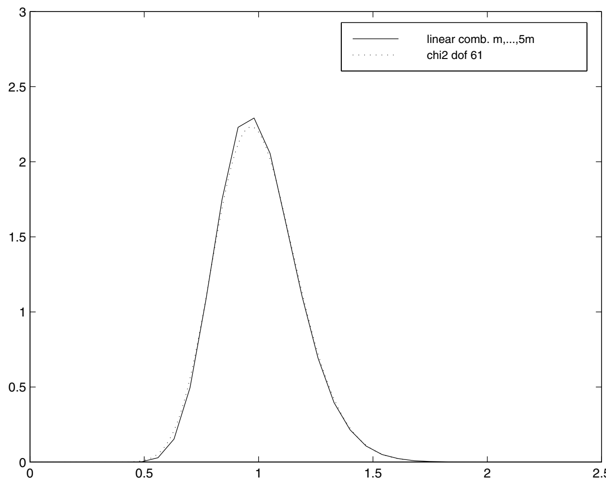
3.1 Confidence Intervals for μ

It follows from the properties given in Eq.s (3), (4), and (11) that

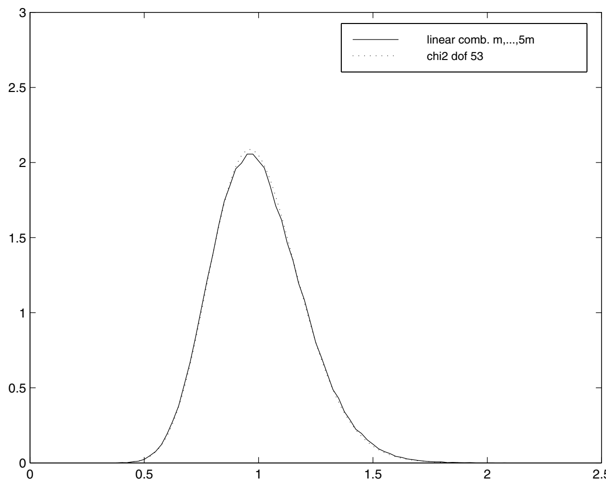
$$\frac{\bar{Y}_n - \mu}{\sqrt{\mathcal{A}^{\text{LO}}(f; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)/n}} \;\dot\sim\; t_{\nu_{\text{eff}}}, \tag{12}$$

for sufficiently large m, where t_ν denotes a Student's t random variable with ν d.o.f. Then an approximate $100(1-\beta)\%$ confidence interval for μ is given by

$$\bar{Y}_n \pm t_{1-\beta/2, \nu_{\text{eff}}}\sqrt{\mathcal{A}^{\text{LO}}(f; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)/n}, \tag{13}$$



(a) Unweighted area estimators



(b) Area estimators with quadratic weights f_2

Figure 2: Estimated and fitted p.d.f.'s for various OLCOVEs based on 1,000,000 sample paths of the i.i.d. standard normal process with $m = 512$, and $b = 20$.

where $t_{\omega,\nu}$ denotes the ω-quantile of a Student's t random variable with ν d.o.f.

Example 4 We use 1,000,000 independent sample paths of a first-order autoregressive [AR(1)] process to compute 1,000,000 realizations of $\bar{Y}_n$ and various OLCOVEs. The specific AR(1) process is given by $Y_i = \phi Y_{i-1} + \epsilon_i$, $i = 1, 2, \ldots$, where the ϵ_i's are i.i.d. $\mathsf{Nor}(0, 1 - \phi^2)$, Y_0 is initialized as a standard normal random variate, independently of the ϵ_i's, and $-1 < \phi < 1$. For the current experiment, we take $\phi = 0.9$, a value that yields a moderately highly autocorrelated stationary process. We construct 1,000,000 two-sided 90% confidence intervals for μ from Eq. (13), where the corresponding value ν_{eff} for the effective degrees of freedom is given in Table 2. Then we estimate the coverage probabilities for the two-sided 90% confidence intervals by determining the proportion of the confidence intervals that actually contain $\mu = 0$. These results are presented in Table 3.

Table 3: Estimated coverage probabilities of two-sided 90% confidence intervals for μ from various OLCOVEs based on 1,000,000 sample paths of the AR(1) process with $\phi = 0.9$, $m = 1024$, and $b = 20$.

	$\mathcal{A}^{\text{LO}}(f_0; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$	$\mathcal{A}^{\text{LO}}(f_2; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$
$\boldsymbol{M}_1$	0.8958	0.8997
$\boldsymbol{M}_2$	0.8961	0.8997
$\boldsymbol{M}_3$	0.8960	0.8999
$\boldsymbol{M}_4$	0.8962	0.8998
$\boldsymbol{M}_5$	0.8962	0.8999

From Table 3, we observe that the achieved empirical coverage probabilities do not differ substantially from the targeted coverage probability, 0.90, thus indicating that the confidence interval procedure works for large-enough batch size m.

3.2 Confidence Intervals for σ^2

Assuming $\mathsf{E}[\mathcal{A}^{\text{LO}}(f; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)] = \sigma^2$ and using Eq. (11), we see that an approximate $100(1-\beta)\%$ confidence interval for σ^2 is given by

$$\frac{\nu_{\text{eff}}\, \mathcal{A}^{\text{LO}}(f; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)}{\chi^2_{1-\beta/2,\nu_{\text{eff}}}} \le \sigma^2 \le \frac{\nu_{\text{eff}}\, \mathcal{A}^{\text{LO}}(f; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)}{\chi^2_{\beta/2,\nu_{\text{eff}}}}, \quad (14)$$

where $\chi^2_{\omega,\nu}$ denotes the ω-quantile of the χ^2 distribution with ν degrees of freedom.

Example 5 We use the OLCOVEs computed from the 1,000,000 independent sample paths of the AR(1) process from Example 4 to construct two-sided 90% confidence intervals for σ^2. The two-sided confidence intervals for σ^2 are given by Eq. (14), where the corresponding ν_{eff} is from Table 2. We obtain the estimated coverage probabilities for the two-sided 90% confidence intervals as presented in Table 4.

Table 4: Estimated coverage probabilities of two-sided 90% confidence intervals for σ^2 from various OLCOVEs based on 1,000,000 sample paths of the AR(1) process with $\phi = 0.9$, $m = 1024$, and $b = 20$.

	$\mathcal{A}^{\text{LO}}(f_0; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$	$\mathcal{A}^{\text{LO}}(f_2; \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$
$\boldsymbol{M}_1$	0.8697	0.8891
$\boldsymbol{M}_2$	0.8708	0.8895
$\boldsymbol{M}_3$	0.8706	0.8902
$\boldsymbol{M}_4$	0.8730	0.8887
$\boldsymbol{M}_5$	0.8719	0.8906

In obtaining Eq. (14), we have assumed that all the estimators are unbiased for σ^2. However, we know that

$\mathcal{A}^{\text{LO}}(f_0;\, \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$ is moderately biased. Thus, our assumption causes the coverage probabilities to be a bit off for the $\mathcal{A}^{\text{LO}}(f_0;\, \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$ entries in Table 4. On the other hand, the empirical coverage probabilities do not differ substantially from the targeted coverage probability for the $\mathcal{A}^{\text{LO}}(f_2;\, \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\alpha}^\star)$ estimators.

4. SUMMARY AND RECOMMENDATIONS

The goal of this paper has been to study a new class of confidence interval estimators for μ and σ^2 arising from a stationary simulation output process. The new estimators are based on linear combinations of overlapping estimators, where each constituent of the linear combination uses a different batch size.

We have shown that the distributions of the OLCOVEs can be approximated quite accurately by scaled χ^2 distributions with appropriate degrees of freedom (at least for the specific AR(1) process we considered). We have applied this approximation to construct confidence intervals for the parameters μ and σ^2 based on various OLCOVEs; and we have conducted Monte Carlo studies to see how these confidence intervals perform when applied to simple stochastic processes. For the confidence intervals for μ, we found that the achieved coverage probability is practically the same as the targeted coverage probability (for large enough batch sizes). Regarding the confidence intervals for σ^2, we found that actual coverage probabilities based on first-order unbiased estimators of σ^2 do not differ substantially from the targeted coverage probabilities for batch sizes that are large enough.

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