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A TEST OF RANDOMNESS BASED ON THE DISTANCE BETWEEN CONSECUTIVE RANDOM NUMBER PAIRS

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ABSTRACT

One of the most popular methods of generating "random" numbers for use in a computer program employs a Lehmer random number generator. If constructed properly, these generators will yield streams of numbers that will appear random but can also be easily replicated. However, Marsaglia (1968) discovered a potential flaw in these random number generators, namely, if consecutive n-tuples of the generated numbers are plotted in n-dimensional space, the points may fall into a small number of hyperplanes. In this paper, we compare the theoretical distribution of the distances to the actual distances produced by a specific random number generator in order to devise a statistical test of randomness for the validity of the random number generator.

1 INTRODUCTION

Since random numbers play a pivotal role in Monte Carlo simulation, discrete-event simulation and bootstrapping, there is a need for random number generators with good statistical properties. Some of the most common types of random number generators are of the class of Lehmer generators. These generators are based on Lehmer's algorithm which defines an iterative equation that can be used to produce a stream of random numbers (Lehmer 1951; Leemis and Park 2006). A Lehmer generator requires three input parameters, m, a and x_0. The *modulus* parameter m is a fixed large prime integer. The *multiplier* parameter a is a fixed positive integer less than m. The *initial seed* x_0 can be any positive integer less than m and can vary from experiment to experiment. Lehmer's algorithm produces integers in the range $(1, m-1)$ via the iterative equation $x_{i+1} = ax_i \bmod m$ for $i = 0, 1, \ldots$. For proper choices of a and m (Law and Kelton 2000), the period length of the generator is $m-1$. To convert the x_i's into U(0,1) pseudo-random variates, each is divided by the modulus m, $u_i = x_i/m$, yielding a rational number in the set $\{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}\}$.

It is known that the numbers produced by a Lehmer generator are not truly random, but with carefully chosen m and a it is possible to generate output that will be "random enough" from a statistical point of view. Lehmer generators are also considered "good" generators because their output can be replicated, they're portable, efficient and thoroughly documented. However, a disturbing amount of regularity inherent in the output of these generators was discovered by Marsaglia (1968). He observed a lattice structure when consecutive random numbers were plotted as overlapping ordered n-tuples (i.e, $(x_0, x_1, x_2, \ldots, x_n)$, $(x_1, x_2, \ldots, x_{n+1}), \ldots$). An example of this structure for consecutive pairs of random numbers can be seen in Figure 1, which was created using a full-period Lehmer generator with modulus $m = 401$ and multiplier $a = 23$.

While it is true that this lattice structure does not appear to be random at all, it may be the case that there is still a degree of randomness hidden within it. This randomness is not to be found in the structure of the lattice, however, but in the order in which the points are generated. More specifically, the empirical distribution of the distance between consecutive random number pairs should be close to the theoretical distance between consecutive random number pairs for a good random number generator. We develop a test of randomness based on the distance between consecutive random number pairs.

The paper is organized as follows. The distribution of the distance between consecutive points and associated goodness-of-fit tests is discussed in Section 2. Section 3 shows results obtained by testing Lehmer generators with $m = 401$. Section 4 discusses possibilities for future research. Section 5 contains conclusions.

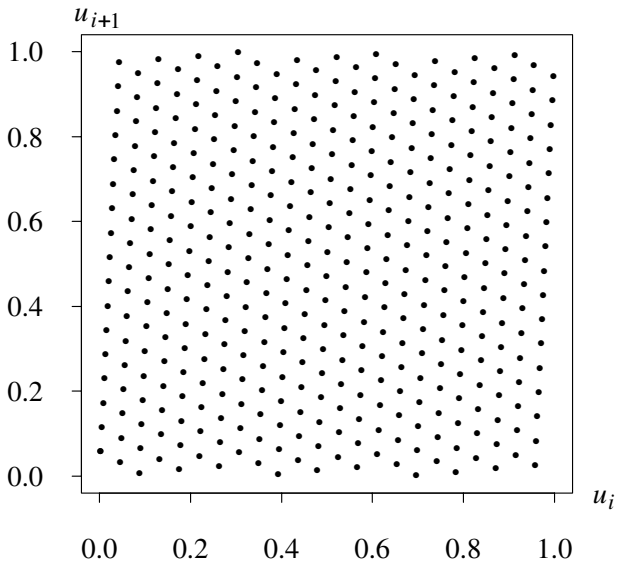

Figure 1: An Example of the Lattice Structure Observed by Marsaglia

2 THE TEST FOR RANDOMNESS

A random number generator is supposed to produce a stream of numbers that appears to be random despite the fact that they are generated by a deterministic function. Even though Marsaglia discovered that a plot of the n-tuples constructed from consecutive random numbers results in a lattice structure, the order in which the points are generated will affect the generator's statistical properties. That is, we hope to observe that points generally are not generated in order along a hyperplane nor is there necessarily a regular pattern of jumping between hyperplanes. With this in mind, we compare the distribution of the distances between consecutive points (ordered pairs) and the distribution of the distances between truly random points on the unit square. The assumption being made is that generators that yield a distribution of distances similar to the purely random points will be better random number generators.

2.1 Overlapping vs. Non-Overlapping Pairs

When considering the distance between consecutive pairs of random numbers, the first question that must be addressed is whether to consider overlapping or non-overlapping ordered pairs. When considering overlapping pairs, a set of three consecutive random numbers such as x_i, x_{i+1}, x_{i+2} can be used to construct the two points (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}). With non-overlapping pairs, it becomes necessary to introduce a fourth random number, x_{i+3}, in order to construct the two points (x_i, x_{i+1}) and (x_{i+2}, x_{i+3}). Both approaches are valid and are discussed in this paper. The mathematics involved with the non-overlapping pairs is significantly easier than that required for the overlapping pairs and for that reason the non-overlapping case will be discussed first. The advantage of the non-overlapping pairs is that we may assume that the four random numbers are independent and thus the two points they represent are also independent. When considering overlapping pairs the three random numbers generated are also assumed to be independent; however the points represented are not independent because the second coordinate of the first point is also the first coordinate of the second point.

When working with non-overlapping pairs, it must be noted that analysis is only performed on half of the possible ordered pairs at a time and then merged together at the end. The two sets of ordered pairs that are examined are $\{(x_0, x_1), (x_2, x_3), (x_4, x_5), \ldots, (x_{m-3}, x_{m-2})\}$ and $\{(x_1, x_2), (x_3, x_4), \ldots, (x_{m-2}, x_{m-1})\}$, where the first set consists of all ordered pairs with the first coordinate having an even index and the second set consists off all ordered pairs with the first coordinate having an odd index. The distances are calculated between consecutive points in the same set and then combined to form a single distribution. When the two sets are joined for the analysis, we revert to the overlapping pairs case. It should be noted that in both cases, the same number of U(0, 1) random numbers are used.

2.2 The Theoretical Distributions

2.2.1 Non-Overlapping Pairs

Under the assumption that the random number pairs contain non-overlapping coordinates, the calculation of the theoretical distribution of the distances between consecutive pairs of randomly generated numbers is straightforward. If we assume that X_1, X_2, X_3, X_4 are i.i.d. U(0, 1) random variables, then we must simply find the distribution of the distance between (X_1, X_2) and (X_3, X_4) given by $D = \sqrt{(X_1 - X_3)^2 + (X_2 - X_4)^2}$. Since D represents a distance between two points, it is obvious that $D > 0$ (the inequality is strict because the Lehmer algorithm cannot generate two identical pairs unless $m = 3$). Furthermore, since the longest distance possible is across the diagonal of the unit square, $D < \sqrt{2}$ (the inequality is strict because the Lehmer algorithm cannot generate the numbers 0 or 1).

An easy way to find the distribution of D is to use the Maple-based APPL programming language (Glen et al. 2001). This avoids a rather cumbersome hand calculation. The theoretical probability density function (pdf) of D is

$$f(x) = \begin{cases} 2x(x^2 - 4x + \pi) & 0 < x \le 1 \\ \frac{-2x\left((2+x^2)\sqrt{x^2-1}+4-4x^2\right)}{\sqrt{x^2-1}} + & \\ \quad \frac{-2x\left(2\sqrt{x^2-1}\arcsin\left(\frac{x^2-2}{x^2}\right)\right)}{\sqrt{x^2-1}} & 1 < x < \sqrt{2}, \end{cases}$$

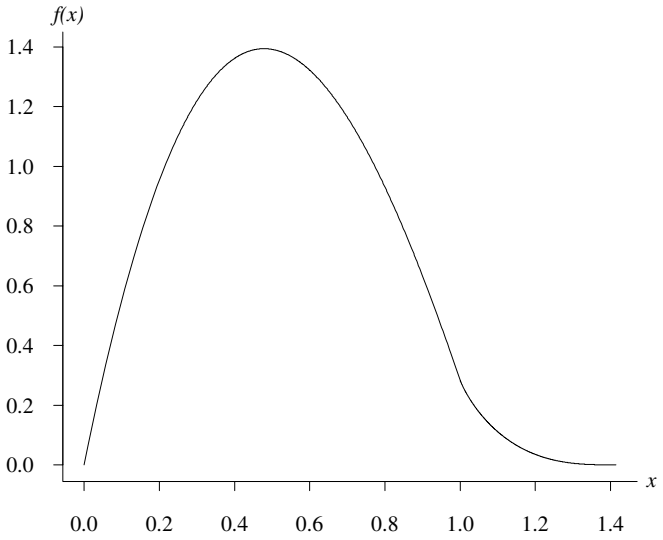

Figure 2: The PDF of the Theoretical Distribution of the Distances Between Points in the Non-Overlapping Case

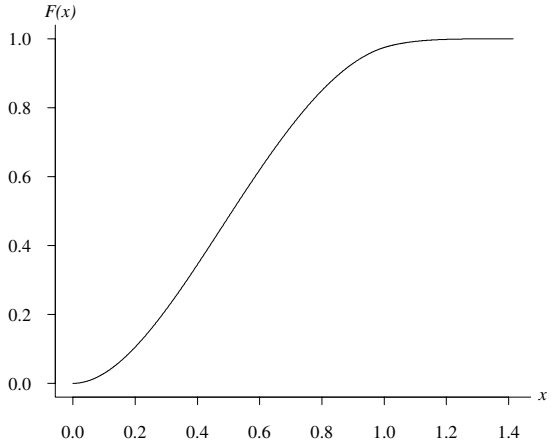

Figure 3: The CDF of the Theoretical Distribution of the Distances Between Points in the Non-Overlapping Case

which is a continuous function since $\lim_{x\uparrow 1} f(x) = \lim_{x\downarrow 1} f(x) = 2(\pi - 3)$. The graph in Figure 2 is a plot of the pdf.

The corresponding cumulative distribution function (cdf) is

$$F(x) = \begin{cases} \frac{1}{2}x^4 - \frac{8}{3}x^3 + \pi x^2 & 0 < x \le 1 \\ \frac{\left(-2+12x^2 \arcsin\frac{x^2-2}{x^2} - 3x^4 + 12x^2\right)}{-6} + & \\ \frac{-24x^2+24-16(x^2-1)(x^2+1)}{-6\sqrt{x^2-1}} & 1 < x < \sqrt{2}. \end{cases}$$

The graph in Figure 3 is a plot of the cdf. The APPL code used to generate these functions is given in Appendix A of Duggan et al. (2005).

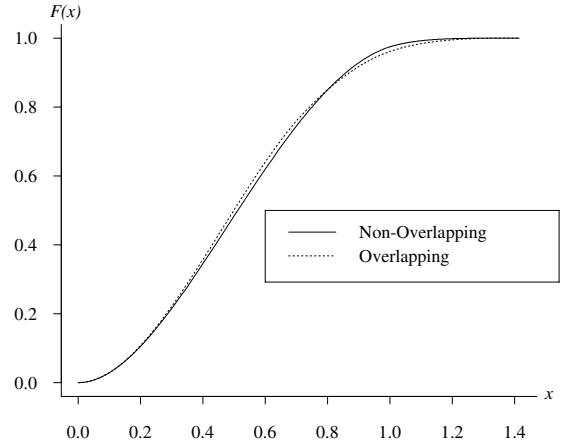

Figure 4: Comparison of the CDFs for the Overlapping and Non-Overlapping Cases

2.2.2 Overlapping Pairs

The calculations involved in determining the distribution of the distances between points when we allow overlapping pairs are significantly more complex than in the previous case. We begin by assuming that X_1, X_2, X_3 are i.i.d. U(0, 1) random variables and we wish to find the distribution of $D = \sqrt{(X_1 - X_2)^2 + (X_2 - X_3)^2}$. The calculations required to determine the cdf of the distances are now much more complicated due to the dependency that is introduced by the overlapping coordinate; the details of these calculations are found in Appendix B of Duggan et al. (2005). Unfortunately, the integrals cannot be evaluated analytically and so are calculated numerically using Maple. The graph in Figure 4 shows the cdf of the overlapping case along with the non-overlapping case. The two cdfs are similar, but there is a noticeable difference between them.

2.3 Goodness-of-Fit Tests

We now have a theoretical probability distribution against which we can compare the distances between the consecutive ordered pairs constructed by a Lehmer generator. The first step is to convert the distances between points into an empirical distribution function (edf) $\hat{F}(x)$, defined by

$$\hat{F}(x) = \frac{N(x)}{n},$$

where n is the number of data values (distances) collected and $N(x)$ is the number of those values which do not exceed x. With the edf defined, it is now possible to perform a hypothesis test, namely:

$$H_0 : \hat{F}(x) = F_0(x)$$

$$H_1 : \hat{F}(x) \neq F_0(x),$$

where $F_0(x)$ is the theoretical cdf calculated in the previous section.

Three common techniques that can be used for this hypothesis test are the Kolmogorov–Smirnov (KS) test, the Cramer–von Mises (CVM) test and the Anderson–Darling (AD) test (see Lawless 2003, pages 467–469 and Law and Kelton 2000, pages 367–369). All three tests can be used to determine a test statistic for conducting the hypothesis test. One advantage of these tests is that all three have relatively straightforward computational formulas. We will conduct all three tests at the $\alpha = 0.05$ level of significance. Based on the results of the tests, we can classify each random number generator into one of three categories:

- Good — the null hypothesis was not rejected in any test;
- Suspect —the null hypothesis was rejected in one or two of the tests;
- Bad — the null hypothesis was rejected in all three tests.

It is important to note that our classification for a generator is based solely on the method described immediately above. A generator should not be considered a "good" generator in a broader sense unless it passes multiple tests of randomness (see Chapter 3 of Knuth 1998, for a description of several tests of randomness). Appendices C, D and E of Duggan et al. (2005) contain the computational formulas and C++ code used to generate the test statistics for the KS, CVM and AD tests, respectively.

3 EXAMPLES

Now that a test has been developed to determine whether the observed distribution of the distances between consecutive randomly generated ordered pairs is the same as the theoretical distribution of truly random points, some actual Lehmer generators can be tested. We investigated the set of Lehmer random number generators that use the modulus $m = 401$, chosen because these generators are small enough to look at a graphical representation of the distances between points, yet large enough for the results to be meaningful.

Once a modulus is determined, we must choose which multiplier(s) to use for testing. Leemis and Park (2006) show that there are 160 full-period multipliers for $m = 401$ and that one of these full-period multipliers is $a = 23$. A simple implementation of their Algorithm 2.2.2 (page 52) with an input of 23 will yield all 160 full-period multipliers; this implementation can be found in Appendix F of Duggan et al. (2005). Since this is a rather small generator, we test the generator using every possible full-period multiplier.

3.1 The Test

In order to see how the Lehmer generators with $m = 401$ performed with respect to the randomness of the distances between consecutive points, we wrote a C++ program that used a list of all of the full-period multipliers associated with $m = 401$ as input. For each of the multipliers, the generator was invoked to produce a full period of random numbers, i.e., a permutation of the set of numbers $\left\{\frac{1}{401}, \frac{2}{401}, \ldots, \frac{400}{401}\right\}$. These numbers were stored in an array for calculating the distance between consecutive ordered pairs based on the specific case being examined. The distances were then sorted into an array which was passed through the KS, CVM and AD tests. The final output of the program was the list of multipliers that resulted in the null hypothesis being rejected based on each test as well as the number of multipliers which resulted in the null hypothesis not being rejected by each of the tests.

3.2 General Results

We ran the program discussed in the previous section for both the overlapping and non-overlapping cases. Some of the results were quite surprising. We were not surprised by the fact that there were some multipliers that resulted in the null hypothesis being rejected by all three tests; these were classified as bad. It was also the case that many multipliers resulted in the null hypothesis not being rejected by any of the tests; these were classified as good. The interesting cases occurred when a multiplier resulted in rejection by only one or two of the three tests; these were classified as suspect. It should be noted, however, that it was a much more common occurrence for a multiplier to be considered suspect in the non-overlapping case than in the overlapping case. There was much more agreement between tests in the overlapping case, though the reason for this is not yet clear.

One thing that is clear is that it is very difficult to determine how a specific multiplier will perform based on a single test since not all of the multipliers performed the same on all three tests. It is also difficult to predict how a multiplier will perform between the two cases as it was not uncommon for the multiplier to be considered bad in the overlapping case but good in the non-overlapping case or vice versa. Table 1 shows how many multipliers caused the null hypothesis to be rejected by each of the tests at $\alpha = 0.05$; we note that even when the number of times the null hypothesis was rejected is the same, the actual multipliers causing the rejection are different. Table 1 indicates that the test of randomness for the overlapping pairs case for $m = 401$ yielded more definitive results (e.g., fewer multipliers identified as "suspect"). In order to better understand the geometry and intuition associated with the

Table 1: Test Results for the 160 Full-Period Multipliers Associated with $m = 401$

Case	Number failed KS	Number failed CVM	Number failed AD	Number good generators	Number suspect generators	Number bad generators
Non-Overlapping	32	32	32	116	20	24
Overlapping	18	16	20	140	4	16

hypothesis test, we chose four full-period multipliers to analyze in depth: $a = 23, 3, 96$ and 143.

3.3 Four Representative Multipliers

After looking through the results generated by the program discussed earlier, we chose four multipliers that seemed representative of most others. We classified each of these multipliers based on the scheme outlined in Section 2.3 and then constructed visual representations to illustrate the geometry associated with the test. For each case we will present the results obtained for each of the multipliers $a = 23, 3, 96$ and 143.

3.3.1 Results for Non-Overlapping Pairs

The results of the hypothesis tests based on the results of the goodness-of-fit tests discussed earlier for the four full-period multipliers are found in Table 2.

Figure 5 shows twelve graphs. The top row of graphs shows the points formed by grouping consecutive random numbers into ordered pairs. The graphs in the second row show the line segments connecting the consecutive points whose lengths are required for the hypothesis test. The last row of graphs shows the theoretical cdf (solid line) along with the edf (dotted line) for distances between consecutive points. The four multipliers are represented by the four columns.

3.3.2 Results for Overlapping Pairs

As with the non-overlapping pairs, we calculated the test statistics for each of the goodness-of-fit tests and then performed hypothesis tests with these statistics. The results of the hypothesis tests for overlapping pairs are found in Table 3.

The associated graphs that are analogous to those described in the previous section appear in Figure 6.

3.3.3 Discussion of Results

As we have seen from results in previous sections, the results of our test are not always consistent between the non-overlapping and overlapping cases. It is sometimes the case that a multiplier will yield statistically better (more random) results when we have overlapping pairs. However, it is also sometimes true that the opposite holds, that is, the results were better for the non-overlapping pairs. The reason this happens is not clear.

The multipliers $a = 23$ and $a = 96$ produce similar edf's in both cases, though only $a = 23$ behaves consistently between Tables 2 and 3. The multiplier $a = 23$ appears to be a very good multiplier based on the results of our test of randomness for both the overlapping and non-overlapping pairs. It does appear though that the line segment graph for the non-overlapping pairs seems to show a slightly higher degree of randomness. The multiplier $a = 96$ was classified as bad for non-overlapping pairs, but good for overlapping pairs. In the case of the non-overlapping pairs, the line segments appear to be slightly longer than they should be, but that is not a problem for the overlapping case. There is some deviation between the empirical and theoretical distributions, but this difference is not statistically significant.

The multiplier $a = 3$ yielded interesting results for both cases. The points generated with this multiplier fell on significantly fewer hyperplanes, only three compared to 23, 20 and 16 with the other multipliers. It is not unreasonable to assume that more hyperplanes indicate a better random number generator, but according to the tests in the non-overlapping case, $a = 3$ performed better than $a = 96$ and $a = 143$ even though 96 and 143 formed 20 and 16 hyperplanes, respectively. There is some indication, however, of a potential problem in the non-overlapping case with $a = 3$, and that is the two notches in the edf that occur around 0.3 and 0.6. The most probable explanation for this is that the hyperplanes are about 0.3 units apart and moving to a different hyperplane causes these notches. When we look at the overlapping pairs, however, we see a totally different result. In this case the line segments show a significant degree of regularity. Also, the notches in the edf are more defined and the line segments are unusually short. Without a doubt, $a = 3$ was the worst multiplier tested when overlapping pairs are considered.

The multiplier $a = 143$ also provided intriguing results. When used with non-overlapping pairs, $a = 143$ was classified as a bad multiplier. The distances between points for this multiplier are unusually high, resulting in an edf that falls significantly below the theoretical cdf and the classification of 143 as a bad multiplier. The graph of the line segments also shows a fairly distinct and regular hub and spoke structure which is consistent with this multiplier's lack of randomness. However, when the results of the overlapping pairs are examined, the graph is

Table 2: Results for Non-Overlapping Pairs

Multiplier	KS	CVM	AD	Classification
$a = 23$	Fail to reject H_0	Fail to reject H_0	Fail to reject H_0	Good
$a = 3$	Fail to reject H_0	Reject H_0	Reject H_0	Suspect
$a = 96$	Reject H_0	Reject H_0	Reject H_0	Bad
$a = 143$	Reject H_0	Reject H_0	Reject H_0	Bad

Table 3: Results for Overlapping Pairs

Multiplier	KS	CVM	AD	Classification
$a = 23$	Fail to reject H_0	Fail to reject H_0	Fail to reject H_0	Good
$a = 3$	Reject H_0	Reject H_0	Reject H_0	Bad
$a = 96$	Fail to reject H_0	Fail to reject H_0	Fail to reject H_0	Good
$a = 143$	Fail to reject H_0	Fail to reject H_0	Fail to reject H_0	Good

completely different. In the overlapping pairs case, we do not see the hub and spoke structure at all and also, even though there are some structural components that do not appear random, there is much more variation in the lengths of the line segments. For this reason the multiplier $a = 143$ shows significant improvement and is classified as a good multiplier in the overlapping case.

The lesson associated with Figures 5 and 6 is clear: the distance between parallel hyperplanes (as in Table 9.3 on page 436 of Fishman 2001) should not be used as the *sole* basis for a global test of randomness. The order in which the pairs are generated is a secondary significant factor. Although the distance between the hyperplanes for the generator with $a = 143$ is about average among the four generators examined, its performance in our test with non-overlapping pairs was by far the worst of the four.

4 FUTURE WORK

There are many different directions that future research on this topic could proceed. The first possibility is to test the larger modulus, more frequently used random number generators to see if they pass our test of randomness. For $m = 2^{31} - 1$, this would include random number generators such as $a = 16807$ (Park and Miller 1988), $a = 48271$ (Leemis and Park 2006), $a = 630\,360\,016$ (Law and Kelton 2000, page 412), $a = 742\,938\,285$, $a = 950\,706\,376$, $a = 1\,226\,874\,159$, $a = 62\,089\,911$, and $a = 1\,343\,714\,438$ (Fishman 2001, page 436). This would be a rather large study, however, since both the non-overlapping and overlapping cases should be considered, and the number of pairs to be used in the goodness-of-fit test is arbitrary. It would also be possible to test non-Lehmer generators, such as additive linear generators, composite generators, or quadratic generators — one of which may prove to be better.

Improvements could also be made on the theoretical distribution of the distances for the overlapping pairs case. Since the integrals necessary to calculate the cdf cannot be evaluated analytically, a numerical approach must be used. Perhaps the values of the cdf could be calculated on the fly as opposed to being stored in an array and referenced when needed.

5 CONCLUSIONS

While all Lehmer generators produce pseudo-random numbers that fall on hyperplanes, some of these sequences are better than others due to the order in which they fall on the hyperplanes. In this paper we have proposed a test of randomness for a Lehmer random number generator based on treating consecutive pairs of random numbers as points in the unit square. First, the empirical distribution of the distances between consecutive points is calculated. To test for randomness, this edf is compared with the theoretical distribution, which is found with the help of APPL in the nonoverlapping pairs case and computed numerically in the overlapping pairs case. With the empirical ($\hat{F}(x)$) and theoretical ($F_0(x)$) distribution functions known, the hypothesis $H_0 : \hat{F}(x) = F_0(x)$ is tested against $H_1 : \hat{F}(x) \neq F_0(x)$. The test statistic and critical value for this hypothesis test can be calculated by using any of a number of tests including the Kolmogorov–Smirnov, Cramer–von Mises and Anderson–Darling tests. As an application of this process, we tested all of the full period Lehmer generators with $m = 401$ and discussed the results of the tests for a few representative multipliers.

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REFERENCES

Duggan, M. J., J. H. Drew, and L. M. Leemis. 2005. A Test of Randomness Based on the Distance Between Consecutive Random Number Pairs. Technical Report,

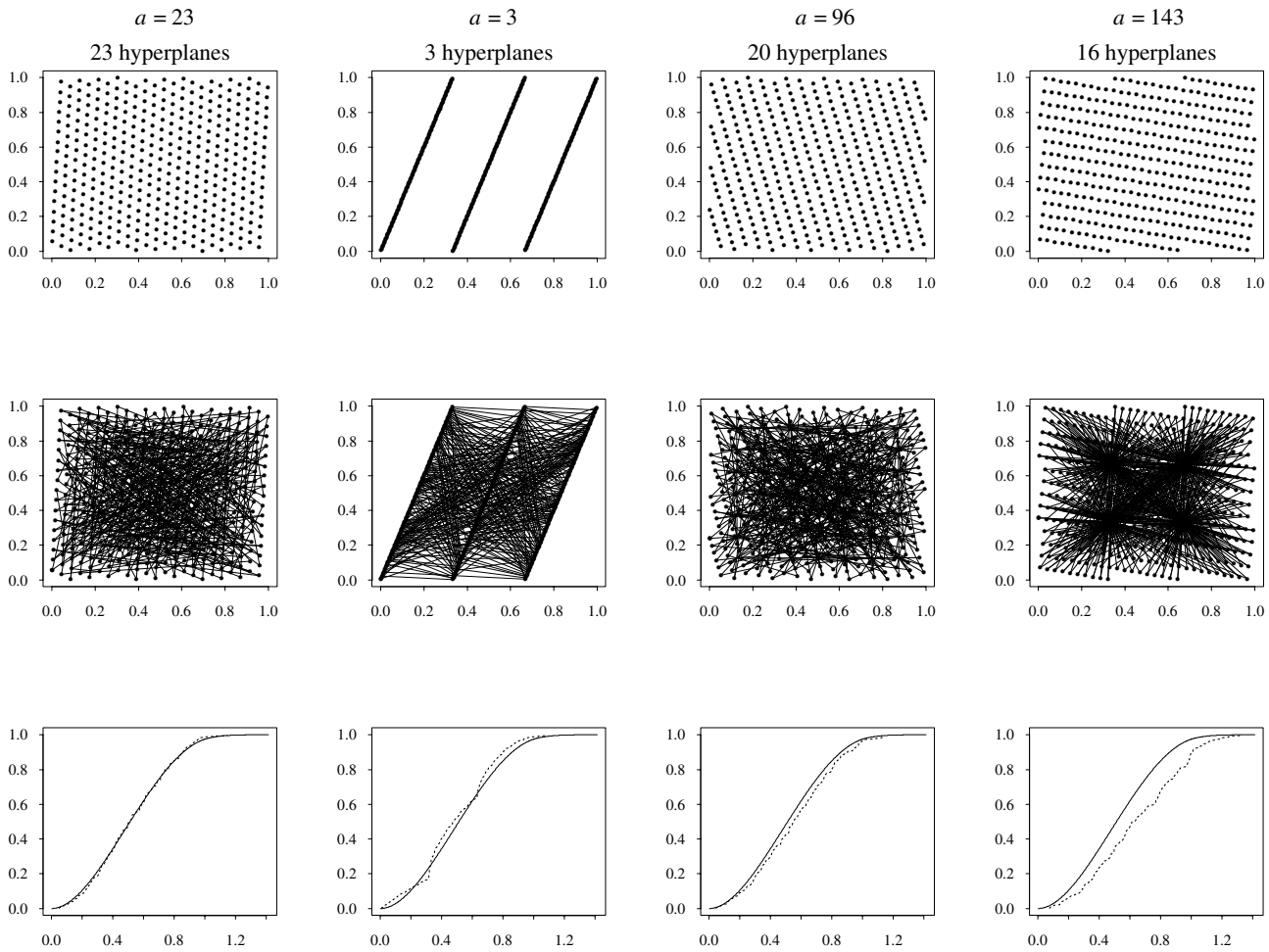

Figure 5: Graphs for Non-Overlapping Pairs

Mathematics Department, The College of William & Mary, Williamsburg, Virginia.

Fishman, G. S. 2001. *Discrete-Event Simulation: Modeling, Programming, and Analysis*, New York: Springer.

Glen, A., D. L. Evans, and L. M. Leemis. 2001. APPL: A Probability Programming Language. *The American Statistician* 55: 156–166.

Knuth, D. E. 1998. *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*. 3rd ed. Reading: Addison–Wesley.

Law, A. M. and D. W. Kelton. 2000. *Simulation Modeling and Analysis*. 3rd ed. New York: McGraw–Hill.

Lawless, J. F. 2003. *Statistical Models and Methods for Lifetime Data*. 2nd ed. New York: Wiley.

Leemis, L. M. and S. K. Park. 2006. *Discrete-Event Simulation: A First Course*. forthcoming. Upper Saddle River, New Jersey: Prentice–Hall.

Lehmer, D. H. 1951. Mathematical Methods in Large-Scale Computing Units. In *Proceedings of the 2nd Symposium on Large-Scale Calculating Machinery*, 141–146. Cambridge: Harvard University Press.

Marsaglia, G. 1968. Random Numbers Fall Mainly in the Planes. In *Proceedings of the National Academy of Sciences*, 61: 25–28.

Park, S. K. and K. W. Miller. 1988. Random Number Generators: Good Ones Are Hard To Find. *Communications of the ACM* 31: 1192–1201.

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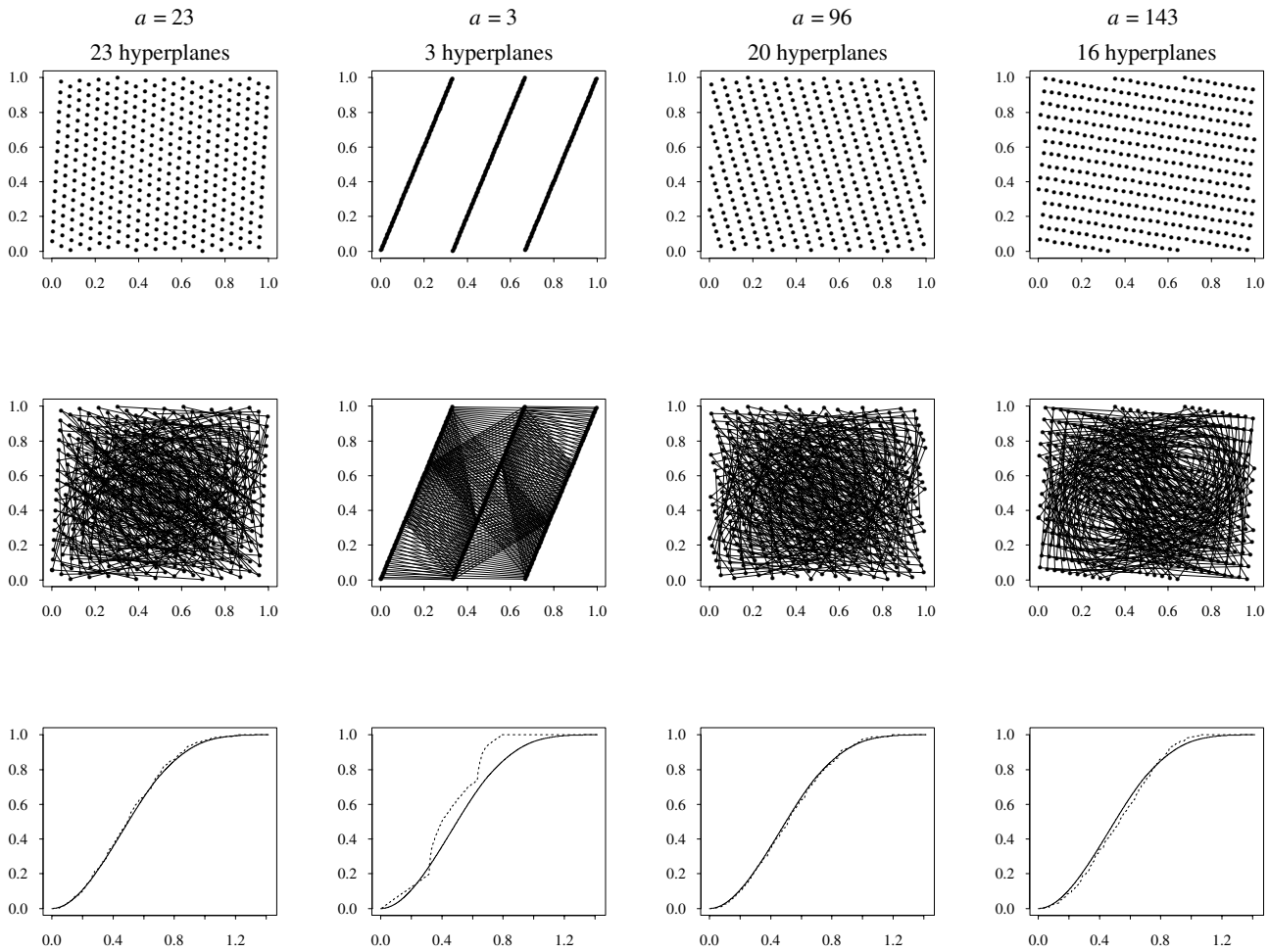

Figure 6: Graphs for Overlapping Pairs

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