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MONTE CARLO METHODS FOR AMERICAN OPTIONS

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ABSTRACT

We review the basic properties of American options and the difficulties of applying Monte Carlo valuation to American options. Recent progress on the Least Squares Monte Carlo (LSM) method is described, including the use of quasi-random sequences in LSM. A particle approach to evaluation of American options is formulated. Conclusions and prospects for future research are discussed.

1 INTRODUCTION

American options are derivative securities for which the holder of the security can choose the time of exercise. In an American put, for example, the option holder has the right to sell an underlying security for a specified price K (the strike price) at any time between the initiation of the agreement ($t = 0$) and the expiration date ($t = T$). The exercise time τ can be represented as a stopping time; so that American options are an example of optimal stopping time problems.

Valuation of American options presents at least two difficulties. First, there is a singularity in the option characteristics at the expiration time. For American puts and calls on equities with dividends, a thorough analysis of this singularity was performed by Evans, Kuske, and Keller (2002). These results are briefly described in Section 3.

A second difficulty occurs for Monte Carlo valuation of American options, the main subject of this paper. Monte Carlo methods are required for options that depend on multiple underlying securities or that involve path dependent features. Since determination of the optimal exercise time depends on an average over future events, Monte Carlo simulation for an American option has a “Monte Carlo on Monte Carlo" feature that makes it computationally complex.

In this paper, we review several methods for overcoming this difficulty with American options. The first, developed by Broadie and Glasserman (1997) and presented in Section 4 involves two branching processes, the first of which provides an upper bound and the second a lower bound on option price. The second method, presented in Section 5, is a Martingale optimization formula developed in Rogers (2002) that provides a dual formulation of the Monte Carlo valuation formula and leads naturally to an upper bound on the option price. The third (Section 6) is the Least Squares Monte Carlo (LSM) derived by Longstaff and Schwartz (2001). Finally we described work by the authors on use of quasi-random sequences in LSM (Chaudhary 2003a) in Section 7.

A brief introduction to the salient features of American options is given in Section 2 and a discussion of conclusions and prospects for future research is described in Section 8.

2 AMERICAN OPTIONS

In this section we describe some of the basic features of American options. These include the Black-Scholes PDE and the risk-neutral valuation formula for option price, the optimal exercise boundary, and the “Monte Carlo on Monte Carlo" difficulty.

Consider an equity price process $S(t)$ that follows an exponential Brownian motion process according to the following stochastic differential equation

$$dS = \mu S dt + \sigma S d\omega \tag{1}$$

in which μ and σ are the average growth rate and volatility (both assumed to be constant) and $\omega = \omega(t)$ is standard Brownian motion.

The option payout function is $u(S, t)$. A path dependent option is one for which $u(S, t)$ depends on the entire path $\{S(t') : 0 < t' < t\}$; whereas a simple (non-path dependent) option has $u(S, t) = u(S(t), t)$. For a simple European option the payout may only be collected at the final time so that it is $f(T) = u(S(T), T)$. For a simple American option, exercise may be at any time before T so that the payout is $f(\tau) = u(S(\tau), \tau)$ in which τ is an optimally chosen stopping time. The reason τ is a stopping time is that the decision of whether to exercise at time t can only depend on the values of S up to and including t.

Examples of simple payout function are a call, for which $u = \max(S-K, 0)$, and a put, for which $u = \max(K-S, 0)$. Examples of path dependent payouts are the Asian option u_A and the lookback u_L given by

$$\begin{aligned} u_A &= U((t-t_0)^{-1}\int_{t_0}^{t} S(t')dt') \\ u_L &= U(\max_{t_0<t'<t} S(t')) \end{aligned}$$

in which U is some function such as the call or put payout. In u_A and u_L, the lower time limit t_0 could be 0 or it could be $t - \Delta$.

The early exercise boundary is the set in time and state space on which exercise of the American option is optimal. For a simple option, this is just a curve $S = S^*(t)$ in the space (S, t). For a path dependent security, the exercise decision depends on more that $S(t)$ and t, so that the early exercise boundary is more complicated.

In their classic papers, Black and Scholes (1973) and Merton (1973) described two methods for valuation of derivative securities. The first is the Black-Scholes PDE. For an American option with value F, the Black-Scholes PDE is

$$F_t + rSF_S + \sigma^2 S^2 F_S S = rF$$

in which r is the risk-free rate of return. The “final condition" is

$$F(S, T) = u(S, T)$$

and the boundary conditions on the free boundary $S = S^*(t)$ are

$$\begin{aligned} F &= u \\ F_S &= u_S. \end{aligned}$$

The second method, which is applicable to path-dependent options and other derivatives for which the PDE is either unavailable or intractable, is the risk-neutral valuation formula

$$F(S, t) = \max_{\tau} E'[e^{-r(\tau-t)}u(S(\tau), \tau)|S(t) = S] \qquad (2)$$

in which E' is the risk-neutral expectation, for which the growth rate μ in (1) is replaced by r. The maximum is taken over all stopping times τ with $t < \tau < T$. This is the formula to which Monte Carlo quadrature can be applied.

This risk-neutral valuation approach provides a stochastic characterization of the early exercise boundary. Consider the exercise decision at a point (S, t). The value of early exercise is just the payoff $u(S, t)$. The expected value of deferred exercise is $\tilde{F}$ given by

$$\tilde{F} = \max_{\tau} E'[e^{-r(\tau-t)}u(S(\tau), \tau)|S(t) = S]. \qquad (3)$$

The holder of the option will choose to exercise if $u > \tilde{F}$, so that

$$F = \max(u(S, t), \tilde{F}) \qquad (4)$$

and $u(S^*(t), t) = \tilde{F}$ on early exercise boundary.

A lower bound on the American option price follows from the formula (4). Let τ' be any stopping time and let F' be the price using this stopping time; i.e.

$$F' = E'[e^{-r(\tau'-t)}u(S(\tau'), \tau')|S(t) = S]$$

then

$$F \geq F'.$$

3 ASYMPTOTICS FOR AMERICAN PUTS AND CALLS WITH DIVIDENDS

When the early exercise boundary $S = S^*(t)$ hits the final boundary $t = T$, there is a singularity in the exercise boundary shape, which is characteristic of many free boundary problems. In addition, $S^*(T)$ (the intersection of the early exercise boundary and the final boundary) may differ from K (the exercise boundary on the final boundary).

While these properties have long been recognized, the detailed asymptotics of the singularity in the early exercise boundary were not analyzed until recently. Evans, Kuske, and Keller (2002) derived the shape of the early exercise boundary for American put and call with dividends by two alternative methods: asymptotics for an integral equation formulation and matched asymptotics for the Black-Scholes PDE. The dividends are assumed to payout at a continuous rate D. The early exercise boundary $S_P^*(t)$ for the American put and $S_C^*(t)$ for the American call satisfy the following:

$$S_P^*(t) = \begin{cases} K + c_1\sqrt{(T-t)\log[1/(T-t)]} & \text{if } 0 \leq D < r \\ K + c_2\sqrt{(T-t)\log[1/(T-t)]} & \text{if } D = r \\ (r/D)(K + c_3\sqrt{T-t}) & \text{if } D > r \end{cases}$$

$$S_C^*(t) = \begin{cases} K + c_1\sqrt{(T-t)\log[1/(T-t)]} & \text{if } D > r \\ K + c_2\sqrt{(T-t)\log[1/(T-t)]} & \text{if } D = r \\ (r/D)(K + c_3\sqrt{T-t}) & \text{if } 0 \leq D < r \end{cases}$$

in which c_1, c_2, c_3 are constants that depend on σ, D and r. Note that for $D > r$, $S_P^*(T) = (r/D)K < K$ and for $D < r$, $S_C^*(T) = (r/D)K > K$ which shows that the exercise boundary on the final boundary is not on the early exercise curve. Also as $D \to 0$, the early exercise boundary for the American call goes away to infinity.

4 BRANCHING PROCESSES

The “Monte Carlo on Monte Carlo" property can be seen in the decision formula (4). Consider a simulated path and a point $(S(t), t)$ on that path. In order to decide whether to exercise at that point, one must evaluate the expectation in (3). This in turns requires continuation from $(S(t), t)$ on many paths. Therefore this direct Monte Carlo simulation of the American option requires a set of continuously branching paths, which is computationally intractable.

Broadie and Glasserman (1997) consider a Bermudan option; i.e., an option in which exercise can occur at any one of a discrete number $d+1$ times $t_0, \ldots, t_d$. They constructed two branching processes, each with b branches at each exercise time. The first process provides an upper estimate F_u and the second a lower estimate F_ℓ, on average; i.e.

$$E[F_\ell] \le F \le E[F_u]. \tag{5}$$

In addition, both processes converge to the correct price as the branching number b and the number of paths N increase; i.e.

$$\lim_{b\to\infty,\ N\to\infty} F_\ell = \lim_{b\to\infty,\ N\to\infty} F_u = F.$$

On the other hand, this construction is computationally complex with CPU time that scales like $O(Nb^d)$.

In both processes the price is determined by “rolling-back" on the branched paths. At the final time, exercise is determined by whether the payout is positive or not. Consider a time t_k before the final time and suppose that the price has been found for all times t_m with $m > k$. The price F_k at a point (S_k, t_k) is determined as in (4). Set

$$\tilde{F}'_k = E'_{S_k,t_k}[e^{-r(t_{k+1}-t_k)} u(S_{k+1}, t_{k+1})] \tag{6}$$

and then

$$F_k = \max(u(S_k, t_k), \tilde{F}_k). \tag{7}$$

In (6), the expectation is the empirical average over a chosen set of branches that continue from (S_k, t_k).

The difference between the upper and lower processes is which paths are used in the expectation of (6). In the upper process all of the branches are used. Since the early exercise decision uses knowledge of the future for the finite set of branching paths, then the price estimate F_u is biased high. This gives the upper estimate in (5).

For the lower process, at each decision points, one of the branches is designated to be the continuation branch. The average in (6) is determined using the other $b-1$ branches. The value of this empirical average is independent of the continuation branch, but since the average is approximate, the resulting exercise decision is suboptimal. Therefore the resulting price estimate F_ℓ is biased low. This gives the lower estimate in (5).

5 MARTINGALE OPTIMIZATION

Rogers (2002) derived a formula for the American option price that is dual to the formula in (2):

$$F(0) = \min_M E'[\max_{0<t'<T} (e^{-rt'} u(t') - M(t'))] \tag{8}$$

in which the min is taken over all martingales for which $M(0) = 0$. Similar formulas were derived by Anderson and Broadie (2001) and Kogan and Haugh (2001).

By insertion of a (non-optimal) martingale M into (8), one gets a upper bound on F. This has been carried out for various choices of M in Andersen and Broadie (2001), Kogan and Haugh (2001), Lamper and Howison (2002), Rogers (2002). Chaudhary (2003b) has used this to form an approximate method for hedging the American option.

6 LEAST SQUARES MONTE CARLO (LSM)

Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (1999) introduced a new approach to Monte Carlo evaluation of American options by replacing the future expectation by a least squares interpolation. The method starts with N random paths (S_n^k, t_n) for $1 \le k \le N$ and $t_n = ndt$. Valuation is performed by rolling-back on these paths.

Suppose that $F_{n+1}^k = F(S_{n+1}^k, t_{n+1})$ is known. For points (S_n^k, t_n) set $X = S_n^k$ the current equity value and $Y = e^{-rdt} F(S_{n+1}^k, t_{n+1})$ the value of deferred exercise. Then perform regression of Y as a function of the polynomials $X, X^2, \ldots, X^m$ for some small value of m; i.e. approximate Y^k by a least squares fit of these polynomials in X. Use this regressed value as an approximation to $\tilde{F}$ in (3) and use this value in deciding whether to exercise early.

Longstaff and Schwartz have applied this method to puts, Asian options, swaps, swaptions and other options with excellent results for small m.

7 QUASI-MONTE CARLO FOR LSM

In their LSM paper, Longstaff and Schwartz (2001) suggested that their method might be improved by the use of quasi-random points. There are two potential difficulties with this extension of the method: the problem is high dimensional and the prices along different paths in the LSM method are correlated, both of which can be problematic for quasi-Monte Carlo.

Quasi-random sequences are a deterministic alternative to random or pseudo-random points. The distribution of quasi-random points is much more uniform that than of random points, because of correlations between the points that are designed to keep them from clumping. As a result, Monte Carlo quadrature in d dimensions using N quasi-random points can converge at a rate $N^{-1}(\log N)^d$, as opposed to convergence at rate $N^{-1/2}$ for random points (Caflisch 1998). The exponent d for the log indicates that the advantages of the method can breakdown for large dimension.

Chaudhary (2003a) implemented a Brownian bridge (BB) construction for the paths in the LSM method. As seen in earlier examples (Caflisch, Morokoff, and Owen 1997, Caflisch 1998), this can reduce or remove the high dimensionality difficulty for quasi-Monte Carlo quadrature of path dependent securities. In addition, the BB method shows that the memory requirements of the LSM method can be significantly reduced. The potential difficulty with correlations between the paths did not turn out to be much of a problem, perhaps because the true correlations are via the early exercise boundary which is deterministic.

8 CONCLUSIONS

Our intention in writing this paper is to describe the difficulties involved in applying Monte Carlo evaluation to American options, as well as several recent methods that are quite promising for overcoming these difficulties. Here are some directions that we believe to be promising for future research.

The singularity in the early exercise boundary at the final time has been well characterized by Evans, Kuske, and Keller (2002), at least for call and put options. The information in these asymptotic results could be valuable in improving Monte Carlo simulations. In particular, Caflisch and Goldenfeld (2004) have formulated a particle method for solution of American options, in which the main determination of the solution comes from this singularity.

For Martingale optimization, there is not yet a good method for choosing the Martingale in order to get a good approximation. In particular, one might hope to find an iterative method, in which an approximate Martingale would be modified at each step in order to improve the approximation over that of the previous step.

The LSM method with random or quasi-random sequences has been shown to work well on a good selection of examples, but it still needs to be validated for more complicated examples, such as American Asian options with a moving window over which the average is taken. Partial results for this problem are found in Bilger (2003).

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