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APPROXIMATING FREE EXERCISE BOUNDARIES FOR AMERICAN-STYLE OPTIONS USING SIMULATION AND OPTIMIZATION

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ABSTRACT

Monte Carlo simulation can be readily applied to asset pricing problems with multiple state variables and possible path dependencies because convergence of Monte Carlo methods is independent of the number of state variables. This paper applies Monte Carlo simulation to the problem of determining free exercise boundaries for pricing American-style options. We use a simulation-optimization method to identify approximately optimal exercise thresholds that are defined by a minimal number of parameters. We demonstrate that asset prices calculated using this method are comparable to those found using other numerical asset pricing methods.

1 INTRODUCTION

Monte Carlo simulation is a popular method for pricing financial options and other derivative securities. Simulation uses random sampling, rather than enumeration implicit in lattice and finite-difference methods, so it can be more easily applied to problems with multiple state variables and possible path dependencies. Convergence of Monte Carlo methods is independent of the number of state variables, whereas convergence in lattice methods is exponential in the number of state variables; thus, simulation is particularly advantageous when the underlying asset follows a process that produces difference equations that are difficult or impossible to solve analytically.

Boyle (1977) offered Monte Carlo simulation as an alternative to numerical integration and finite difference approach methods for valuing European options on financial assets. Under this method, the distribution of terminal stock values is determined by the process generating future stock price movements; this series in turn determines the future distribution of terminal option values. To obtain an estimate of the option value, a number of sample values are picked at random from the distribution describing the terminal values of the option. In turn, these terminal values are discounted and averaged over the number of trials. Charnes (2000) adapts Boyle's technique for use with various exotic options and also demonstrates variance reduction techniques to increase the precision of estimates of option values obtained by simulation.

Applying Monte Carlo simulation to pricing of American-style options remains a challenging problem. Proper valuation of American-style options is more difficult than pricing European options because these options can be exercised on multiple dates. The complexity in using simulation lies in applying a forward-based procedure to a problem that requires a backward method to solve. Pricing an American-style security requires an appropriate estimation of the early exercise rule for the decisions available in American-style derivative contracts.

Barraquand and Martineau (1995) developed a numerical method for valuing American options with mutiple underlying sources of uncertainty which uses Monte Carlo simulation. Their technique relies on partitioning the state space of possible exercise opportunities into a tractable number of cells, then computing an optimal Cash Flow Management strategy (CMS) that is constant over each cell. The option value is based on the CMS with the maximum value.

Grant *et al.* (1997) consider how to incorporate optimal early exercise in the Monte Carlo method by linking forward-moving simulation and backward-moving dynamic programming through an iterative search process. They simplify the problem by optimizing the option value with respect to a piece-wise linear early exercise hurdle, albeit at the expense of biasing the option value downward. After the exercise boundary is established at each potential exercise point, the price is estimated in a forward simulation based on the obtained boundaries.

American-style securities can be priced using simulation (Broadie and Glasserman 1997) by developing a "high" and "low" estimator, then using the average to estimate the value of the option. While both estimators are biased, both are also consistent, so as the number of trials in the simulation is increased, the error bounds on the estimate narrow.

Longstaff and Schwartz (2001) present another method for valuing American options with simulation that utilizes least squares regression. First, a number of paths of the underlying asset are randomly generated and the cash flows from a corresponding European option in the last period are generated for each path. In the next to last period, the paths that are "in the money" are selected and the cash flows are discounted to the current period. To estimate the expected cash flows from continuing the option's life conditional on the stock price in the next-to-last period, the discounted option payoffs are regressed on basis functions of the stock price. With this conditional expectation function, the value of immediate exercise in the next to last period and the value from continuing the option can be compared. Using the optimal decision, the cash flow matrix for the next-to-last period is generated and the process is repeated. Given the sample paths, a stopping rule is created for each sample path. These cash flows are then discounted to the current period and averaged over all paths to estimate the option value.

Fu *et al.* (2001) introduces a simulation-based approach that parameterizes the early exercise curve and casts the valuation problem as an optimization problem of maximizing the option value with respect to the associated parameters. This approach simultaneously optimizes the option value with respect to a parameter vector by iterative updates via a stochastic approximation algorithm. This approach is compared with two dynamic programming techniques (Tilley 1993, Grant *et al.* 1997) and the stochastic mesh and simulated tree methods of Broadie and Glasserman (1997a, 1997b, 1998) on a test bed of several American-style options. Wu and Fu (2003) gives further details of the application of this technique to American-Asian options.

Additional implementations of Monte Carlo simulation for pricing American-style options are described by Bossaerts (1989), Fu (1995), Fu and Hu (1995), Carriere (1996), Raymar and Zwecher (1997), and Ibanez and Zapatero (2004).

This paper addresses the ongoing challenge of developing a flexible framework for paramterizing the early exercise boundary for American-style financial options. Such a framework should provide the correct value for the option so that it can be appropriately be used for efficient management of risk. The framework must also provide an exercise rule for the option in terms of observable stochastic variables, e.g. stock prices. Glasserman (2003) notes that an approximate boundary is often adequate to provide a good estimate of the option value. Thus, we can develop an exercise rule for an American-style option by using just a few parameters to define the optimal exercise boundary.

We study several formulations for an approximately optimal exercise boundary required to value and manage a financial option, and parameterize the boundaries using a simulation-optimization method. Because we use relatively few parameters, we define a procedure that uses only forward simulation to identify approximately optimal parameters. Additionally, we parameterize a "random exercise region" to understand the sensitivity of the option value to the exact placement of the exercise boundary.

The remainder of the paper is organized as follows. Section 2 defines the notation required to implement the simulation-optimization approach for determining optimal exercise boundaries. Section 3 defines two types of exercise boundaries and the "random exercise region" used to price American-style options. Section 4 implements the simulation-optimization approach by determining optimal exercise boundaries and regions for pricing an American call option on a stock paying continuous dividends. Section 5 gives a summary and conclusions.

2 SIMULATION-OPTIMIZATION APPROACH

2.1 Overview

The simulation-optimization method relies on a discounted cash flows model to determine the value of the American-style option. The inputs used in the discounted cash flows model are classified as follows:

1. Decision variables — used to parameterize the early exercise boundary or region and can be adjusted to increase option value as required.
2. Stochastic inputs — random variables with known or estimated probability distributions.
3. Deterministic inputs — based on established benchmarks or option features.

We construct a simulation-optimization component which interacts with the discounted cash flow model by selecting different combinations of the decision variables and generating random simulation trials using the stochastic assumptions. The simulation-optimization component tracks the mean discounted cash flows from the option for each combination of the decision variables to determine the optimal decision rule.

2.2 Notation

This section defines variables that will be used throughout the remainder of the paper.

We use the simulation-optimization method to price American-style options by assuming that exercise is restricted to the discrete points $t_j,\ j = 0, 1, ..., N$.

The stochastic assumptions are as follows:

$$
\begin{aligned}
S_{t_j}^{(k)} &= \text{Value of asset } k \text{ in period } t_j \\
\sigma_k &= \text{Volatility of asset } k \\
\delta_k &= \text{Continuous dividend paid on asset } k.
\end{aligned}
$$

In many cases, we will define σ_k and δ_k to be deterministic, but we can easily adapt the model to handle stochastic volatility and/or stochastic dividend rates. Similarly, we can define the initial value of asset k, $S_{t_0}^{(k)}$, to be either deterministic or stochastic.

Deterministic assumptions are defined as follows:

$$
\begin{aligned}
K &= \text{Strike price of the option} \\
N &= \text{Number of exercise dates, including the expiration date} \\
r &= \text{The risk-free rate of return for the period from } t_0 \text{ to } t_N \text{ (representing continuous annual returns)} \\
\Delta_{t_j}^{(k)} &= \text{Dividend paid on asset } k \text{ at time } t_j \\
T &= \text{Time to expiration of the option (in years).}
\end{aligned}
$$

If we are valuing an option on only one underlying asset, we drop the (k) superscript on all variables. The remaining variables are deterministic, given a specific instantiation of the decision variables and stochastic assumptions:

$$
\begin{aligned}
Y_{t_j} &= \text{Optimal exercise boundary in period } t_j \\
D_{t_j} &= \text{Indicator variable representing comparison of asset price to linear threshold in period } t_j \\
A_{t_j} &= \text{Indicator variable representing whether option has been exercised prior to period } t_j.
\end{aligned}
$$

2.3 Scatter Search

The optimal decision rule is determined by considering many possible combinations of the decision variables which parameterize the exercise boundary or region. We use a scatter search algorithm to select decision variable scenarios and obtain an approximately optimal solution without testing a complete enumeration of the possible combinations of the decision variables (for more information on the scatter search algorithm, see Glover *et al.* (1996)).

Using the scatter search approach requires simulating only forward paths of the underlying asset values, without any backward recursion to price the option. We find that searching a limited number of combinations of the decision variables leads to approximately optimal values that are very close to or the same as those found by other methods.

3 FREE EXERCISE BOUNDARIES

To specify fewer decision variables than exercise dates to price an American-style option, we can fit a threshold as a function of time, thus only optimizing the parameters for this function, as opposed to a threshold for each period. In this section, we discuss possible exercise boundaries.

3.1 Piece-Wise Linear

We suppose that the optimal exercise boundary for an American-style option is a smooth curve, but that this curve can be approximated by a piece-wise linear function with two sections. The decision variables required to implement the two-piece linear threshold are

$$
\begin{aligned}
a_1 &= \text{Point on horizontal axis representing the start of the second piece of the threshold;} \\
& \quad a_1 = t_j \text{ for exactly one } j,\ 1 \leq j \leq N-1 \\
b_1 &= \text{Point on vertical axis representing the start of the second piece of the threshold} \\
b_2 &= \text{Point on vertical axis representing the start of the first piece of the threshold, i.e. the } y\text{-intercept of the threshold at } t_0.
\end{aligned}
$$

The piece-wise linear threshold is then defined by the following function:

$$
Y_{t_j}(j) = \begin{cases} b_2 + \frac{b_1 - b_2}{a_1} \cdot j, & 0 \leq j \leq a_1 \\ \left(b_1 - \frac{K - b_2}{N - a_1} \cdot a_1\right) + \frac{K - b_2}{N - a_1} \cdot j, & a_1 < j \leq N. \end{cases} \tag{1}
$$

Depending on the type of option to be priced, i.e. call or put, different constraints are placed on the decision variables. These constraints ensure that the piece-wise linear threshold is either monotonically increasing or decreasing. A graphical representation of the piece-wise linear threshold for an American call option on a dividend-paying asset is shown in Figure 1.

3.2 Bézier Curve

A cubic Bézier Curve is defined by four points. The *origin endpoint* is defined as (x_0, y_0) and the *destination endpoint* is defined as (x_3, y_3). The *control points* are (x_1, y_1) and (x_2, y_2). Two equations define points on the curve and both are evaluated at an arbitrary number of values of m between 0 and 1. The first equation yields the values of t_j

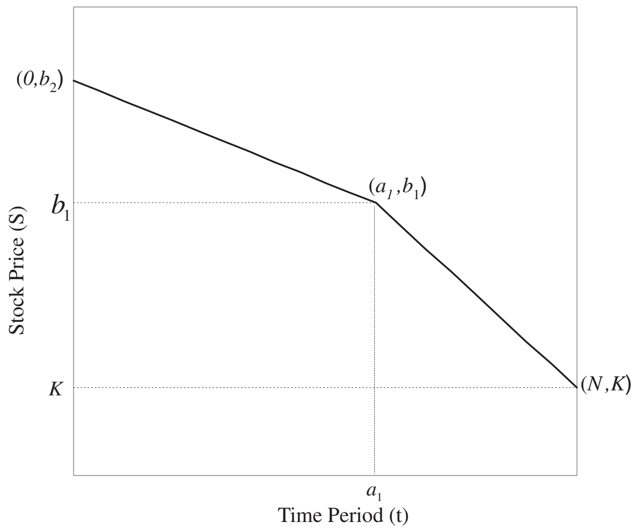

Figure 1: Piece-Wise Linear Threshold for an American Call Option on a Single Asset Paying Continuous Dividends

(the x-coordinates for points on the curve):

$$t_j(m) = a_x \cdot m^3 + b_x \cdot m^2 + c_x \cdot m + t_0 \,. \qquad (2)$$

The values of m can chosen to ensure that points between t_0 and t_N are selected.

The second equation yields the values of Y_{t_j} (the y-coordinates for points on the curve):

$$Y_{t_j}(m) = a_y \cdot m^3 + b_y \cdot m^2 + c_y \cdot m + Y_{t_0} \,. \qquad (3)$$

The coefficients required are the following functions of the control points and endpoints:

$$\begin{aligned} c_x &= 3(x_1 - x_0) \\ b_x &= 3(x_2 - x_1) - c_x \\ a_x &= x_3 - x_0 - c_x - b_x \\ c_y &= 3(y_1 - y_0) \\ b_y &= 3(y_2 - y_1) - c_y \\ a_y &= y_3 - y_0 - c_y - b_y \,. \end{aligned}$$

In the simulation-optimization model, $x_0 = 0$, $x_3 = N$, and $y_3 = K$. The other decision variables for the control points and endpoints are identified in the simulation-optimization routine, subject to constraints as required. A graphical representation of the Bézier curve threshold for an American call option on a dividend-paying asset is shown in Figure 2.

3.3 Random Exercise Region

Glasserman (2003) notes that for many options, the option value is not very sensitive to the exact position of the exercise boundary and that a rough approximation to the boundary gives an approximately optimal option price. We approximate a region over which the option value is continuous. If the price of the underlying asset falls into this region, the owner can apply any desired hold/exercise strategy, without changing the expected payoff of the option.

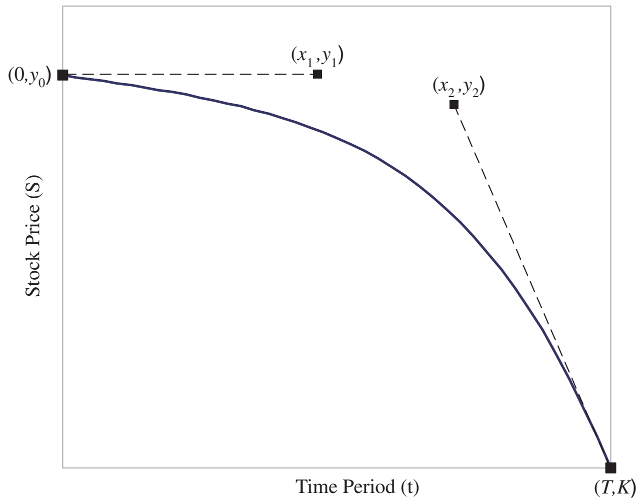

Figure 2: Bézier Curve Threshold for an American Call Option on a Single Asset Paying Continuous Dividends

We define this region as the space bounded by two piece-wise linear thresholds. The decision variables required to implement the random exercise region are the same as those required for the piece-wise linear threshold, plus three additional decision variables defined as

a_2 = Point on horizontal axis representing the start of the second piece of the second threshold; $a_1 = t_j$ for exactly one j, $1 \leq j \leq N-1$

b_3 = Point on vertical axis representing the start of the second piece of the second threshold.

b_4 = Point on vertical axis representing the start of the first piece of the second threshold, i.e., the y-intercept of the threshold at t_0.

A graphical representation of the random exercise region for an American call option on a dividend-paying asset is shown in Figure 3.

The lower piece-wise linear threshold of the random exercise region is defined as in (1), but will be denoted as $Y^L_{t_j}$ in period j. The upper piece-wise linear threshold bounding the random exercise region is defined by the following function:

$$Y^U_{t_j}(j) = \begin{cases} b_4 + \frac{b_3 - b_4}{a_2} \cdot j, & 0 \leq j \leq a_2 \\ \left(b_3 - \frac{K-b_4}{N-a_2} \cdot a_1\right) + \frac{K-b_4}{N-a_2} \cdot j, & a_2 < j \leq N. \end{cases}$$

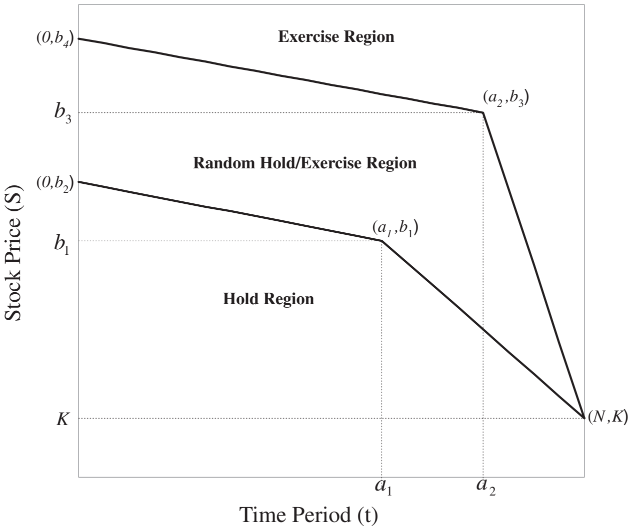

Figure 3: Random Exercise Region for an American Call Option on a Single Asset Paying Continuous Dividends

We will ensure that the upper and lower piece-wise linear thresholds bounding the random exercise region do not intersect by placing linear constraints on the decision variables. These constraints will be specified differently depending on the type of option being valued in a particular application.

4 EXAMPLES

Consider an American call option on a single asset paying continuous dividends, where $\delta = 0.04$, $S_0 = \$100$, and $r = 5\%$. Initially, $\sigma = 20\%$, but we vary σ in some cases to determine the effect on the exercise boundary or region. The values of T and K will be varied in each experiment.

When using the piece-wise linear or Bézier curve thresholds to price the option,

$$D_{t_j} = \mathbf{1}\left\{S_{t_j} \geq Y_{t_j}\right\} ,$$

where $\mathbf{1}\{A\}$ denotes the indicator of event A, i.e. $\mathbf{1}\{A\} = 1$ if event A occurs and $\mathbf{1}\{A\} = 0$ otherwise. The indicator variable $A_{t_0} = D_{t_0}$ and

$$A_{t_j} = \mathbf{1}\left\{\sum_{\ell=0}^{j-1} D_{t_\ell} = 0\right\}$$

for all $j > 0$, $j = 1, ..., N$.

Let $(A)^+$ denote max$[0, A]$. The discounted payoff function for the American call option on a single asset paying continuous dividends is

$$P_0 = \sum_{j=0}^{N} \exp\{-rj \cdot (T/N)\} \cdot D_{t_j} \cdot A_{t_j} \cdot (S_{t_j} - K)^+ .$$

Table 1: Values for American Call Options on a Single Asset Paying Continuous Dividends Obtained Using a Piece-Wise Linear Threshold

K	T	Price	95% Lower	95% Upper	Lattice	Opt. Sim.
\$80	0.5	20.306	20.296	20.316	20.311	96
\$90	0.5	11.879	11.869	11.889	11.877	27
\$100	0.5	5.757	5.747	5.766	5.759	12
\$110	0.5	2.296	2.286	2.306	2.294	78
\$120	0.5	0.758	0.749	0.768	0.759	74
\$80	1.0	21.103	21.093	21.113	21.103	66
\$90	1.0	13.657	13.648	13.667	13.663	34
\$100	1.0	8.104	8.094	8.113	8.106	3
\$110	1.0	4.432	4.422	4.442	4.432	98
\$120	1.0	2.247	2.237	2.257	2.250	3
\$80	2.0	22.579	22.559	22.598	22.743	17
\$90	2.0	16.285	16.265	16.305	16.292	77
\$100	2.0	11.271	11.252	11.291	11.280	3
\$110	2.0	7.589	7.569	7.608	7.585	3
\$120	2.0	4.975	4.955	4.995	4.968	3
\$80	3.0	24.012	23.993	24.032	24.115	82
\$90	3.0	18.172	18.153	18.192	18.232	38
\$100	3.0	13.394	13.375	13.414	13.552	3
\$110	3.0	9.855	9.836	9.875	9.936	3
\$120	3.0	7.206	7.187	7.226	7.197	72

4.1 Piece-Wise Linear Threshold

To determine the piece-wise linear exercise boundary, we use the simulation-optimization routine and maximize the value of P_0 over all possible values of a_1, b_1, and b_2, subject to the constraints $b_2 \geq b_1$ and $b_2 \leq S_{max}$, where S_{max} is the maximum stock price observed on a trial simulation of the value of the underlying asset.

The values of American call options on an asset paying continuous dividends (where $\delta = 0.04$, $S_0 = \$100$, $\sigma = 20\%$, and $r = 0.05$) are shown for various values of K and T in Table 1, which also notes the number of the simulation that identified the optimal decision variable values. The lattice values are taken from Fu *et al.* (2001) and are obtained using 500 time steps. In each scenario, twenty potential early exercise dates are used. The simulation-optimization method captures the lattice value within a 95% confidence interval on all options where $T \leq 2.0$ (standard error of estimate is \$0.01 or less in all cases). For options where $T = 3.0$, the option is slightly undervalued for options where $K \leq \$110$. By using additional exercise dates for options with longer maturities, the option value can be increased.

Consider the option in Table 1 where $K = \$110$ and $T = 0.5$. Figure 4 shows four approximately optimal piece-wise linear exercise boundaries that give the same option value. Using a binomial lattice with 500 time steps, this option is valued at \$2.294 (Fu *et al.* 2001). Using simulation

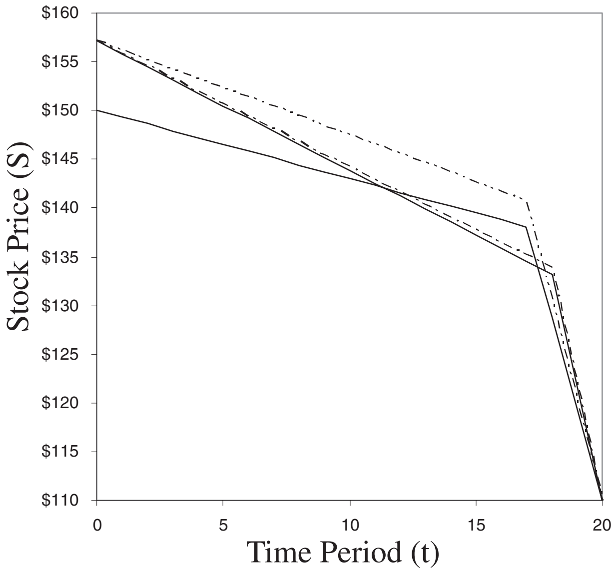

Figure 4: Multiple Piece-Wise Linear Early Exercise Thresholds that Yield the Same Option Value for an American Call Option on an Asset Paying Continuous Dividends

with each exercise boundary, the 95% confidence intervals all contain the lattice value at a precision of $0.01 or less standard error of estimate. In Section 4.3, we define the optimal exercise region for this option to understand the continuity of the option value across the boundary.

4.2 Bézier Curve Threshold

To determine the parameters for the Bézier curve threshold, we use the simulation-optimization routine and maximize the value of P_0 over all possible values of x_1, x_2, y_0, y_1, and y_2, subject to the constraints $y_0 \geq y_1 \geq y_2$, $y_0 \leq S_{max}$, $y_2 \geq K$, and $x_2 \geq x_1$. The constraints ensure that the Bézier curve is monotonically non-increasing. The constants a_x, b_x, c_x, a_y, b_y, and c_y are calculated from the optimal values of x_1, x_2, y_0, y_1, and y_2. The coordinates for the Bézier curve are calculated using equations (2) and (3). We solve for the values of $m_1, ...m_{40}$ such that $t_j(m_1) = 1, t_j(m_2) = 2, ..., t_j(m_{40}) = 40$. The values established for $m_1, ...m_{40}$ are then used to calculate Y_{t_j} for $j = 1, ..., 40$. Thus, we are calculating the option value based on 40 potential early exercise dates.

For the test option with $K = \$110$ and $T = 0.5$, the optimal values of the decision variables are $x_1 = x_2 = 32$ and $y_0 = y_1 = y_2 = S_{max} = 167.30$. With $x_3 = 40$ (which is the number of early exercise dates) and $y_3 = K = \$110$, this creates the Bézier curve threshold shown in Figure 5 overlayed on the various optimal piece-wise linear thresholds.

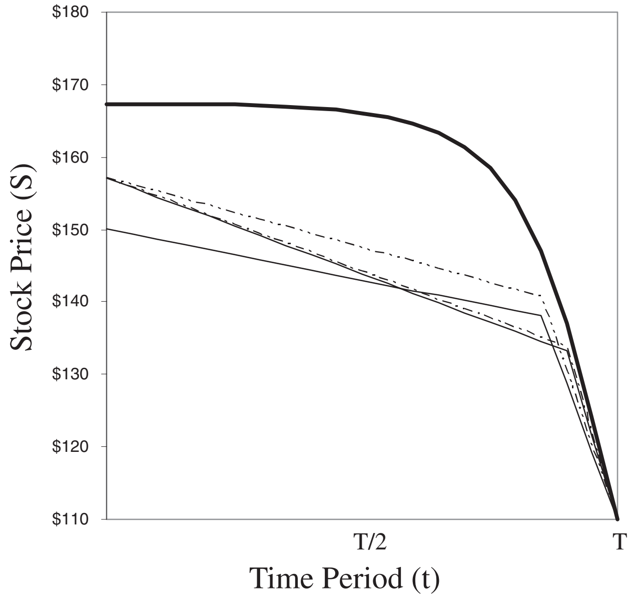

Figure 5: Bézier Curve Threshold for an American Call Option on a Single Asset Paying Continuous Dividends Overlayed on the Piece-Wise Linear Thresholds from Figure 4

Using the Bézier curve threshold with the simulation-optimization method yields an option value of $2.290 with a 95% confidence interval of [$2.270, $2.310].

4.3 Random Exercise Region

To price the American call option on a single stock paying continuous dividends using the random exercise region, we define

$$D_{t_j} = \mathbf{1}\left\{S_{t_j} \geq Y_{t_j}^U\right\} .$$

An additional indicator variable is defined as

$$R_{t_j} = \mathbf{1}\left\{\left(Y_{t_j}^L \leq S_{t_j} \leq Y_{t_j}^U\right) \cap \left(U_{t_j} \leq E_{t_j}\right)\right\} ,$$

where U_{t_j} are i.i.d. Uniform[0, 1] random variates and E_{t_j} is a random exercise threshold established for each t_j, $j = 0, ..., n$. The second criteria used to determine if $R_{t_j} = 1$ is an arbitrary random exercise rule and could be replaced by any other rule created by the owner of the option.

The indicator variable $A_{t_0} = D_{t_0}$ and

$$A_{t_j} = \mathbf{1}\left\{\sum_{\ell=0}^{j-1}(D_{t_\ell} + R_{t_\ell}) = 0\right\}$$

for all $j > 0$, $j = 1, ..., N$.

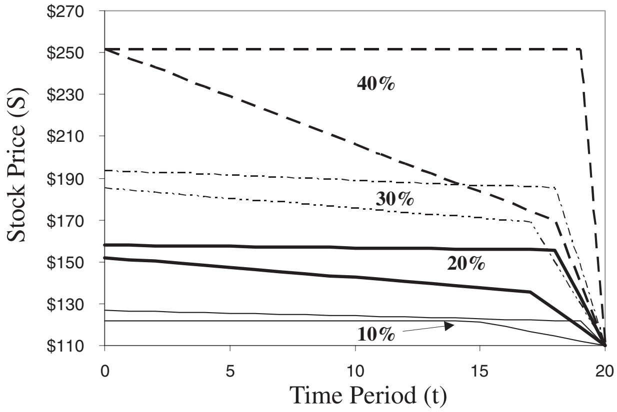

Figure 6: Random Exercise Region for an American Call Option on a Single Asset Paying Continuous Dividends

The discounted payoff function for the American call option on a single asset paying continuous dividends—using the random exercise region—is

$$P_{R0} = \sum_{j=0}^{N} \exp\{-rj\cdot(T/N)\}\cdot(D_{t_j}+R_{t_j})\cdot A_{t_j}\cdot(S_{t_j}-K)^+ .$$

The difference in the payoff function above and the one established for the piece-wise linear payoff function is that the option is exercised if the stock price exceeds the upper piece-wise linear boundary, or if the stock price falls between the lower and upper piece-wise linear boundaries and the random exercise criterion is met.

To determine the parameters for the random exercise region, we use the simulation-optimization routine and maximize the value of P_{R0} over all possible values of a_1, a_2, and b_1, ..., b_4, subject to the constraints $b_4 \geq b_3 \geq b_2 \geq b_1$, $b_4 \leq S_{max}$, and $a_2 \geq a_1$. The constraints ensure that the upper and lower-boundaries of the random exercise region are monotonically non-increasing and that the boundaries do not cross. The boundaries could coincide, which would reduce the model to the piece-wise linear model.

The random exercise region for the test option with $K = \$110$ and $T = 0.5$ is shown in Figure 6, along with the random exercise regions for similar options with volatility parameters of $\sigma = 10\%$, $\sigma = 30\%$, and $\sigma = 40\%$. For the case where $\sigma = 20\%$, the decision variable values which define the random exercise region are $a_1 = 17$, $a_2 = 18$, $b_1 = 135.75$, $b_2 = 152.09$, $b_3 = 155.69$, and $b_4 = 158.03$. This region is shown in Figure 7 overlayed on the various piece-wise linear exercise thresholds depicted in Figure 4. These values were obtained by using a value of $E_{t_j} = 0.5$ for all $j = 0, ..., 20$.

Using the random exercise region with the simulation-optimization method yields an option value of \$2.283 with a 95% confidence interval of [\$2.263, \$2.303]. Using the same values for the decision variables defining the random exercise region, we also value the option using values for E_{t_j} of 0.25 and 0.75 for all $j = 0, ..., N$. A comparison of option values for each random exercise rule is shown in Table 2. The value shown for the "PW Linear" method corresponds to the value for the option with the same parameters in Table 1.

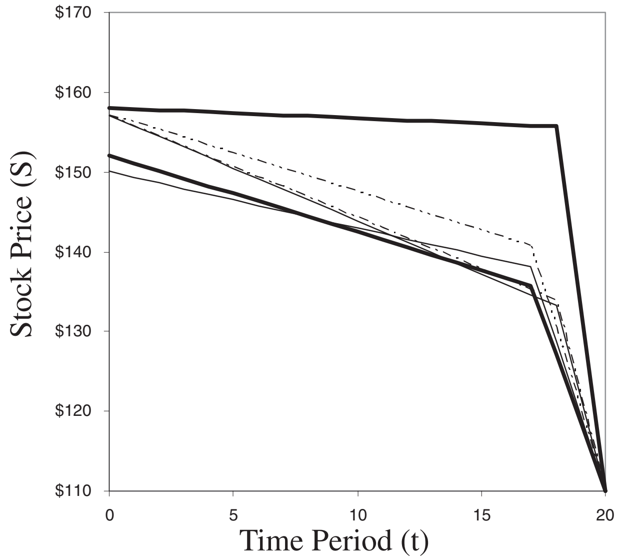

Figure 7: Random Exercise Region for an American Call Option on a Single Asset Paying Continuous Dividends Overlayed on Piece-Wise Linear Thresholds from Figure 4

Table 2: Values for American Call Options on a Single Asset Paying Continuous Dividends

Method	E_{t_j}	Price	95% Lower	95% Upper
PW Linear	—	2.296	2.286	2.306
Random	0.25	2.286	2.266	2.305
Random	0.50	2.283	2.263	2.303
Random	0.75	2.282	2.262	2.302
Lattice	—	2.294	—	—

5 CONCLUSIONS

We have demonstrated a method for parameterizing optimal exercise boundaries for pricing American-style options that uses simulation and optimization. By employing a scatter search approach and defining boundaries parameterized by just a few parameters we are able to use only forward simulation to identify the approximately optimal exercise boundary. The prices found for American options on a single asset paying continuous dividends are found to be comparable to those found using lattice methods. In future work, we plan to apply these methods to other American-style and exotic options.

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