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SIMULATION OF COHERENT RISK MEASURES

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ABSTRACT

In financial risk management, a coherent risk measure equals the maximum expected loss under several different probability measures, which are analogous to systems in ranking and selection. Here it is the best system's expected value and not identity that is of interest. We explore the correctness and computational efficiency of simulated confidence intervals for a maximum of several expectations.

1 INTRODUCTION

Artzner et al. (1999) introduced coherent risk measures as a recommendation for a basis of financial risk management. Authors including Jaschke and Küchler (2001) and Staum (2004) applied coherent risk measures to the problem of pricing derivative securities. The practice of financial risk management and derivative security pricing frequently involves simulation. With this application in mind, we develop a sequential (multi-stage) simulation algorithm which generates a fixed-width, two-sided confidence interval for a coherent risk measure that is the maximum among k expectations.

Any coherent risk measure ρ has a representation of the form $\rho(Y) = \sup_{\mathbf{P}\in\mathcal{P}} \mathrm{E}_{\mathbf{P}}[-Y/r]$, where Y is the value of a portfolio at a future time horizon, r is a stochastic discount factor which represents the time value of money, and $\mathcal{P}$ is a set of probability measures (Artzner et al. 1999, Prop. 4.1). Equations of a similar form hold for the related problems in derivative security pricing. We simplify the problem somewhat by assuming that the set $\mathcal{P}$ has only a finite number k of elements. This assumption will often hold, for instance, when the decision maker has designed the coherent risk measure (or the underlying acceptance set, in the case of derivative security pricing) by specifying k probability measures. If it does not hold, it may nonetheless hold for an approximation to the original problem, in which the new $\mathcal{P}$ has k elements whose convex hull contains the original $\mathcal{P}$. Financial simulations typically require large samples, which allows us to assume further, for purposes of theoretical analysis, that sample averages will be approximately normally distributed. Also let $X := -Y/r$ and $\mu_i := \mathrm{E}_{\mathbf{P}_i}[X]$. We now focus on inference for $\max_{i=1,\ldots,k} \mu_i$ based on data points $X_{ij} \sim \mathcal{N}(\mu_i, \sigma_i^2)$ which are independent of each other, and where the means and variances are all unknown.

The set-up is the same as that studied in the literature on ranking and selection, in which the primary goal is inference not about the maximum, but about the identity of the index attaining the maximum. Indeed, because of this commonality, the results here are applicable to the problem of selecting the best system, if one is also interested in knowing the value of the best system. (This is different from the value of the selected system.)

However, the problem of estimating the maximum is more difficult than that of selecting the best. To see this, we introduce some more notation. Define $[i]$ as the index of the ith smallest mean, $\mu_{[i]}$. Thus, in particular, $\mu_{[k]}$ is the largest: this is the quantity $\max_{i=1,\ldots,k} \mu_i$ we want to estimate. Let $\bar{X}_{[i]}$ be the average of a sample from the population whose mean is $\mu_{[i]}$. The problem features a natural bias: the most obvious estimator $\max_{i=1,\ldots,k} \bar{X}_i$ is an upper bound for, and has a larger expectation than, $\bar{X}_{[k]}$, whose mean is $\mu_{[k]}$. Even maximum likelihood estimation for this problem is not simple and produces remarkable results: see Dudewicz (1971).

Our point of departure is the theorem of Chen and Dudewicz (1976) providing a fixed-width, two-sided confidence interval for the maximum $\mu_{[k]}$, based on a two-stage sampling plan. We also draw on results of Nelson et al. (2001) to analyze a multi-stage simulation with screening: those systems which are very likely not to be the best are discarded so that thereafter computational resources can be devoted to simulating systems that are more likely to be the best.

2 ALGORITHMS WITH GUARANTEED COVERAGE

We use as a standard the two-stage procedure of Chen and Dudewicz (1976). In the first stage, it samples n_0 observations from each system. It then estimates the standard deviation of each system, and uses this to determine how many additional observations are required for each system to attain a minimum coverage guarantee for the confidence interval. In the second stage, it samples this additional data.

For simplicity of presentation, we first consider a two-stage algorithm with screening. It is a modification of the Chen-Dudewicz procedure, in which we screen out those systems which prove sufficiently uncompetitive in the first stage. We sample only from the remaining systems in the second stage. Subsequently we present a multi-stage algorithm, in which screening takes place between every stage.

To facilitate consistency of notation, henceforth let the first stage be denoted the 0th and the second be denoted the 1st. Let

$$\bar{X}_i := \frac{1}{n_0}\sum_{j=1}^{n_0} X_{ij} \quad \text{and} \quad S_i^2 := \frac{1}{n_0}\sum_{j=1}^{n_0}(X_{ij}-\bar{X}_i)^2$$

be the stage 0 sample average and sample variance. Then $\bar{X}_{[i]}$ is the stage 0 sample average associated with the population whose mean is $\mu_{[i]}$. Let the total number of samples from system i taken by the end of the stage 1 be N_i; this is specified later in Equation (6). Define the stage 1 sample average

$$\bar{\bar{X}}_i := \frac{1}{N_i}\sum_{j=1}^{N_i} X_{ij}.$$

Finally, let F_ν be the t distribution with $\nu := n_0 - 1$ degrees of freedom.

We want a two-sided confidence interval of the form

$$\left(\bar{\bar{X}}_{(k)} - a, \bar{\bar{X}}_{(k)} + b\right) \tag{1}$$

with error bounds

$$\Pr\left[\mu_{[k]} \le \max_{i\in I} \bar{\bar{X}}_i - a\right] \le \alpha \tag{2}$$

and

$$\Pr\left[\mu_{[k]} \ge \max_{i\in I} \bar{\bar{X}}_i + b\right] \le \beta \tag{3}$$

and having fixed width $L := a+b$. The reason for specifying the confidence for the lower and upper confidence limits separately is the asymmetry of the financial problem. It may be considered worse to underestimate risk than to overestimate it; or worse to set the price of a derivative security too low, thus incurring losses, than to set it too high, thus failing to make sales. If so, one would choose $\beta < \alpha$.

2.1 A Two-Stage Algorithm

To begin with, choose a width L and confidence levels $1-\alpha$ and $1-\beta$. There is also freedom to choose the first-stage sample size n_0 and to decompose the upper confidence level as $1-\beta = (1-\beta_0)(1-\beta_1)$ where β_0 is the error bound allocated to screening and β_1 is the error bound allocated to mean estimation. Let

$$a = L\frac{F_\nu^{-1}((1-\alpha)^{1/k})}{F_\nu^{-1}((1-\alpha)^{1/k}) + F_\nu^{-1}(1-\beta_1)} \tag{4}$$

and

$$b = L\frac{F_\nu^{-1}(1-\beta_1)}{F_\nu^{-1}((1-\alpha)^{1/k}) + F_\nu^{-1}(1-\beta_1)}. \tag{5}$$

Take the stage 0 sample of X_{ij} for $i = 1,\ldots,k$ and $j = 1,\ldots,n_0$. Compute the sample averages $\bar{X}_i$ and variances S_i^2.

Construct the set

$$I := \left\{i | \forall j \ne i, \bar{X}_i \ge \bar{X}_j - W_{ij}\right\}$$

where

$$W_{ij} := F_\nu^{-1}\left((1-\beta_0)^{1/(k-1)}\right)\sqrt{(S_i^2+S_j^2)/n_0}.$$

This is the set of systems which are not too unlikely to be the best, in the sense of not being statistically dominated by some other system at stage 0. Every $i \notin I$ has been screened out.

For all $i \in I$, let the sample size by the end of stage 1 be

$$N_i := \max\left\{n_0, \left\lceil\left(\frac{S_i F_\nu^{-1}(1-\beta_1)}{b}\right)^2\right\rceil\right\} \tag{6}$$

and sample X_{ij} for $i \in I$, $j = n_0+1,\ldots,N_i$. Compute the stage 1 sample averages $\bar{\bar{X}}_i$, choose the greatest, and from it compute the confidence interval as in (1).

There is a tension in choosing n_0. If it is too large, then excessive resources are spent, as one may wish to have $N_i < n_0$, which is impossible. If it is too small, then there is insufficient information to screen out poor systems. This motivates the introduction of a multi-stage algorithm, which provides multiple opportunities to screen out poor systems.

2.2 A Multi-Stage Algorithm

In this algorithm, there are m screening stages and one final estimation stage. The upper confidence level decomposes as $1-\beta = \prod_{\ell=0}^{m}(1-\beta_\ell)$ where β_m is for the final estimation stage and $\beta_0, \ldots, \beta_{m-1}$ are for the m screening stages.

Stage 0 is the same as in the previous subsection, with sample size n_0 for each system. Construct in the same way the set I of systems that are not screened out. We need at this point to compute the total sample sizes $N_i(\ell)$ for system i achieved by the end of each stage $\ell > 0$. There is substantial freedom to do this.

We choose to do so on the following principles. First, the standard error of the sample average should be equal for all systems that have not been screened out. Second, this standard error should decrease by a constant factor C between each stage $1, \ldots, m$. Third, the final sample size should be (much as in the previous subsection)

$$N_i(m) = \max\left\{n_0, \left\lceil \left(\frac{S_i F_v^{-1}(1-\beta_m)}{b}\right)^2 \right\rceil\right\}.$$

To satisfy these, use

$$N_i(\ell) = \left\lceil n_0 \left(C^{\ell-1}\frac{S_i}{\min_{j\in I} S_j}\right)^2 \right\rceil$$

where

$$C = \left(\frac{F_v^{-1}(1-\alpha_m)\min_{j\in I} S_j}{b\sqrt{n_0}}\right)^{1/(m-1)}.$$

After each stage $\ell = 1, \ldots, m$, compute the sample averages $\bar{X}_i(\ell) := \sum_{j=1}^{N_i(\ell)} X_{ij}/N_i(\ell)$ for those systems i that have not been screened out, i.e. $i \in I(\ell-1)$ where the screening procedure is defined by

$$I(\ell) := \left\{i | \forall j \in I(\ell-1)\setminus\{i\}, \bar{X}_i(\ell) \geq \bar{X}_j(\ell) - W_{ij}(\ell)\right\}$$

where $I(0) = I$ and

$$W_{ij}(\ell) := F_v^{-1}\left((1-\beta_\ell)^{1/(k-1)}\right)\sqrt{\frac{S_i^2}{N_i(\ell)} + \frac{S_j^2}{N_j(\ell)}}.$$

For ease of theoretical analysis, the preceding formula uses stage-1 sample variances; they are not updated for purposes of computing screening thresholds.

In the end, the confidence interval is as in (1), with final sample average $\bar{\bar{X}}_i = \bar{X}_i(m)$.

3 COMPUTATIONAL RESULTS

We test the performance of our algorithms in pricing a basket put option. This is a derivative security whose payoff at a terminal time T is $\max\{0, K - w^\top S(T)\}$ where K is a contractually specified strike price, w is a vector of weights, and $S(T)$ is the vector of terminal prices of the securities in the basket. The basket put is the right to sell the basket of securities for the strike price K at time T. If the underlying security price vector S obeys the Black-Scholes model, the basket put's price should be its expected discounted payoff.

Under this model, the price vector S follows multivariate geometric Brownian motion with drift r, the risk-free interest rate, and some covariance matrix Σ. That is, $\ln S_j(T) = \ln S_j(0) + (r - \|\sigma_j\|^2/2)T + A_j Z$ where A is a matrix satisfying $AA^\top = \Sigma$ and Z is a multivariate standard normal random vector. The short-term interest rate r is observable, and there are standard methods for calibrating the individual underlying securities' volatility magnitudes $\|\sigma_j\|$, whether from historical data or by fitting to observable prices of market-traded options on the underlying securities. However, estimation of the correlation matrix poses a greater problem. Unfortunately, the crucial quantity $\|w^\top \Sigma\|$, the volatility of the basket, depends significantly on the correlation matrix. There may be a range of plausible correlations and thus a range of plausible prices for the basket put.

We consider an example in which the basket is a weighted average of three security prices with weights $w_1 = 0.5$, $w_2 = 0.3$, and $w_3 = 0.2$. The initial security prices are all 100, and the strike price $K = 85$. The interest rate r =5% and the volatilities have magnitudes $\|\sigma_1\| = 40\%$, $\|\sigma_2\| = 30\%$, and $\|\sigma_3\| = 20\%$. To account for uncertainty about correlations, we use the $k = 4^3 = 64$ probability measures produced by allowing each of three correlations to be 0.2, 0.35, 0.55, or 0.75. Although the payoff in this example is far from normally distributed, the sample averages were approximately normally distributed, and the minimum coverage guarantees for the confidence limits held in all of our computational experiments, which include 300 independently simulated confidence intervals.

We report in Tables 1 and 2 efficiency improvements for this example, expressed as the ratio of the average number of samples required by the procedure of Chen and Dudewicz (1976) to the average number required by our algorithms. The results are reported for the two-stage algorithm with various choices of n_0, the initial (stage 0) sample size, and for the multi-stage algorithm with 30 stages and $n_0 = 1000$. For each of four choices of confidence interval width, the best efficiency improvement of a two-stage algorithm is highlighted in bold type.

In all experiments, one fifth of the error is allocated to the upper confidence limit, and four fifths to the lower confidence limit. For example, in the results of Table 1 for a 99%

confidence interval, the probability that the true maximum mean exceeds the upper confidence level is guaranteed to be no more than $\beta = 0.2\%$, while the probability that it falls below the lower confidence level is guaranteed to be no more than $\alpha = 0.8\%$.

For ease of interpretation, we specify the confidence interval width L relative to the true value $\mu_{[k]}$, as estimated in advance by a very precise simulation. To assign L equal to a fraction of an estimate of $\mu_{[k]}$ after stage 0 would introduce additional complications. In financial applications, there is often a previous simulation with similar parameters, which can supply a value of L giving approximately the desired relative precision.

Table 1 uses levels of confidence and precision appropriate for a derivative pricing problem. The error probability bound $\beta = 0.2\%$ is very low because offering to sell a derivative security at a low price can lead to large losses, which can be tolerated only infrequently. We consider confidence interval widths of 0.1% to 1% of the true value, which are comparable to or slightly smaller than typical bid-ask spreads. That is, at greater widths, one would be unable to quote competitive prices. Lesser widths would be unnecessarily precise.

Table 1: Efficiency Improvement, 99% Confidence

Width of CI	0.1%	0.2%	0.5%	1%
2-stage, $n_0 = 50000$	9.7	9.1	9.1	7.8
2-stage, $n_0 = 100000$	13	14	12	**8.3**
2-stage, $n_0 = 200000$	22	17	**14**	6.3
2-stage, $n_0 = 500000$	**39**	**29**	9.8	3.4
2-stage, $n_0 = 1000000$	35	22	6.0	1.7
multi-stage, $m = 30$	**43**	**42**	**36**	**27**

Table 2 is appropriate for a risk management problem requiring lower confidence and precision. Risk management is more a matter of decisions internal to a firm, so there are no customers to take advantage of violations of the upper confidence limit in the 1% of cases where it occurs, or whose business is lost when the upper confidence limit is too far above the true value.

Table 2: Efficiency Improvement, 95% Confidence

Width of CI	0.5%	1%	2%	5%
2-stage, $n_0 = 5000$	2.6	2.7	2.6	2.3
2-stage, $n_0 = 10000$	4.0	4.1	3.8	**2.5**
2-stage, $n_0 = 20000$	6.3	5.8	**4.8**	2.0
2-stage, $n_0 = 50000$	9.6	**7.8**	4.1	1.0
2-stage, $n_0 = 100000$	**14**	7.6	2.7	0.5
2-stage, $n_0 = 200000$	12	5.0	1.5	0.3
multi-stage, $m = 30$	**36**	**24**	**13**	**4.5**

These tables both show that the performance of the two-stage algorithm depends significantly on the initial sample size n_0. When n_0 is small, increasing it tends to lead to improved screening, as more information at stage 0 allows more suboptimal systems to be screened out. If n_0 becomes too large, computational resources are wasted on poor systems that could have been screened out earlier and on systems with low standard deviation, for which one would have liked to set $N_i < n_0$ if this were possible—see Equation (6). However, because many financial simulations are repeated with parameters only slightly different from those at the previous repetition, a good value of n_0 may well be known in advance.

Nonetheless, the performance of the multi-stage algorithm is entirely superior in the examples here. It overcomes limitations of the two-stage algorithm by using a small initial sample size $n_0 = 1000$, but continuing screening at subsequent stages. There seems to be little problem in choosing the multi-stage algorithm's parameters for an entirely unfamiliar simulation, which makes it superior to the two-stage algorithm. The following investigation of the sensitivity of the multi-stage algorithm's performance to its parameters is done at 95% confidence and for a confidence interval width of 5%.

Figure 1 shows that the efficiency of the multi-stage algorithm has low local sensitivity to the number of stages.

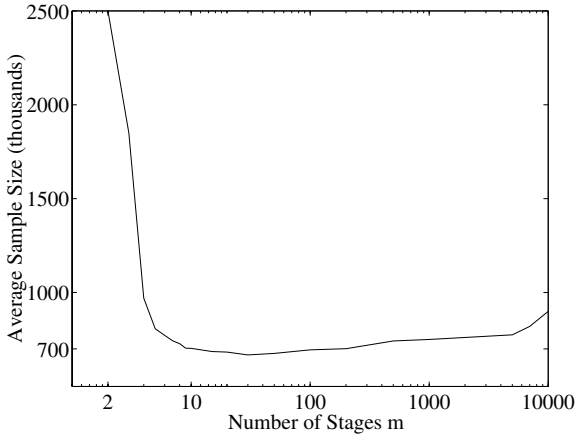

Figure 1: Number of Stages and Efficiency

Figure 2 shows that the impact of initial sample size n_0 on the algorithm's efficiency is not negligible, but is not as dramatic as it is for the two-stage algorithm. Varying n_0 from 200 to 2000 caused efficiency to change by less than 5%. However, $n_0 = 1000$ is not very close to optimal, but noticeably too large, if the required precision is low and the variances are much smaller (say, one tenth as large) relative to the differences in expectations. Still, the n_0 problem is much less severe than for the two-stage algorithm: $n_0 = 1000$ is close to optimal for a fairly wide range of variances and confidence interval widths L.

The performance of the multi-stage algorithm also has little local sensitivity to the decomposition of the upper confidence level $1-\beta$ into confidence $1-\beta_m$ for estimation

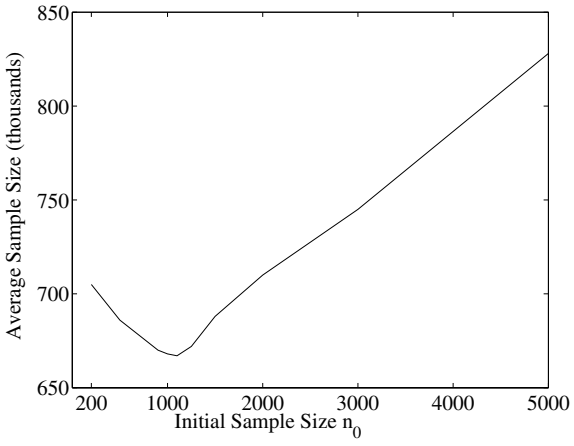

Figure 2: Initial Sample Size and Efficiency

and $1-\beta_\ell$ for screening at stage $\ell = 0, \ldots, m-1$. In the examples reported here, we have chosen $\beta_0 = \cdots = \beta_{m-1}$ and $\prod_{\ell=0}^{m-1}(1-\beta_\ell) = 1-\beta/5$, but as Figure 3 shows, there is little change in performance for nearby values of the overall screening confidence level. Allocating too little of the error to screening makes it very difficult to screen out systems; allocating too little of the error to estimation inflates the required final sample size $N_i(m)$ for a system $i \in I(m-1)$ that is never screened out.

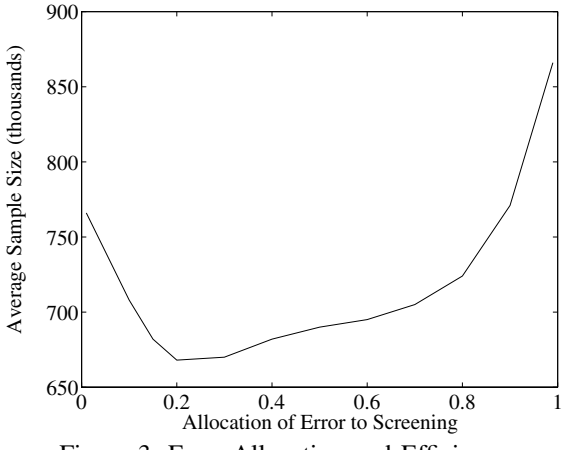

Figure 3: Error Allocation and Efficiency

4 PROOF OF MINIMUM COVERAGE

We present a proof for the two-stage algorithm. A generalization shows that the error bounds (2) and (3) hold for the multi-stage algorithm too.

4.1 Lower Confidence Limit

The basis for bounding

$$p := \Pr\left[\mu_{[k]} \geq \max_{i\in I} \bar{\bar{X}}_i - a\right] \geq 1-\alpha$$

is the motivating observation that $\max_{i\in I} \bar{\bar{X}}_i \leq \max_{i=1,\ldots,k} \bar{\bar{X}}_i$. Even for a system $j \notin I$, i.e. which has been screened out, $\bar{\bar{X}}_j$ is defined on the probability space, although we do not simulate it. So we have

$$\begin{aligned} p &\geq \Pr\left[\mu_{[k]} \geq \max_{i=1,\ldots,k} \bar{\bar{X}}_i - a\right] \\ &= \Pr\left[\forall i = 1,\ldots,k,\ \bar{\bar{X}}_i \leq \mu_{[k]} + a\right] \\ &\geq \Pr\left[\forall i = 1,\ldots,k,\ \bar{\bar{X}}_i \leq \mu_i + a\right] \end{aligned}$$

because $\mu_i \leq \mu_{[k]}$. Using independence,

$$p \geq \prod_{i=1}^{k} \Pr\left[\bar{\bar{X}}_i \leq \mu_i + a\right] = \prod_{i=1}^{k} \Pr\left[\frac{\bar{\bar{X}}_i - \mu_i}{S_i/\sqrt{N_i}} \leq \frac{a\sqrt{N_i}}{S_i}\right].$$

From (4) and (5), $a = bF_\nu^{-1}((1-\alpha)^{1/k})/F_\nu^{-1}(1-\beta_1)$, while from (6), $b\sqrt{N_i}/S_i \geq F_\nu^{-1}(1-\beta_1)$. Therefore $a\sqrt{N_i}/S_i \geq F_\nu^{-1}((1-\alpha)^{1/k})$ so

$$p \geq \prod_{i=1}^{k} F_\nu\left(F_\nu^{-1}((1-\alpha)^{1/k})\right) = 1-\alpha.$$

4.2 Upper Confidence Limit

The probability of interest is

$$\begin{aligned} q &:= \Pr\left[\mu_{[k]} \leq \max_{i\in I} \bar{\bar{X}}_i + b\right] \\ &\geq \Pr\left[\mu_{[k]} \leq \bar{\bar{X}}_{[k]} + b, [k] \in I\right] \\ &= \Pr\Big[\mu_{[k]} \leq \bar{\bar{X}}_{[k]} + b, \\ &\qquad \forall j \neq k\ \bar{X}_{[k]} \geq \bar{X}_{[j]} - W_{[k],[j]}\Big]. \end{aligned} \quad (7)$$

Define

$$Z_k := \frac{\bar{\bar{X}}_{[k]} - \mu_{[k]}}{\sigma_{[k]}/\sqrt{N_{[k]}}}$$

and for $j \neq k$,

$$Z_j := \frac{(\bar{X}_{[k]} - \bar{X}_{[j]}) - (\mu_{[k]} - \mu_{[j]})}{\sqrt{(\sigma^2_{[k]} + \sigma^2_{[j]})/n_0}}.$$

The probability (7) can be rewritten as $\Pr[\cap_{i=1}^{k} E_i]$ where the event E_k is that $-Z_k \le b\sqrt{N_{[k]}}/\sigma_{[k]}$ and for $j \ne k$, the event E_j is that $-Z_j \le (W_{[k],[j]}+\mu_{[k]}-\mu_{[j]})/\sqrt{(\sigma^2_{[k]}+\sigma^2_{[j]})/n_0}$.

Now we need to condition on the first-stage sample variances, because they appear in the event E_j for $j \ne k$ through $W_{[k],[j]}$, and also determine the sample sizes $N_{[k]}$, which is present in E_k. Let $\mathcal{F}$ represent the information in $(S_1^2, \ldots, S_k^2)$. The conditional distribution of each Z_i is normal with mean 0. Their joint conditional distribution is such that each $\mathrm{Cov}[Z_i, Z_j|\mathcal{F}] > 0$. By Slepian's inequality (Hochberg and Tamhane 1987, Thm. A2.2.1),

$$\Pr[\cap_{i=1}^{k} E_i|\mathcal{F}] \ge \Pr[E_k|\mathcal{F}] \prod_{j\ne k} \Pr[E_j|\mathcal{F}].$$

Taking expectations,

$$\begin{aligned}\Pr[\cap_{i=1}^{k} E_i] &\ge \mathrm{E}\left[\Pr[E_k|\mathcal{F}] \prod_{j\ne k} \Pr[E_j|\mathcal{F}]\right] \\ &\ge \Pr[E_k] \prod_{j\ne k} \Pr[E_j]\end{aligned}$$

where the second line follows from Kimball's inequality (Hochberg and Tamhane 1987, Thm. A2.2.6).

The appendix of Nelson et al. (2001) shows that the product over $j \ne k$ is greater than or equal to $1-\beta_0$. This relates to the probability of correct screening:

$$\Pr[[k] \in I] \ge \prod_{j\ne k} \Pr[E_j] \ge 1-\beta_0.$$

The first factor

$$\begin{aligned}\Pr[E_k] &= \Pr\left[-\frac{\bar{\bar{X}}_{[k]}-\mu_{[k]}}{\sigma_{[k]}/\sqrt{N_{[k]}}} \le \frac{b\sqrt{N_{[k]}}}{\sigma_{[k]}}\right] \\ &= \Pr\left[\mu_{[k]} \le \bar{\bar{X}}_{[k]} + b\right] \\ &= \Pr\left[-\frac{\bar{\bar{X}}_{[k]}-\mu_{[k]}}{S_{[k]}/\sqrt{N_{[k]}}} \le \frac{b\sqrt{N_{[k]}}}{S_{[k]}}\right],\end{aligned}$$

and by (6), $b\sqrt{N_{[k]}}/S_{[k]} \ge F_\nu^{-1}(1-\beta_1)$, so this probability is at least $F_\nu\left(F_\nu^{-1}(1-\beta_1)\right) = 1-\beta_1$. This relates to the probability of coverage without screening:

$$\begin{aligned}\Pr\left[\mu_{[k]} \le \max_{i=1,\ldots,k} \bar{\bar{X}}_i + b\right] &\ge \Pr\left[\mu_{[k]} \le \bar{\bar{X}}_{[k]} + b\right] \\ &= \Pr[E_k] \ge 1-\beta_1.\end{aligned}$$

Putting all the pieces together, and using $1-\beta = (1-\beta_0)(1-\beta_1)$, we conclude that (3) holds.

5 CONCLUSIONS

We have introduced a multi-stage screening and selection algorithm for producing a simulated confidence interval for the maximum of several expectations. To choose good values of the algorithm's parameters (number of stages, initial sample size, and error allocation) does not require precise knowledge of the problem's characteristics; this and superior efficiency are advantages of the multi-stage algorithm over the two-stage algorithm. For the financial application of simulating a coherent risk measure of a basket put option, this algorithm was between 4.5 and 43 times faster than the procedure of Chen and Dudewicz (1976). The efficiency improvement is greater when the required levels of confidence and precision are higher, in which case it is possible for substantial screening to occur while the algorithm runs.

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REFERENCES

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath. 1999. Coherent measures of risk. *Mathematical Finance* 9: 203–228.

Chen, H. J., and E. J. Dudewicz. 1976. Procedures for fixed-width interval estimation of the largest normal mean. *Journal of the American Statistical Association* 71: 752–756.

Dudewicz, E. J. 1971. Maximum likelihood estimates for ranked means. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 35: 283-297.

Hochberg, Y., and A. C. Tamhane. 1987. *Multiple Comparison Procedures.* New York: John Wiley & Sons.

Jaschke, S., and U. Küchler. 2001. Coherent risk measures and good-deal bounds. *Finance and Stochastics* 5: 181–200.

Nelson, B. L., J. Swann, D. Goldsman, and W. Song. 2001. Simple procedures for selecting the best simulated system when the number of alternatives is large. *Operations Research* 49: 950–963.

Staum, J. 2004. Fundamental theorems of asset pricing for good deal bounds. *Mathematical Finance* 14: 141–161.

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